2M1 Mathematics Coursework 2
Submit work in mailbox labelled 'Dodson' in MSS/N2 by end of Week 11

Please read the following notes carefully CTJD December 12, 2001

1. This coursework will be marked and the mark awarded will contribute towards the end of semester assessment. Marks will be reduced by 10% per day for coursework handed in late.
2. Work should be submitted on A4 sheets attached together. Do not write in pencil or red ink. Please print your full name clearly on each sheet of paper together with your department and course.
3. Calculators are permitted, except where stated. However, you should show your method of working — do not just write down the answer.
4. Answer all 4 questions, they carry equal marks.

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1. Consider the Earth as a sphere with radius taken as one unit length and the equatorial plane coinciding with the plane of x, y coordinates; the north and south poles are at $(0, 0, \pm 1)$. The Greenwich meridien is the circle of longitude 0^o passing through Greenwich and the point $(1, 0, 0)$ is just off the south coast of Ghana. Sketch a sphere showing these points and **by constructing rotation matrices** find the (x, y, z) coordinates of New Orleans, which lies at 90^oW, 30^oN.

2. **(a)** Let A and B be the following matrices

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -2 & 2 \\ -3 & 1 & 2 \end{pmatrix}$$

Find A^t, $2AB$ and $AB - BA$.

(b)

(i) Suppose that k is a real number, and

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & k & 0 \\ k & 1 & 0 \end{pmatrix}$$

(ii) For what values of k is A invertible?

(iii) Find A^{-1} when $k = 0$.

3. Solve the following partial differential equations for $u(t, x)$ using partial integration

(a) $u_{tx} = 5, \quad u_t(0, t) = e^{-t}, \; u(x, 0) = -1 + \log x.$

(b) $u_t = 2 + \sin(e^x), \quad u(x, 0) = \sin^{-1} x.$

(c) $u_{tt} = tx, \quad u(x, 0) = e^x, \; u_t(x, 0) = \cos x.$

(d) $u_{xxx} = 0, \quad u(0, t) = t, \; u_x(0, t) = e^t, \; u_{xx}(0, t) = \cos t.$

4. **(a)** Show that the 1-D wave equation

$$\frac{1}{4} u_{xx} = u_{tt} \quad \text{for} \quad 0 \le x \le 2 \text{ and } \; t \ge 0$$

has a solution given by $u(x, t) = \sin \frac{3\pi x}{2} \cos \frac{3\pi t}{4}$.

What were the boundary conditions at $x = 0$ and $x = 2$?

(b) For the inhomogeneous flow equation

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = kt^2, \quad \text{given initial values} \quad u(x, 0) = \sin x, \; 0 \le x \le \pi,$$

where k is a positive constant, use the method of characteristics to find a solution $u(x, t)$ for

$$0 \le x \le \pi, \; t \ge 0.$$

Solutions

1. The coordinates are given by rotations of $+30^o$ round the y-axis followed by -90^o round the z-axis:

$$\begin{pmatrix} \cos 90 & \sin 90 & 0 \\ -\sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30 & 0 & -\sin 30 \\ 0 & 1 & 0 \\ \sin 30 & 0 & \cos 30 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Sketch:

2.(a) We have

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -2 & 2 \\ -3 & 1 & 2 \end{pmatrix}$$

Now A^t is the *transpose* of A, i.e. A with the rows and columns exchanged (or *reflected*, if you will). Thus

$$A^t = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Now, we calculate AB and BA as follows:

$$AB = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & -2 & 2 \\ -3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 7 & 3 \\ 9 & -7 & 4 \\ 1 & 5 & 6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -2 & 2 \\ -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 9 & 0 \\ 6 & -6 & 8 \\ 1 & 8 & -5 \end{pmatrix}$$

So, we have

$$2AB = 2 \begin{pmatrix} -7 & 7 & 3 \\ 9 & -7 & 4 \\ 1 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -14 & 14 & 6 \\ 18 & -4 & 8 \\ 2 & 10 & 12 \end{pmatrix}$$

and for the last part

$$AB - BA = \begin{pmatrix} -7 & 7 & 3 \\ 9 & -7 & 4 \\ 1 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 9 & 0 \\ 6 & -6 & 8 \\ 1 & 8 & -5 \end{pmatrix} = \begin{pmatrix} -10 & -2 & 3 \\ 3 & -1 & -4 \\ 0 & -3 & 11 \end{pmatrix}$$

2.(b) A (square) matrix is invertible if, and only if, its determinant is not zero. So, we calculate the determinant of A by expanding along the top row:

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & k & 0 \\ k & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} k & 0 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ k & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & k \\ k & 1 \end{vmatrix}$$

$$= 1(k.0 \ - \ 0.1) - 0(2.0 \ - \ 0.k) + 2(2.1 \ - \ k.k) = 4 - 2k^2$$

So A is invertible so long as $4 - 2k^2 \neq 0$, i.e. $k^2 \neq 2$, $k \neq \pm\sqrt{2}$.

Now, when $k = 0$, A becomes:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and the determinant of A will be 4 (i.e. $4 - 2k^2$ with $k = 0$). We compute the cofactors as follows:

$$A_{11} = (0.0\ -\ 0.1) = 0 \qquad A_{12} = -(2.0\ -\ 0.0) = 0 \qquad A_{13} = (2.1\ -\ 0.0) = 2$$
$$A_{21} = -(0.0\ -\ 2.1) = 2 \qquad A_{22} = (1.0\ -\ 2.0) = 0 \qquad A_{23} = -(1.1\ -\ 0.0) = -1$$
$$A_{31} = (0.0\ -\ 2.0) = 0 \qquad A_{32} = -(1.0\ -\ 2.2) = 4 \qquad A_{33} = (1.0\ -\ 0.2) = 0$$

So, we put these values in a matrix, and divide by the determinant:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{pmatrix}$$

and we can check the result:

$$A\,A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{4} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as required.

3.

(a) $u_{tx} = 5, \quad u_t(0,t) = e^{-t},\ u(x,0) = -1 + \log x.$
$u_t = 5x + f(t) = 5x + e^{-t}$
$u = 5xt - e^{-t} + g(x) = 5xt - e^{-t} + \log x.$

(b) $u_t = 2 + \sin(e^x), \quad u(x,0) = \sin^{-1} x.$
$u = 2t + t\sin(e^x) + g(x) = 2t + t\sin(e^x) + \sin^{-1} x.$

(c) $u_{tt} = tx, \quad u(x,0) = e^x,\ u_t(x,0) = \cos x.$
$u_t = \frac{t^2}{2}x + f(x) = \frac{t^2}{2}x + \cos x$, so $u = \frac{t^3}{6}x + t\cos x + g(x) = \frac{t^3}{6}x + t\cos x + e^x.$

(d) $u_{xxx} = 0, \quad u(0,t) = t,\ u_x(0,t) = e^t,\ u_{xx}(0,t) = \cos t.$
$u_{xx} = f(t) = \cos t$
$u_x = x\cos t + g(t) = x\cos t + e^t$
$u = \frac{x^2}{2}\cos t + xe^t + h(t) = \frac{x^2}{2}\cos t + xe^t + t.$

4. (a)

$$\frac{1}{4}u_{xx} = u_{tt} \qquad \text{for} \quad 0 \le x \le 2 \text{ and } \ t \ge 0$$

This PDE has a solution given by $\ u(x,t) = \sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4}$, since

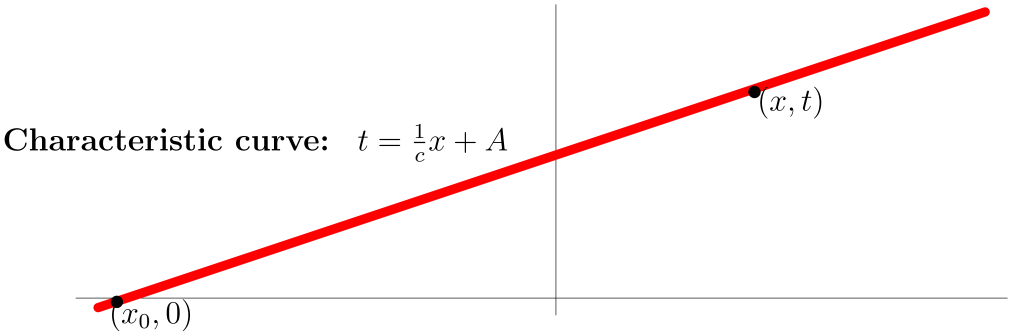
$$\begin{aligned}
\partial_x \sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4} &= \frac{3\pi}{2}\cos\frac{3\pi x}{2}\cos\frac{3\pi t}{4} \\
u_{xx} = \partial_{xx} \sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4} &= -\frac{9\pi^2}{4}\sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4} \\
\partial_t \sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4} &= -\frac{3\pi}{4}\sin\frac{3\pi x}{2}\sin\frac{3\pi}{4} \\
u_{tt} = \partial_{tt} \sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4} &= -\frac{9\pi^2}{16}\sin\frac{3\pi x}{2}\cos\frac{3\pi t}{4}, \quad \text{so } \frac{1}{4}u_{xx} = u_{tt}.
\end{aligned}$$

Boundary conditions, by substitution:

$$\begin{aligned}
u(0,t) &= \sin\frac{3\pi 0}{2}\cos\frac{3\pi t}{4} = 0 \\
u(2,t) &= \sin\frac{3\pi 2}{2}\cos\frac{3\pi t}{4} = 0.
\end{aligned}$$

(b)

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = kt^2, \quad \text{given initial values} \quad u(x,0) = \sin x,\ 0 \le x \le \pi,$$



On a characteristic curve :

$$\begin{aligned}
\frac{dt}{1} &= \frac{dx}{c} = \frac{du}{kt^2} \\
t &= \frac{1}{c}x + A, \quad \text{and } u = \int kt^2\,dt = \frac{kt^3}{3} + B \\
(x_0, 0) &= (-Ac, 0) \;\; \text{and so } x_0 = (x - ct) \\
u(x_0, 0) &= B = \sin x_0 = \sin(x - ct).
\end{aligned}$$

Hence, solution is

$$u(x,t) = \frac{kt^3}{3} + \sin(x - ct).$$