Mathematics 117
Lecture Notes for Curves and Surfaces Module

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These notes supplement the lectures and provide practise exercises. We begin with some material you will have met before, perhaps in other forms, to set some terminology and notation. Further details on unfamiliar topics may be found in, for example Cohn [3] for algebra, Dodson and Poston [5] for linear algebra, topology and differential geometry, Gray [6] for curves, surfaces and calculations using the computer algebra package *Mathematica*, and Wolfram [10] for *Mathematica* itself. Several on-line hypertext documents are available to support this course [1].

1 Sets and maps

A **function** or **map** from a set X to a set Y is a set of ordered pairs from X and Y (pairs like (x, y) are the coordinates in the graph of the function) satisfying the **uniqueness of image** property:

for all $x \in X$, there exists a **unique** $y \in Y$ that is related to the given x

Then we usually write $y = f(x)$ or just $y = fx$, and $f : X \to Y : x \mapsto f(x)$.
A map $f : X \to Y$ may have any or none of the following properties:

injectivity (1 to 1)	$f(x) = f(y)$ implies $x = y$
surjectivity (onto)	im $f = Y$; denoted $f : X \twoheadrightarrow Y$
bijectivity (both)	injectivity and surjectivity

We shall use sometimes the following common abbreviations:

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural, integer, rational, real, complex numbers.
$x \in V$	x is a member of set V.
$x \notin V$	x is not a member of set V.
$\exists\, x \in V$	There exists at least one member x in V.
$\forall\, x \in V$	For all members of V.
$W \subseteq V$	W is a subset of set V: so $(\forall\, x \in W)\ x \in V$.
$\{x \in V \mid p(x)\}$	The set of members of V satisfying property p.
$\emptyset$	The empty set.
$f : V \to W$	f is a map or function from V to W.
$f : x \mapsto f(x)$	f sends a typical element x to $f(x)$.
dom f	Domain of f: the set $\{x \mid \exists\, f(x)\}$.
im f	Image of f: the set $\{f(x) \mid x \in \text{dom } f\}$.
fU for $U \subseteq \text{dom } f$	Image of U by f: the set $\{f(x) \mid x \in U\}$.
$f^{\leftarrow}M$ for $M \subseteq \text{im } f$	Inverse image of M by f: the set $\{x \mid f(x) \in M\}$.
1_X	Identity map on x: the map given by $1_X(x) = x$ for all $x \in X$.
$U \cap V$	Intersection of U and V: the set $\{x \mid x \in U \text{ and } x \in V\}$.
$U \cup V$	Union of U and V: the set $\{x \mid x \in U \text{ or } x \in V \text{ or both}\}$.
$V \setminus U$	Complement of U in V: the set $\{x \in V \mid x \notin U\}$.
$f \circ g$	Composite of maps: apply g then f.
$\sum_{i=1}^{n} x_i$	Sum $x_1 + x_2 + \cdots + x_n$.
$\prod_{i=1}^{n} x_i$	Product $x_1 x_2 \cdots x_n$.
$\Rightarrow$	Implies, then.
$\Leftrightarrow$	Implies both ways, if and only if.
$a \times b$	Vector cross product of two vectors.
$a \cdot b$	Scalar product of two vectors.
$\|\|a\|\|$	Norm, $\sqrt{a \cdot a}$, of a vector.

2 Euclidean space $\mathbb{E}^3$

In geometry it is convenient and helpful to distinguish between $\mathbb{R}^3$, the **vector space** or **linear space** of triples of real numbers, and Euclidean space $\mathbb{E}^3$, the **point space** of triples of real numbers. Intuitively, we can think of a vector in $\mathbb{R}^3$ as an arrow corresponding to the *directed line* in $\mathbb{E}^3$ from one point (the blunt end of the vector arrow) to another point (the sharp end of the vector arrow). In this course we shall be concerned only with three dimensional $\mathbb{E}^3$ but the basic definitions of points, difference vectors and distances are the same for all $\mathbb{E}^n$ with $n = 1, 2, 3, \ldots$; of course, in dimensions higher than 3, the extra directions will arise from other features than ordinary space—such as time, temperature, pressure etc. The important fact to hang onto is that $\mathbb{E}^3$ consists of points represented by coordinates $p = (p_1, p_2, p_3)$ while the directed difference between a pair of such points p, q is a vector $\overrightarrow{pq}$ or $\overline{q-p}$ with components $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$. In modern mathematics, it is customary to omit the overbar or overarrow when writing vectors and this will be our usual practice; we identify vectors with their sets of components and points with their sets of coordinates.

Our main interest in this course is to develop the geometry of curves and surfaces in $\mathbb{E}^3$. The basic ideas are very simple: a curve is a continuous image of an interval and a surface is a continuous image of a product of intervals; in each case the intervals may be open or closed or neither.

2.1 Difference vectors and distances

The **difference map** gives the vector arrow from one point to another and is defined by

$$\text{difference} : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}^3 : (p, q) \mapsto v = \overrightarrow{pq}.$$

The **distance map** takes non-negative real values and is defined by

$$\text{distance} : \mathbb{E}^3 \times \mathbb{E}^3 \to [0, \infty) : (p, q) \mapsto ||\overrightarrow{pq}||$$

here, $||\ ||$ denotes the operation of taking the **norm** or absolute value of the vector, defined by

$$||(q_1 - p_1, q_2 - p_2, q_3 - p_3)|| = +\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

Then we can view $\mathbb{E}^3$ as the set of points representing ordinary space, together with the standard Euclidean angles and Pythagorean distances and $\mathbb{R}^3$ provides the vectors of **directed differences** between points. Not all books make this distinction so you need to be prepared to encounter the unstated identification $\mathbb{E}^3 = \mathbb{R}^3$. Often, we use the coordinates (x, y, z) for points in $\mathbb{E}^3$ and denote by $\mathbb{E}^2$ the set of points in $\mathbb{E}^3$ with $z = 0$ and then we abbreviate $(x, y, 0)$ to (x, y).

The **standard unit sphere** $\mathbb{S}^n$ in a Euclidean n-space is the set of points unit distance from the origin; we shall often use $\mathbb{S}^1$ in $\mathbb{E}^2$ and $\mathbb{S}^2$ in $\mathbb{E}^3$.

3 Curves

Given a point $p \in \mathbb{E}^3$ and a vector $v \in \mathbb{R}^3$, there is always a unique point $q \in \mathbb{E}^3$ such that $\overrightarrow{pq} = v$; intuitively, q is at the point of the arrow v when its tail is at p. Then we can write $q = p + v$ and the **line segment** from p (in the direction v) to q is given in parametric form by

$$L : [0, 1] \to \mathbb{E}^3 : t \mapsto p + tv$$

or equivalently

$$L : [0, 1] \to \mathbb{E}^3 : t \mapsto (p_1, p_2, p_3) + t(v_1, v_2, v_3)$$

and we say that the **tangent vector**, or **velocity vector** of this line at $t \in [0, 1]$ is v. Thus, we can write this as the derivative, which is the limit of differences:

$$D_t L = \lim_{h \to 0} \frac{L(t + h) - L(t)}{h} = \frac{dL}{dt} = v$$

because L is a linear function of t, since p and v are constant.

As we shall see, a curve may have a nonlinear dependence on its parameter and a velocity vector that varies in magnitude and direction, so curves are natural generalisations of line segments.

3.1 Tangent vector, speed and acceleration vector

A **curve** in $\mathbb{E}^3$ with parameter t satisfying $a \leq t \leq b$ is a continuous map

$$\alpha : [a,b] \to \mathbb{E}^3 : t \mapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \tag{1}$$

Note that α is a map and its image, (also called track or path), $\alpha([a,b])$, is a subset of $\mathbb{E}^3$; we keep these concepts distinct. The curve starts at the point $\alpha(a)$ and ends at the point $\alpha(b)$. Sometimes, one or both endpoints of the curve are absent; so in general the domain of a curve may be an interval of any kind.

For our purposes, we shall suppose that our curves are **differentiable**, in the sense that the components, $\alpha_1, \alpha_2, \alpha_3$, are real functions of t possessing derivatives of all orders—so no corners like those in the graph of $|x|$. The **tangent vector** or **velocity** of α is the vector valued map $D_t\alpha = \alpha'$ which in components is given by

$$\alpha' : [a,b] \to \mathbb{R}^3 : t \mapsto (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)) \tag{2}$$

and its **speed** is the absolute value of the velocity vector. The **acceleration** of α is the vector $\alpha'' = (\alpha')'$, given by the derivative of the velocity. Observe that the velocity and acceleration vectors are attached to the curve and change as the parameter moves the point of attachment.

We are particularly interested in **regular** curves which are differentiable and have nowhere zero velocity (they are always going somewhere, not stopped); then we make calculations easier if we choose the parameter set $0 < s < L$ to make α' a unit vector for all s, and L is actually the total length of the curve.

The **length** of the curve (1) is defined as the integral of the speed over the domain $[a,b]$

$$\text{Length}[\alpha] = \int_{[a,b]} \|\alpha'(t)\| = \int_a^b \sqrt{\alpha_1'(t)^2 + \alpha_2'(t)^2 + \alpha_3'(t)^2}\; dt \tag{3}$$

and it is independent of reparametrization. Unit speed curves are parametrized by arc length because then $\|\alpha'(t)\| = 1$ for all t. In general, it is difficult to calculate analytically the arc length of a given curve—because of the presence of the square root of sums of functions in the integrand; the same is true for other calculations for curves and surfaces but computer algebra software can help[1].

4 Plane Curves

A **plane curve** is a curve that lies in some plane in $\mathbb{E}^3$. If the curve lies in the $z = 0$ plane, then we may write the curve with just two components in the form

$$\alpha : [a,b] \to \mathbb{E}^2 : t \mapsto (\alpha_1(t), \alpha_2(t)) \tag{4}$$

In general, of course, whether a curve lies in a plane is not obvious from its equation; we shall construct in the section on space curves a function called **torsion** that measures departure from planarity for general curves in $\mathbb{E}^3$.

On regular plane curves, we can measure the curvature as the rate of change of the direction of a unit tangent vector with arc length. We call this the **signed curvature** $\kappa 2$, of α, defined by

$$\kappa 2[\alpha](t) = \frac{\alpha''(t) \cdot J\alpha'(t)}{\|\alpha'(t)\|^3}, \tag{5}$$

[1] On this course we use the computer algebra package *Mathematica* [10] to perform calculus and create graphics for curves and surfaces; Gray [6] provides the necessary *Mathematica* input to plot and study virtually all named curves and surfaces and perform analytic calculations on them—including the solution of geodesic equations on the surfaces and construction of curves with prescribed curvature and torsion. The necessary files can be found via the web server
`http://www.ma.umist.ac.uk/kd/mmaprogs/AREADMEFILE`
An important source of curves is from solutions of ordinary differential equations. Gray et al. [7] provided a definitive text on this subject, together with associated *Mathematica* functions to solve ordinary differential equations analytically and numerically, and plot families of solutions. The necessary files can be found via the web server `http://www.ma.umist.ac.uk/kd/ode/ode.htm`

in which J is the linear operator

$$J : \mathbb{R}^2 \to \mathbb{R}^2 : (p,q) \mapsto (-q,p), \quad \text{so always } J\alpha' \cdot \alpha' = 0. \tag{6}$$

At points where the curvature $\kappa 2$ is nonzero, $1/\kappa 2[\alpha]$ is called the **radius of curvature** of α. The rate at which the angular direction Θ of a regular plane curve changes can be calculated by differentiating its normalized velocity vector and we find that this coincides with the curvature for unit speed curves. Denote the speed by $v(t) = ||\alpha'(t)|| > 0$, then we deduce:

$$\frac{\alpha'(t)}{v(t)} = (\cos(\Theta(t), \sin(\Theta(t)) \tag{7}$$

$$\frac{v(t)\alpha''(t) + v'(t)\alpha'(t)}{v(t)^2} = \Theta'(t)(-\sin(\Theta(t), \cos(\Theta(t)) \tag{8}$$

$$J\alpha'(t) = v(t)(-\sin(\Theta(t), \cos(\Theta(t)) \tag{9}$$

$$\kappa 2(t) = \frac{\Theta'(t)}{v(t)}, \quad \text{using (6) and (5).} \tag{10}$$

So, we have proved the following for the case of constant $v = 1$ in (10):

Theorem 4.1 (Curvature of Plane Curves) *A regular unit speed plane curve has curvature $\kappa 2$ given by the rate of change with arc length of the angular direction of its tangent vector.*

In fact, $\kappa 2$ gives a complete classification of regular plane curves, up to a Euclidean motion:

Theorem 4.2 (Fundamental Theorem of Plane Curves) *Two regular plane curves defined on the same interval with the same curvature $\kappa 2$, can be transformed into one another by application of a translation and an orthogonal transformation.*

Gray [6] proves this classification theorem and studies applications in detail, giving many examples and a *Mathematica* algorithm for drawing a curve in $\mathbb{E}^2$ with specified curvature.

Exercises on plane curves

1. Show that a parametric equation for the line segment from $(3,5)$ to $(6,1) \in \mathbb{E}^2$ is given by

   $$\ell : [0,1] \to \mathbb{E}^2 : t \mapsto (3,5) + t(3,-4).$$

   Note that the line segment is the **image** of the map ℓ. What is the tangent vector of ℓ? Show that it has zero acceleration and zero curvature.

2. Find another parametric equation for the line segment in the previous example, but having a tangent vector with half the magnitude of that used for ℓ.

3. Clearly, there is a well-defined continuous curve given by

   $$h : [-1,1] \to \mathbb{E}^2 : t \mapsto (t, |t|). \tag{11}$$

   What happens to its tangent vector at $t = 0$? Show that the curve has a well-defined length.

4. Show that for all $\theta \in [0, 2\pi]$ the matrix

   $$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

   when applied to the coordinates of a curve (viewed as a column vector) rotates the curve through angle θ in the plane, that is, round the z-axis. Find a suitable θ value that rotates the curve in the previous question through 30^o; give an explicit equation for the rotated curve.

5. Why does the rotation matrix leave the length of a curve unaltered? What does the rotation matrix do to the curvature of a curve?

6. Can a reflection in the line $y = x$ be a rotation?

7. Find θ for the rotation matrix corresponding to the linear operator
$$J : \mathbb{R}^2 \to \mathbb{R}^2 : (p, q) \mapsto (-q, p)$$
and show that J gives an anticlockwise rotation of $\pi/2$ in $\mathbb{R}^2$.

8. Apply a rotation

9. Express the equation for the parabola $y = 3x^2 + 2$ in parametric form and find its velocity, speed and acceleration. Show that the same image can be obtained by another curve with velocity vector the negative of that of the first one.

10. Verify that a curve defining the unit circle in $\mathbb{S}^1 \subset \mathbb{E}^2$ with centre at the origin O given by the set
$$\mathbb{S}^1 = \{p \in \mathbb{E}^2 | \ \mathrm{dist}(p, O) = 1\}$$
has a parametric equation given by
$$f : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto (\cos t, \sin t).$$
Find its velocity, speed, acceleration, curvature and length. Show that the same image can be obtained by another curve with velocity vector the negative of that of the first one.

11. For a unit speed plane curve α, show that the acceleration is related to the curvature by $\alpha'' = \kappa 2[\alpha] J\alpha'$.

12. Find an expression for $\kappa 2[\alpha](t)$ in terms of its component functions.

13. Investigate the hyperbola
$$f : [-1, 1] \to \mathbb{E}^2 : t \mapsto (\cosh t, \sinh t).$$

14. Plot the limaçon (French name for slug—why?) given by
$$l : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto ((2\cos t + 1)\cos t, (2\cos t + 1)\sin t)$$
and show that the curve passes twice through the origin in different directions, which emphasises why we have to specify the parameter value and not the point on the curve when we require the velocity vector.

15. Find a parametric equation for the equator of the sphere $\mathbb{S}^2$, and for a perpendicular circle of longitude.

Implicitly defined plane curves
We know that some curves are defined implicitly, like the unit circle,
$$x^2 + y^2 - 1 = 0 \qquad (12)$$
However, for $f(x, y) = 0$ to define a parametrized curve near some point (x_0, y_0) where f is zero, it is sufficient for f to have at least one of its partial derivatives nonzero there.
Exercises on implicit curves

1. Use the implicit function theorem to prove this assertion.

2. Investigate the sets of zeros of the following function and a slightly perturbed version
$$f(x, y) = x^3 + y^3 - 3xy \qquad (13)$$
$$f^*(x, y) = x^3 + y^3 - 3xy - 0.01 \qquad (14)$$
for which Gray [6] gives graphs on pages 59 and 60.

5 Space Curves

In 3-space we take advantage of the usual vector algebra operations available on $\mathbb{R}^3$ to study the curvature (departure from linearity) and torsion (departure from planarity) of curves in space. Since we are interested in curves with nonzero speed everywhere, we can always reparametrize to have unit speed; then the parameter coincides with arc length along the curve, often denoted by s.

5.1 Curvature, torsion and the Frenet-Serret equations

Consider a unit speed space curve β, so its parameter is arc length s and we shall write $\dot{\beta} = \frac{d\beta}{ds}$

$$\beta : [a, b] \to \mathbb{E}^3 \tag{15}$$

The **curvature** of β is the norm of its acceleration

$$\kappa[\beta](s) = \|\ddot{\beta}(s)\| \tag{16}$$

It is easy to show that the velocity vector $\dot{\beta}$ is perpendicular to the acceleration vector $\ddot{\beta}$ by differentiating $(\dot{\beta} \cdot \dot{\beta}) = 1$. So if we take their cross product we get a vector perpendicular to both; we have only three dimensions and so the derivative of the new vector must be expressible in terms of the others. In this way, three, mutually perpendicular unit vectors $\{T, N, B\}$ arise at each point: $T = \dot{\beta}$, $N = \dot{T}/\kappa$ and $B = T \times N$. These vector functions along the curve β with curvature κ are controlled by the famous

Frenet-Serret equations for unit-speed curves:

$$\begin{aligned} \dot{T} &= \kappa N \qquad \text{recall that we have } \kappa > 0 & (17)\\ \dot{N} &= -\kappa T + \tau B & (18)\\ \dot{B} &= -\tau N & (19) \end{aligned}$$

Here, N is the **principal normal**, B is the **binormal** and τ is the **torsion**. $\{T, N, B\}$ is called the **Frenet frame field** along β, and consists of three mutually perpendicular unit vectors—a triad that moves along the curve with T pointing always forward. We can easily solve for τ as follows:

$$\begin{aligned} \dot{\beta} &= T, \qquad \ddot{\beta} = \dot{T} = \kappa N & (20)\\ \dot{\beta} \times \ddot{\beta} &= \kappa B, \qquad ||\dot{\beta} \times \ddot{\beta}|| = \kappa & (21)\\ \dddot{\beta} &= \dot{\kappa} N + \kappa \dot{N} = \dot{\kappa} N + \kappa(-\kappa T + \tau B) & (22)\\ \dot{\beta} \times \ddot{\beta} \cdot \dddot{\beta} &= \kappa^2 \tau. & (23) \end{aligned}$$

For a regular curve α with arbitrary speed $\sqrt{\alpha' \cdot \alpha'} = ||\alpha'|| = v > 0$, we have $\beta(s(t)) = \alpha(t)$, so $\dot{\beta} s' = \dot{\beta} v = \alpha'$ and $T' = \dot{T} v$, then

$$T = \alpha'/v, \qquad B = T \times N, \qquad B = \frac{\alpha' \times \alpha''}{||\alpha' \times \alpha''||} \tag{24}$$

and the **Frenet-Serret equations for arbitrary-speed curves**:

$$\begin{aligned} T' &= v\kappa N \qquad \text{recall that we have } \kappa > 0 & (25)\\ N' &= -v\kappa T + v\tau B & (26)\\ B' &= -v\tau N & (27) \end{aligned}$$

where

$$\begin{aligned} \kappa &= \frac{||\alpha' \times \alpha''||}{v^3} & (28)\\ \tau &= \frac{\alpha' \times \alpha'' \cdot \alpha'''}{||\alpha' \times \alpha''||^2}. & (29) \end{aligned}$$

5.2 Exercises

Here, β is the unit speed curve in equation (15).

1. Show that the helix

$$\gamma : [0, 10] \to \mathbb{E}^3 : s \mapsto (2\cos(\frac{s}{\sqrt{5}}), 2\sin(\frac{s}{\sqrt{5}}), \frac{s}{\sqrt{5}}) \tag{30}$$

   is a unit speed curve and has constant curvature and torsion.

2. Why do we always have $\kappa[\beta] \geq 0$?

3. For all s, $\ddot{\beta}(s) \cdot \dot{\beta}(s) = 0$; so the acceleration is always perpendicular to the acceleration along unit-speed curves. What about $\alpha'(t) \cdot \alpha''(t)$ on arbitrary speed curves?

4. Derive the Frenet-Serret equations for an arbitrary-speed regular curve α.

5. Viviani's curve[2] is the intersection of the cylinder $(x-a)^2+y^2=a^2$ and the sphere $x^2+y^2+z^2=4a^2$ and has parametric equation:

$$\alpha:[0,4\pi]\to\mathbb{E}^3:t\mapsto a(1+\cos t,\sin t,2\,\sin\frac{t}{2}).$$

   Show that it has curvature and torsion given by

$$\kappa(t)=\frac{\sqrt{13+3\,\cos t}}{a\,(3+\cos t)^{\frac{3}{2}}}\quad\text{and}\quad\tau(t)=\frac{6\,\cos\frac{t}{2}}{a(13+3\,\cos t)}.$$

6. Investigate the following curves for $n=0,1,2,3$

$$\gamma:[0,2\pi\sqrt{6}]\to\mathbb{E}^3:s\mapsto(\sqrt{6}\cos(\frac{s}{\sqrt{6}}),\sqrt{\frac{3}{2}}\sin(\frac{s}{\sqrt{6}}),\frac{\sqrt{3}}{2}\sin(\frac{ns}{\sqrt{6}}))\qquad(31)$$

7. Show that for all $\theta\in[0,2\pi]$ the matrix

$$R_z(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

   when applied to the coordinates of a curve in $\mathbb{E}^3$ rotates the curve through angle θ in the (x,y)-plane, that is, round the z-axis. Find a matrix $R_y(\theta)$ representing rotation round the y-axis and hence obtain explicitly the result of rotating the curves in the previous question by 60^o round the y-axis.

8. On plane curves, $\tau=0$ everywhere and we use the signed curvature $\kappa 2$, defined by

$$\kappa 2[\alpha](t)=\frac{\alpha''(t)\cdot J\alpha'(t)}{\|\alpha'(t)\|^3},\qquad\text{where } J \text{ is the linear operator }\; J:\mathbb{R}^2\to\mathbb{R}^2:(p,q)\mapsto(-q,p).\qquad(32)$$

   Show how $\kappa 2$ is related to κ for a general planar curve in $\mathbb{E}^3$, not necessarily in the $z=0$ plane.

9. Give an equation of a regular curve in $\mathbb{E}^2$ and then apply a rotation out of the $z=0$ plane. Show that for this rotated curve the torsion is zero and find the radius of curvature $1/\kappa 2$.

10. Investigate a selection of named curves from Gray [6, 1].

5.3 Classification of curves

Regular space curves with nonzero curvature are classified by their curvature and torsion, up to a Euclidean transformation (translation plus reflection and/or rotation):

Theorem 5.1 (Fundamental Theorem of Space Curves) *Two space curves defined on the same interval with the same torsion and nonzero curvature can be transformed into one another by application of a translation and a Euclidean transformation.*

Gray [6] proves this classification theorem and studies applications in detail, giving many examples and a *Mathematica* algorithm for drawing a curve in $\mathbb{E}^3$ with specified curvature and torsion.

6 Regular Surfaces

The notion of a surface is a generalization of that of a curve: a surface is a continuous image of a *product* of (any kinds of) intervals; in practice, we make surfaces like a patchwork quilt, from a collection of overlapping pieces. Recall that in our study of curves we concentrated on **regular**

[2] An animated Frenet-Serret frame graphic for this curve is given at `http://www.ma.umist.ac.uk/kd/mmaprogs/viviani.gif`

curves—which had nowhere zero derivatives for component functions. There are three parts to this: existence of the continuous curve, existence of its tangent vector at all points, nowhere zero speed. This meant that, locally, the curve was homeomorphic to open subintervals of its domain and had a well-defined tangent vector at all parameter values so no corners (tangent vectors are undefined at corners as we saw in (11) but globally the curve may have had self-intersections.

We seek the corresponding generalization to define regular surfaces such that locally they are homeomorphic to open sets (called **coordinate patches** or just **patches**) in $\mathbb{E}^2$ and with a well-defined tangent plane at every point (the apex of a cone does not have a well-defined tangent plane). To achieve this, we need the maps defining our surface to be injective and with rank 2 Jacobians on the patches; so our patches are not allowed to generate self-intersections in their images. This means that the 2×2 matrix of partial derivatives on the patch must have nonzero determinant.

6.1 Coordinates

Denote by

$$x : U \to \mathbb{E}^3 : (u, v) \mapsto (x_1, x_2, x_3) \tag{33}$$

the **coordinate patch map** for some surface M in $\mathbb{E}^3$. Then the partial derivatives of x are given by

$$x_u : U \to \mathbb{E}^3 \quad : \quad (u, v) \mapsto (\partial_u x_1, \partial_u x_2, \partial_u x_3) \tag{34}$$

$$x_v : U \to \mathbb{E}^3 \quad : \quad (u, v) \mapsto (\partial_v x_1, \partial_v x_2, \partial_v x_3) \tag{35}$$

At any point $p \in U$, the Jacobian of x has rank 2 if and only if at that point we have x_u, x_v linearly independent or equivalently the non-zero determinant

$$\begin{vmatrix} x_u \cdot x_u & x_u \cdot x_v \\ x_v \cdot x_u & x_v \cdot x_v \end{vmatrix} \neq 0. \tag{36}$$

The **arc length** function s of a curve α lying in the image of the patch map x in (33) satisfies

$$\frac{ds}{dt} = \sqrt{E\,\frac{du}{dt}^2 + 2F\,\frac{du}{dt}\frac{dv}{dt} + G\,\frac{dv}{dt}^2} \tag{37}$$

$$\text{with} \qquad E = x_u \cdot x_u, \quad F = x_u \cdot x_v, \quad G = x_v \cdot x_v \tag{38}$$

$$\text{equivalently} \qquad ds^2 = E\,dU^2 + 2F\,du\,dv + G\,dv^2 \tag{39}$$

The functions E, F, G are called the coefficients of the **first fundamental form** or of the **Riemannian metric** induced on $M \subset \mathbb{E}^3$.

Obviously, when we need more than one patch to define the surface (as for a sphere—why?, try wrapping a ball with paper!) then we want the change of patch maps to be bijective and differentiable with differentiable inverses—that is, *diffeomorphisms* on their overlaps. This allows us to define differentiability on a surface in terms of differentiability of components on local patches.

The image of a patch together with the induced map from the surface is called a **chart**; the collection of charts used to define the surface is called an **atlas** and any given surface may have many different choices of atlases. Many of our constructions generalize further from dimension 2 to arbitrary dimension n—giving **n-manifolds** [5]. As is often the case in mathematics, the big step is from one to more than one—from two to many is usually straightforward, until the many becomes infinite. If you want to try to imagine the 3-sphere $\mathbb{S}^3$, which sits in 4-space, think of it as the unit complex circle (so its coordinates are pairs of complex numbers (z, w)) in 2-dimensional complex space:

$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 |\ |z|^2 + |w|^2 = 1\}$$

which is clear as a generalization of the other form of the 1-sphere (circle):

$$\mathbb{S}^1 = \{z \in \mathbb{C} |\ |z|^2 = 1\}.$$

6.2 Exercises

1. Investigate the cup and saddle surfaces given by these patch maps:

$$f : (u, v) \to (u, v, u^2 + v^2)$$

$$g : (u, v) \to (u, v, u^2 - v^2)$$

Find two regular curves that pass through the origin perpendicularly in these surfaces.

2. Find the functions E, F, G for local patches on a plane, a cylinder, a sphere and a saddle; satisfy yourself that they generalize the Euclidean distance function and Pythagoras's theorem.

3. Show that the following conical surface is not regular:

$$c : [-1, 1] \times [-1, 1] \to \mathbb{E}^2 : (u, v) \mapsto (u, v, 2 - \sqrt{u^2 + v^2}) \tag{40}$$

You should show that curves through the apex do not have tangent vectors there.

4. Verify that a parametric equation for the unit 2-sphere in $\mathbb{E}^3$

$$\mathbb{S}^2 = \{p \in \mathbb{E}^3 | \ \mathrm{dist}(p, O) = 1\}$$

is given by

$$g : [0, 2\pi] \times [-\pi/2, \pi/2] :\to \mathbb{E}^3 : (u, v) \mapsto (\cos v \cos u, \cos v \sin u, \sin v).$$

Find a parametric equation for the equator of this sphere.

5. Find a parametric equation for a sphere of radius a.

6. Find a parametric equation for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with lengths of its principal semi-axes having values a, b, c.

7. The unit 2-sphere $\mathbb{S}^2$ has an atlas consisting of two charts

$$\{(U_N, \phi_N), \ (U_S, \phi_S)\}$$

where U_N consists of $\mathbb{S}^2$ with the north pole (n.p.) removed, U_S consists of $\mathbb{S}^2$ with the south pole (s.p.) removed, and the chart maps are stereographic projections. Thus, if $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centered at the origin then:

$$\phi_N : \mathbb{S}^2 \setminus \{n.p.\} \longrightarrow \mathbb{E}^2 : (x, y, z) \longmapsto \frac{1}{1+z}(x, y)$$

$$\phi_S : \mathbb{S}^2 \setminus \{s.p.\} \longrightarrow \mathbb{E}^2 : (x, y, z) \longmapsto \frac{1}{1-z}(x, y).$$

What are the patches corresponding to these charts?

8. The same type of atlas works also for $\mathbb{S}^n$, which is an example of an n-dimensional generalization of surfaces.

9. $\mathbb{R}^n$ has an atlas consisting of just one chart, the identity map.

10. Find another atlas for $\mathbb{S}^2$ consisting of projections of six hemispheres onto three perpendicular planes through the origin.

11. Find atlases for the cylinder $\mathbb{S}^1 \times (0, 1)$, and for the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

12. Gray [6] gives a parametrization of, among many other surfaces, a Möbius strip:

$$\mu : (0, 2\pi) \times (-0.3, 0.3) \to \mathbb{R}^3 : (u, v) \mapsto (\cos u + v \cos(\frac{u}{2}) \cos u, \sin u + v \cos(\frac{u}{2}) \sin u, v \sin(\frac{u}{2})) \tag{41}$$

Find an atlas for this surface.

7 Vector Fields

The tangent vector directions to curves through a point in a regular surface determine there the **tangent space** as a 2-d subspace of directions in $\mathbb{E}^3$; perpendicular to this subspace is the **normal direction** to the surface there. The **derivative** of any map f at a point p is that limit of differences giving the **best linear approximation** to f, at p. Thus, we need vector spaces to define linearity for maps between surfaces and these are automatically present when our surfaces have well-defined tangent planes everywhere.

7.1 Vector field and derivative

At each point p of a surface M in $\mathbb{E}^3$ we construct a vector space T_pM, called the **tangent space to** M **at** p and looking like $\mathbb{R}^2$, from the tangent vectors to curves in M that pass through p. Precisely:

$$T_pM = \{\alpha'(0) \in \mathbb{E}^3 | \alpha \text{ is a curve in } M \text{ starting at } p \in M\}$$

We can think of the tangent space as the plane tangent to the surface at the point p, so it looks like a copy of $\mathbb{R}^2$ with origin at p. A continuous choice of tangent vector on M is called a **tangent vector field** on M; a continuous choice of vector orthogonal to the tangent space is a **normal vector field** on M.

There is only one reasonable way to define the **derivative** D_wV of a vector field V on M in the tangent direction $w \in T_pM$:

$$D_wV = \lim_{t\to 0} \frac{V \circ \alpha(t) - V \circ \alpha(0)}{t} \tag{42}$$

where α is a curve in M beginning at p with $\alpha'(0) = w$.

7.2 Exercises

1. A west wind on the Earth corresponds to a tangent vector field on $\mathbb{S}^2$; vertically pointing vectors correspond to a normal vector field. On the cylinder, $\mathbb{S}^1 \times \mathbb{R}$, radial vectors form a normal vector field and surface vectors parallel to the axis form a tangent vector field.
2. Construct a tangent vector field on $\mathbb{S}^2$, and show that it must be zero somewhere [2, 4].
3. Check that D in (42) is linear in w and V, and find coordinate expressions for D_wV in terms of those for w and V for the cases of a unit tangent vector field and a unit normal vector field on the cylinder $\mathbb{S}^1 \times \mathbb{R}$.

8 Gauss Map and Orientability

We are familiar with a unit normal vector field on a sphere—simply extend each radius vector unit distance out from the surface. In general, such unit normal vector fields are harder to find but for some surfaces we have a quick method: Where x_u, x_v from (36) are linearly independent, their vector cross product defines a normal vector field $x_u \times x_v$; then we need only divide by the norm to obtain a unit normal vector field.

8.1 Gauss map

For surfaces in $\mathbb{E}^3$, the **Gauss map** of the patch map

$$x : U \to \mathbb{E}^3$$

is defined at regular points by

$$G : U \to \mathbb{S}^2 : (u, v) \mapsto \frac{x_u \times x_v}{\|x_u \times x_v\|} \tag{43}$$

If it is possible to make a continuous assignation of a unit normal vector over the whole of a regular surface in $\mathbb{E}^3$, (eg when $x_u \times x_v$ is nowhere zero) then we say that the surface is **orientable**. On orientable surfaces the Gauss map on a patch extends to a continuous map on the whole surface.

For $\hat{N}$ a unit normal vector field, differentiate $\hat{N} \cdot x_u = \hat{N} \cdot x_v = 0$ with respect to u and with respect to v. The **coefficients of the second fundamental form** of x are the real functions

$$e = -\hat{N}_u \cdot x_u, \ f = -\hat{N}_v \cdot x_u \text{ and } g = -\hat{N}_v \cdot x_v. \tag{44}$$

8.2 Exercises

1. On orientable surfaces there are exactly two choices of the continuous unit normal vector field $\hat{N}$.
2. The 2-sphere $\mathbb{S}^2$, the cylinder $\mathbb{S}^1 \times \mathbb{R}$ and the torus $\mathbb{S}^1 \times \mathbb{S}^1$ are all orientable, but a Möbius strip is not.

9 Shape Operator and Curvature on Regular Surfaces

Let $\hat{N}$ be a unit normal vector field defined in a neighbourhood of $p \in M$. Now, clearly $\hat{N}$ will change in different ways in different directions at each point. So, for each point and each direction there, we need to be able to calculate the rate of change of $\hat{N}$. The matrix of partial derivatives does this for us and we collect the various parts into an important map for characterizing the geometry of the surface.

9.1 Curvature

The **shape operator** measures the vectorial rate of change of the unit normal vector field $\hat{N}$ by means of the linear map [8]

$$S : T_pM \to T_pM : y \mapsto -D_y\hat{N} \tag{45}$$

The **normal curvature** of M at p in the direction $y \in T_pM$ is

$$k : T_pM \to \mathbb{R} : y \mapsto \frac{S(y) \cdot y}{\|y\|^2} \tag{46}$$

Now, if we restrict the normal curvature k to unit vectors in T_pM then we get a real-valued map on a set that is essentially $\mathbb{S}^1$, which is closed and bounded, so k achieves its bounds; the extreme values are called **principal curvatures** k_1, k_2, and determine the **principal directions** in the surface. In fact, the principal curvatures turn out to be the **eigenvalues** of the shape operator and the corresponding unit eigenvectors are the principal direction vectors.

Recall from linear algebra that a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ has a 3×3 matrix representation and $|det(A)|$ is the volume (up to sign) of the image under A of a unit cube (in 2-dimensional space $|det(A)|$ is the area up to sign of the image under A of a unit square). The trace of A is the sum of the diagonal elements of A, which is actually also the sum of the eigenvalues (including multiplicities). These functions applied to the shape operator give measures of the curvature of surfaces.

The **Gaussian curvature** K and the **mean curvature** H of a surface $M \subset \mathbb{E}^3$ are defined by

$$K : M \to \mathbb{R} \quad : \quad p \mapsto det(S)_{|p} \tag{47}$$

$$H : M \to \mathbb{R} \quad : \quad p \mapsto \frac{1}{2} trace(S)_{|p} \tag{48}$$

M is **flat** if and only if K is the zero function and is a **minimal surface** if H is the zero function. See Osserman [9] for an easily readable account of minimal surfaces.

Expressions for the Gaussian and mean curvatures in terms of the components of the first (39) and second (44) fundamental forms are

$$K = k_1 k_2 \quad = \quad \frac{eg - f^2}{EG - F^2} \tag{49}$$

$$\text{and} \tag{50}$$

$$H = \frac{1}{2}(k_1 + k_2) \quad = \quad \frac{eG - 2fF + gE}{2(EG - F^2)} \tag{51}$$

which are convenient for calculations.

9.2 Exercises—*Mathematica* will help in doing these

1. Find $\hat{N}, S$ and k in a neighbourhood of the origin for the surfaces:

$$\begin{array}{rcll} \text{plane} & p & : & (u,v) \to (u,v,2) \\ \text{parabolic cup} & f & : & (u,v) \to (u,v,u^2+v^2) \\ \text{saddle} & g & : & (u,v) \to (u,v,u^2-v^2) \\ \text{monkey saddle} & h & : & (u,v) \to (u,v,u^3-3uv^2) \end{array}$$

2. Investigate K and H for the surfaces in the previous example.

3. Show that the following are minimal surfaces:

$$\begin{array}{rcl} \text{helicoid} \quad \ell : [0,4\pi] \times [0,1] \to \mathbb{E}^3 & : & (u,v) \to (v\cos u, v\sin u, 2u) \\ \text{catenoid} \quad f : [0,4\pi] \times [0,1] \to \mathbb{E}^3 & : & (u,v) \to (\cosh v \cos u, \cosh v \sin u, v) \end{array}$$

   You can try to show also that the catenoid is the surface of revolution of a catenary curve:

$$\alpha : [0,1] \to \mathbb{E}^3 : t \mapsto (\cosh t, t).$$

4. Investigate a selection of named surfaces from Gray [6, 1].

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