

Revision Notes On Laplace Transforms

C.T.J. Dodson, Manchester University

As always, trust nothing you read until you have tested it!

Function $f(t)$	Transform $\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$
1	$1/s$
$t^n, \text{ for } n = 0, 1, 2, \dots$	$n!/s^{n+1}$
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$
$t^{-1/2}$	$(\frac{\pi}{s})^{1/2}$
e^{at}	$1/(s-a)$
$\sin \omega t$	$\omega/(s^2 + \omega^2)$
$\cos \omega t$	$s/(s^2 + \omega^2)$
$t \sin \omega t$	$2\omega s/(s^2 + \omega^2)^2$
$t \cos \omega t$	$(s^2 - \omega^2)/(s^2 + \omega^2)^2$
$e^{at} t^n$	$n!/(s-a)^{n+1}$
$e^{at} \sin \omega t$	$\omega / ((s-a)^2 + \omega^2)$
$e^{at} \cos \omega t$	$(s-a) / ((s-a)^2 + \omega^2)$
$\sinh \omega t$	$\omega/(s^2 - \omega^2)$
$\cosh \omega t$	$s/(s^2 - \omega^2)$
Impulse (Dirac δ): $\delta(t-a)$ ($\neq 0$ at $t=a$, else $= 0$)	e^{-as}
Step function: $H_a(t)$ ($= 0$ for $t < a$ and $= 1, t \geq a$)	e^{-as}/s
Delay of g: $H_a(t)g(t-a)$	$e^{-as}\tilde{g}(s)$
Shift of g: $e^{at}g(t)$	$\tilde{g}(s-a)$
Convolution: $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau) d\tau$	$\tilde{g}(s)\tilde{f}(s)$
Integration: $1 * g(t) = \int_0^t g(\tau) d\tau$	$\frac{1}{s}\tilde{g}(s)$
Derivative: y'	$s\tilde{y} - y(0)$
y''	$s^2\tilde{y} - sy(0) - y'(0)$

1. Finding inverse transforms using partial fractions

Given an expression (a Laplace transform) of the form N/D where numerator N and denominator D are both polynomials of s , possibly in the form of factors, and N may be constant; use partial fractions:

- (i) if N has degree equal to or higher than D , divide N by D until the remainder is of lower degree than D
- (ii) For every linear factor like $(as + b)$ in D , write a partial fraction of the form $A/(as + b)$
- (iii) For every repeated factor like $(as + b)^2$ in D write two partial fractions of the form $A/(as + b)$ and $B/(as + b)^2$. Similarly for every repeated factor like $(as + b)^3$ in D write three partial fractions of the form $A/(as + b)$, $B/(as + b)^2$ and $C/(as + b)^3$; and so on.
- (iv) For quadratic factor $(as^2 + bs + c)$ write a partial fraction $(As + B)/(as^2 + bs + c)$.

For repeated quadratic factors write a series of partial fractions as in (iii), but with numerators of the form $(As + B)$ and successive powers of the quadratic factor as the denominators.

With a little more algebra you should in this way be able to write the original expression as a sum of simpler transforms, which are found in your table. Add their inverse transforms together, to get the inverse of the original transform. *Mathematica* and *Maple* have functions to calculate and invert Laplace Transforms.

2. Example: Differential equations involving impulse and step functions.

The equation below represents the dynamics of a point, initially at rest, moving away from the origin along the y -axis under a constant acceleration of value 10 for $0 \leq t < 1$ and an extra impulse acceleration of size 10 is applied at $t = 1$. This is like a simple rocket boost, but can you solve it any other way? Begin by sketching the graph of the acceleration, y'' , to show the step increase.

$$y'' = 10 + 10\delta(t - 1), \quad y(0) = y'(0) = 0.$$

Solution method

Transforming according to the table, we get

$$s^2\tilde{y} - sy(0) - y'(0) = \frac{10}{s} + 10e^{-s} \quad \text{so, rearranging, } \tilde{y} = \frac{10}{s^3} + \frac{10e^{-s}}{s^2} = 5\frac{2}{s^3} + 10e^{-s}\frac{1}{s^2}$$

$$\text{From the table } y(t) = 5t^2 + H(t-1)10(t-1)$$

Finally, by interpreting the step function $H(t-1)$ up to and after $t = 1$, we see that the impulse at $t = 1$ produces what we would expect: a discontinuity in **velocity** at $t = 1$. You should sketch the full solution:

$$y(t) = 5t^2 \quad \text{for } t \leq 1 \quad (\text{so here } y'(t) = 10t)$$

$$y(t) = 5t^2 + 10(t-1) \quad \text{for } t > 1 \quad (\text{so here } y'(t) = 10t + 10).$$

3. Application of Laplace Transform in mathematical statistics

Let f be a probability density function (pdf) for a positive random variable, so f is positive and of unit measure: $\int_0^\infty f(x) dx = 1$. The expectation operator E gives the mean m and variance $\sigma^2 < \infty$ by

$$E[x] = \int_0^\infty x f(x) dx = m \quad \text{and} \quad E[(x - m)^2] = \int_0^\infty x^2 f(x) dx - m^2 = \sigma^2.$$

The Laplace Transform of f is therefore the expectation of e^{-tx}

$$\phi_x(t) = E[e^{-tx}] = \int_0^\infty e^{-tx} f(x) dx. \quad (1)$$

Its n^{th} derivative is $\phi_x^{(n)}(t) = (-1)^n \int_0^\infty x^n e^{tx} f(x) dx$ and so in particular $\phi_x'(0) = -m$, $\phi_x''(0) = \sigma^2 + m^2$.

Lemma If y is another positive random variable, with probability density function g and y is independent of x then from (1) the pdf of $z = x + y$ has Laplace Transform

$$\phi_z(t) = E[e^{-tz}] = E[e^{-tx} e^{-ty}] = E[e^{-tx}]E[e^{-ty}] = \phi_x(t)\phi_y(t)$$

It follows that if x_1, x_2, \dots, x_n are independent random variables all with the same probability density function f , then the pdf of the sum $w = \sum_{j=1}^n x_j$ has Laplace Transform $\phi_w(t) = (\phi_x(t))^n$.

Theorem [Adapted from RG Laha *Ann. Math. Stat.* 25 (1954) 784-787]

Let x_1, x_2, \dots, x_n be independent positive random variables all with the same probability density function f . If the sum $\sum_{j=1}^n x_j$ is independent of the ratio

$$\left(\frac{\sum_{j=1}^n x_j^2}{\left(\sum_{j=1}^n x_j \right)^2} \right), \quad \text{then } f \text{ is a gamma distribution.}$$

Proof By the given hypothesis of independence,

$$E \left[\frac{\sum_{j=1}^n x_j^2}{\left(\sum_{j=1}^n x_j \right)^2} \left(\sum_{j=1}^n x_j \right)^2 e^{t \sum_{j=1}^n x_j} \right] = E \left[\frac{\sum_{j=1}^n x_j^2}{\left(\sum_{j=1}^n x_j \right)^2} \right] E \left[\left(\sum_{j=1}^n x_j \right)^2 e^{t \sum_{j=1}^n x_j} \right].$$

Using the above Lemma and dividing through by ϕ^n ,

$$n\phi_x''/\phi_x = K \left(n\phi_x''/\phi_x + n(n-1) (\phi_x'/\phi_x)^2 \right)$$

Now put $\psi(t) = \log \phi_x(t)$ and simplify to give

$$A\psi'' + B(\psi')^2 = 0, \quad \text{with } \psi(0) = 0, \psi'(0) = m, \psi''(0) = \sigma^2 \quad \text{so } \frac{\psi''(t)}{\psi'(t)^2} = -\frac{\sigma^2}{m^2}. \quad (2)$$

Put $\xi(t) = \psi'(t)$ then rewrite (2) as

$$-\frac{\psi''(t)}{\psi'(t)^2} = \frac{d}{dt} \frac{1}{\xi(t)} = \frac{\sigma^2}{m^2} \quad \text{hence } \xi(t) = \frac{m}{1 + (\sigma^2/m)t}.$$

Integrating $\xi(t) = \psi'(t)$ gives

$$\psi(t) = \log \phi_x(t) = (m^2/\sigma^2) \log[1 + (\sigma^2/m)t] \quad \text{finally } \phi_x(t) = (1 + (\sigma^2/m)t)^{m^2/\sigma^2}. \quad (3)$$

It can be checked that this ϕ_x is the Laplace Transform of the gamma probability density function

$$f(x) = \left(\frac{m}{\sigma^2} \right)^{(m/\sigma)^2} \frac{e^{-mx/\sigma^2}}{\Gamma(m^2/\sigma^2)} x^{(m/\sigma)^2 - 1}. \quad \square$$