Introducing Surfaces

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These notes supplement the lectures and provide practise exercises. We begin with some material you will have met before, perhaps in other forms, to set some terminology and notation. Further details on unfamiliar topics may be found in, for example Cohn [3] for algebra, Dodson and Poston [5] for linear algebra, topology and differential geometry, Gray [6] for curves, surfaces and calculations using the computer algebra package *Mathematica*, and Wolfram [10] for *Mathematica* itself. Several on-line hypertext documents are available to support this course [1].

Introduction

This document briefly summarizes definitions and hints at proofs of principal results for a first course on surfaces, beginning with an informal introduction to Euclidean space. It is intended as an *aide memoire*—a companion to lectures, tutorials and computer lab classes, with exercises and proofs to be completed by the student. Exercises include the statements to be verified—mathematics needs to be done, not just read!

The prereqisites are: elementary knowledge of Euclidean geometry and the definition of $\mathbb{R}^n$, familiarity with vector and scalar product, norms and basic linear algebra
(remember $dim\ dom\ =\ dim\ ker\ +\ dim\ im$?)
some basic topological concepts—compactness, covering space—and elementary group theory—free groups and quotients, presentation by relations, commutator subgroup.

Where possible, we encourage use of computer algebra software to experiment with the mathematics, to perform tedious analytic calculations and to plot graphs of functions that arise in the studies. For this purpose, we shall make use of Gray's book [6]—which contains all of the theory we need for curves and surfaces—and we use the computational packages he provides free in the form of Mathematica NoteBooks via [1]

For general information about the Mathematica software, see Wolfram's book [10] and the website Mathematica. For further study of more general differential geometry and its applications to relativity and spacetime geometry, see Dodson and Poston [5]. For an introduction to algebraic topology see Armstrong [2] and for more advanced topics and their applications in analysis, geometry and physics, see Dodson and Parker [4]. The abovementioned books contain substantial bibliography lists for further reference.

The document you are reading was created with LaTeX and a LaTeX tutorial is available at [1].

1 Sets and maps

A **function** or **map** from a set X to a set Y is a set of ordered pairs from X and Y (pairs like (x, y) are the coordinates in the graph of the function) satisfying the **uniqueness of image** property:

for all $x \in X$, there exists a **unique** $y \in Y$ that is related to the given x

Then we usually write $y = f(x)$ or just $y = fx$, and $f : X \to Y : x \mapsto f(x)$.

A map $f : X \to Y$ may have any or none of the following properties:

injectivity (1 to 1)	$f(x) = f(y)$ implies $x = y$
surjectivity (onto)	im $f = Y$; denoted $f : X \twoheadrightarrow Y$
bijectivity (both)	injectivity and surjectivity

We shall use sometimes the following common abbreviations:

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural, integer, rational, real, complex numbers.
$x \in V$	x is a member of set V.
$x \notin V$	x is not a member of set V.
$\exists\, x \in V$	There exists at least one member x in V.
$\forall\, x \in V$	For all members of V.
$W \subseteq V$	W is a subset of set V: so $(\forall\, x \in W)\ x \in V$.
$\{x \in V \mid p(x)\}$	The set of members of V satisfying property p.
$\emptyset$	The empty set.
$f : V \to W$	f is a map or function from V to W.
$f : x \mapsto f(x)$	f sends a typical element x to $f(x)$.
dom f	Domain of f: the set $\{x \mid \exists\, f(x)\}$.
im f	Image of f: the set $\{f(x) \mid x \in \text{dom } f\}$.
fU for $U \subseteq \text{dom } f$	Image of U by f: the set $\{f(x) \mid x \in U\}$.
$f^{\leftarrow}M$ for $M \subseteq \text{im } f$	Inverse image of M by f: the set $\{x \mid f(x) \in M\}$.
1_X	Identity map on x: the map given by $1_X(x) = x$ for all $x \in X$.
$U \cap V$	Intersection of U and V: the set $\{x \mid x \in U \text{ and } x \in V\}$.
$U \cup V$	Union of U and V: the set $\{x \mid x \in U \text{ or } x \in V \text{ or both}\}$.
$V \setminus U$	Complement of U in V: the set $\{x \in V \mid x \notin U\}$.
$f \circ g$	Composite of maps: apply g then f.
$\sum_{i=1}^{n} x_i$	Sum $x_1 + x_2 + \cdots + x_n$.
$\prod_{i=1}^{n} x_i$	Product $x_1 x_2 \cdots x_n$.
$\Rightarrow$	Implies, then.
$\Leftrightarrow$	Implies both ways, if and only if.
$a \times b$	Vector cross product of two vectors.
$a \cdot b$	Scalar product of two vectors.
$\|\|a\|\|$	Norm, $\sqrt{a \cdot a}$, of a vector.

2 Euclidean Space $\mathbb{E}^n$

We distinguish between $\mathbb{R}^n$ the real **vector space** of n-tuples of real numbers, and $\mathbb{E}^n$ the **affine point space** of n-tuples of real numbers with difference map:

$$\text{difference} : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}^n : (p, q) \mapsto q - p$$

Thus, we view $\mathbb{E}^n$ as the set of points, together with the standard Euclidean (Pythagorean) distance structure and angles, and $\mathbb{R}^n$ as providing the vectors of directed differences between points. Not all books make this distinction so you need to be prepared to encounter the unstated identification $\mathbb{E}^n = \mathbb{R}^n$. Find out more about Euclid at webpage [1].

The derivative of a map $f : \mathbb{E}^m \to \mathbb{E}^n$ at $p \in \mathbb{E}^m$ is the limit of differences that is the best linear approximation to f, at p. Thus, we need vector spaces to define linearity for maps between Euclidean spaces and suitable vector spaces are automatically present at each point of $\mathbb{E}^n$. At each point p in $\mathbb{E}^n$ we construct a vector space $T_p\mathbb{E}^n$, called the **tangent space to** $\mathbb{E}^n$ **at** p, from the directed difference vectors to lines in $\mathbb{E}^n$ that pass through p.

$$T_p\mathbb{E}^n = \{\alpha'(0) \in \mathbb{E}^n | \alpha \text{ is a line in } \mathbb{E}^n \text{ starting at } p \in \mathbb{E}^n\}$$

and α' is the n-tuple of derivatives of the coordinates of α.

Technically, we collect all of the $T_p\mathbb{E}^n$ together in one large product space:

$$T\mathbb{E}^n \cong \mathbb{E}^n \times \mathbb{R}^n$$

called the **tangent bundle** to $\mathbb{E}^n$, which comes equipped with a natural projection map onto its first component to keep track of the points to which tangent vectors are attached [5].

3 Euclidean Space $\mathbb{E}^3$

This is the space of our normal experience and we distinguish between $\mathbb{R}^3$, the **vector space** or **linear space** of triples of real numbers, and Euclidean 3-space $\mathbb{E}^3$, the **point space** of triples of real numbers. Intuitively, we can think of a vector in $\mathbb{R}^3$ as an arrow corresponding to the *directed line* in $\mathbb{E}^3$ from one point (the blunt end of the vector arrow) to another point (the sharp end of the vector arrow).

In this course we shall be concerned only with three dimensional $\mathbb{E}^3$ but the basic definitions of points, difference vectors and distances are the same for all $\mathbb{E}^n$ with $n = 1, 2, 3, \ldots$; of course, in dimensions higher than 3, the extra directions will arise from other features than ordinary space—such as time, temperature, pressure etc. The important fact to hang onto is that $\mathbb{E}^3$ consists of points represented by coordinates $p = (p_1, p_2, p_3)$ while the directed difference between a pair of such points p, q is a vector $\overline{q - p}$ with components $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$. In modern mathematics, it is customary to omit the overbar when writing vectors and this will be our usual practice; we identify vectors with their sets of components and points with their sets of coordinates.

The space $\mathbb{E}^3$ has one particularly important feature: the availability of the vector cross product on $\mathbb{R}^3$, which simplifies many geometrical proofs.

Our main interest in this course is to develop the geometry of curves and surfaces in $\mathbb{E}^3$. The basic ideas are very simple: a curve is a continuous image of an interval and a surface is a continuous image of a product of intervals; in each case the intervals may be open or closed or neither.

Difference vectors and distances The **difference map** gives the vector arrow from one point to another and is defined by

$$\text{difference} : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}^3 : (p, q) \mapsto v = q - p.$$

The **distance map** takes non-negative real values and is defined by

$$\text{distance} : \mathbb{E}^3 \times \mathbb{E}^3 \to [0, \infty) : (p, q) \mapsto ||q - p||$$

here, $||\ ||$ denotes the operation of taking the **norm** or absolute value of the vector, defined by

$$||(q_1 - p_1, q_2 - p_2, q_3 - p_3)|| = +\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

Then we can view $\mathbb{E}^3$ as the set of points representing ordinary space, together with the standard Euclidean angles and Pythagorean distances and $\mathbb{R}^3$ provides the vectors of **directed differences** between points. Not all books make this distinction so you need to be prepared to encounter the unstated identification $\mathbb{E}^3 = \mathbb{R}^3$. Often, we use the coordinates (x, y, z) for points in $\mathbb{E}^3$ and denote by $\mathbb{E}^2$ the set of points in $\mathbb{E}^3$ with $z = 0$ and then we abbreviate $(x, y, 0)$ to (x, y).

The **standard unit sphere** $\mathbb{S}^n$ in a Euclidean n-space is the set of points unit distance from the origin; we shall often use $\mathbb{S}^1$ in $\mathbb{E}^2$ and $\mathbb{S}^2$ in $\mathbb{E}^3$.

4 Group actions

In algebra, geometry and topology we often exploit the fact that important structures arise from families of morphisms that are indexed by a group. For example, rotations in the plane about the origin are indexed by the unimodular group of complex numbers; we say that this group **acts on the plane** and the **orbit** of a point at distance r from the origin is the circle of radius r.

We use in geometry the groups that act on subsets of $\mathbb{E}^n$ while preserving Euclidean distances and angles; these are **groups of isometries** of $\mathbb{E}^n$. They form subgroups of matrix groups. The set of $n \times n$ nonsingular real matrices forms a group $GL(n, \mathbb{R})$, often just written $GL(n)$, the **general linear group**, under matrix multiplication. So does $O(n)$, the subset consisting of **orthogonal** matrices, and its subset $SO(n)$ consisting of those with determinant $+1$.

The **Euclidean group** $E(n)$ consists of all isometries of Euclidean n-space $\mathbb{E}^n$. Isometries can always be written as an ordered pair from $O(n) \times \mathbb{R}^n$ with action on $\mathbb{E}^n$ given by

$$(O(n) \times \mathbb{R}^n) \times \mathbb{E}^n \longrightarrow \mathbb{E}^n : ((\alpha, u), x) \longmapsto \alpha(x) + u$$

and composition

$$(\alpha, u)(\beta, v) = (\alpha\beta, \alpha(v) + u)\,.$$

Thus, topologically $E(n)$ is the product $O(n) \times \mathbb{R}^n$ but algebraically it is not the product group. It is called a **semidirect product** of $O(n)$ and $\mathbb{R}^n$.

Definitions

A group G is said to **act** on a set (for example, a group, vector space, manifold, topological space) X **on the left** if there is a map (for example, homomorphism, linear, smooth, continuous)

$$\alpha : G \times X \longrightarrow X : (g, x) \longmapsto \alpha_g(x)$$

such that $\alpha_{g*h}(x) = \alpha_g(\alpha_h(x))$ and $\alpha_e(x) = x$ for all $x \in X$. Normally, we shall want each $\alpha_g : X \to X$ to be an isomorphism in the category for X; in this case, an **action** is the same as a representation of G *in* the automorphism group of X, or a representation *on* X. We sometimes abbreviate the notation to $g \cdot x$, especially when α is fixed for the duration of a discussion. There is a dual theory of actions on the right; we have to keep the concepts separate because every group acts on itself by its group operation, but it may be different on the right from on the left.

The **orbit** of $x \in X$ under the action α of G is the set

$$G \cdot x = \{\alpha_g(x) \mid g \in G\}\,.$$

It is easy to show that the orbits partition X, so they define an equivalence relation on X:

$$x \sim y \iff \exists\, g \in G \text{ with } \alpha_g(x) = y\,.$$

The quotient object (set, space, *etc.*) is called the **orbit space** and denoted by X/G.

The **stabilizer** or **isotropy subgroup** of x is defined to be the set

$$\mathrm{stab}_G(x) = \{g \in G \mid \alpha_g(x) = x\},$$

and it is always a subgroup of G.

The action is called **transitive** if for all $x, y \in X$ we can find $g \in G$ such that

$$\alpha_g(x) = y \qquad (\text{so also } \alpha_{g^{-1}}(y) = x)\,,$$

free if the only α_g with a fixed point has $g = e$ (the identity of G),
and **effective** if

$$\alpha_g(x) = x \quad (\forall\, x \in X) \implies g = e.$$

Note that an action being transitive is equivalent to it having exactly one orbit, or to its orbit space being a singleton.

The situations of most practical interest are when:

- X is a subset of Euclidean space, a group or vector space—especially $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{R}^n$;
- G, X are **topological groups**, so each has a topology with respect to which its binary operation and the taking of inverses is continuous;
- G is a topological group and X is a topological space;

- G is a **Lie group**, so G has a differentiable structure with respect to which its binary operation and the taking of inverses is smooth, and X is a smooth manifold. Here smooth means all derivatives of all orders exist and are continuous. Important examples of Lie groups are $\mathbb{R}^n$, $GL(n)$ and $\mathbb{S}^1$, where the differentiability arises from that of the underlying real functions.

Exercises on group actions

1. $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$.

2. The **symmetric group** S_n of permutations of n objects is not abelian for $n > 2$.

3. Find a group G consisting of four, 2×2 real matrices such that G acts on the plane $\mathbb{E}^2$. For the case $n = 2$ find discrete subgroups $G < E(2)$ such that $\mathbb{R}^n/G$ is: (i) the cylinder; (ii) the torus.

4. The general linear group $GL(n; \mathbb{R})$ is not abelian if $n > 1$.

5. Prove that $GL(2)$ has a subgroup consisting of rotations in a plane
$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \ \theta \in \mathbb{R} \right\}.$$
This is actually $SO(2)$, the **special orthogonal group** of 2×2 real matrices.

6. Find an isomorphism
$$f : SO(2) \to \{z \in \mathbb{C} | \ |z| = 1\}$$
and give its inverse.

7. Prove that, for all elements a in group G, the map
$$c_a : G \to G : x \mapsto a^{-1}xa$$
is an automorphism; find the inverse of c_a.

8. The group $SO(2)$ of rotations in a plane acts on a sphere $\mathbb{S}^2$ as rotations of angles of longitude. The orbits are circles of latitude and the quotient space by this action is the interval $[-1, 1]$. The action is neither transitive nor free, but it is effective.

9. Prove that $SO(2)$ defines a left action on $\mathbb{E}^2$ by
$$\rho : SO(2) \times \mathbb{E}^2 \to \mathbb{E}^2 : (A, p) \mapsto L_A p$$
where $L_A p$ denotes matrix multiplication of the coordinate column vector p by the matrix A. To establish this you need to show that the map ρ is well-defined and that it satisfies two rules for all $p \in \mathbb{E}^2$ and all $A, B \in SO(2)$, namely

   Product $L_A(L_B p) = L_{AB} p$

   Identity $L_I p = p$

   [In fact, the whole of the general linear group $GL(2)$ acts on $\mathbb{E}^2$.]

10. Prove that the action ρ is effective but neither free nor transitive. Find the orbits under this action of the points on the x-axis of $\mathbb{E}^2$.

11. Prove that the action ρ preserves the scalar product; that is, for all $p, q \in \mathbb{E}^2$ and all $A \in SO(2)$,
$$L_A p \cdot L_A q = p \cdot q$$
Hence deduce that the action preserves Euclidean angles, lengths and areas.

12. Show that
$$L_J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO(2)$$
and find the image under L_J of the unit square in the upper right quadrant of $\mathbb{E}^2$. [Hint: Check the edge vectors.] Find an element $K \in GL(2)$ with $K \notin SO(2)$ and $\det K = -1$. This defines a linear map L_K; compare its effect on the unit square with the image found for L_J.

13. It is clear that $GL(3)$, which acts on $\mathbb{E}^3$, has a subgroup $SO(3)$, consisting of 3×3 real matrices having determinant $+1$. Find three distinct subgroups of $SO(3)$, consisting of rotations around the three coordinate axes, respectively, by finding three group homomorphisms $SO(2) \to SO(3)$ with trivial kernels.

14. Use the subgroups of $SO(3)$ found in the previous exercise, and the parametric equation for the equator of $\mathbb{S}^2$, to show how any other great circle on $\mathbb{S}^2$ can be found by appropriate combinations of rotations of the equator.

15. Find two matrices, R_1 and R_2 from $SO(3)$ which represent, respectively, rotation by $\pi/3$ about the y-axis and rotation by $\pi/4$ about the z-axis; each rotation must be in a right-hand-screw sense in the positive direction of its axis. Find the product matrix $R_1 R_2$ and show that its transpose is its inverse.

5 Regular Surfaces

A strip, with its faces of different colours, twisted into a Möbius band. As you probably know, this yields a non-orientable surface with one face and one edge. Recall that in our study of curves we concentrated on **regular** curves—which had nowhere zero derivatives (ie rank 1 Jacobian) for component functions. This meant that, locally, the curve was homeomorphic to open subintervals of its domain but globally the curve may have had self-intersections. We seek the corresponding generalization to define regular surfaces such that locally they are homeomorphic to open sets (called **coordinate patches** or just **patches**) in $\mathbb{E}^2$. To achieve this, we need the maps defining our surface to be injective and with rank 2 Jacobians on the patches; so our patches are not allowed to generate self-intersections in their images.

Find out more about Jacobi at webpage [1]

Exercises on coordinates

1. Verify that a parametric equation for the unit 2-sphere $\mathbb{S}^2$ in $\mathbb{E}^3$ is given by
$$g : [0, 2\pi] \times [-\pi/2, \pi/2] :\to \mathbb{E}^3 : (u, v) \mapsto (\cos v \cos u, \cos v \sin u, \sin v).$$
Find a parametric equation for the equator of this sphere, and for a perpendicular circle of longitude.

2. Find a parametric equation for a sphere of radius a.

3. Find a parametric equation for an ellipsoid with lengths of its principal semi-axes having values a, b, c.

4. Denote by
$$x : U \to \mathbb{E}^n : (u, v) \mapsto (x_1, x_2, \ldots, x_n) \tag{1}$$
the coordinate patch map for some surface M in $\mathbb{E}^n$. Then the partial derivatives of x are given by
$$x_u : U \to \mathbb{E}^n \quad : \quad (u, v) \mapsto (\partial_u x_1, \partial_u x_2, \ldots, \partial_u x_n) \tag{2}$$
$$x_v : U \to \mathbb{E}^n \quad : \quad (u, v) \mapsto (\partial_v x_1, \partial_v x_2, \ldots, \partial_v x_n) \tag{3}$$

5. At any point $p \in U$, the Jacobian of x has rank 2 if and only if at that point we have x_u, x_v linearly independent or equivalently

$$\begin{vmatrix} x_u \cdot x_u & x_u \cdot x_v \\ x_v \cdot x_u & x_v \cdot x_v \end{vmatrix} \neq 0. \tag{4}$$

6. The arc length function s of a curve α lying in the image of patch map x in (1) satisfies

$$\frac{ds}{dt} = \sqrt{E\frac{du}{dt}^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\frac{dv}{dt}^2} \tag{5}$$

$$\text{with} \qquad E = x_u \cdot x_u, \;\; F = x_u \cdot x_v, \;\; G = x_v \cdot x_v \tag{6}$$

$$\text{equivalently} \qquad ds^2 = EdU^2 + 2Fdudv + Gdv^2 \tag{7}$$

   The functions E, F, G are called the coefficients of the **first fundamental form** or of the **Riemannian metric** induced on $M \subset \mathbb{R}^n$. Find out more about Riemann [1].

7. Find the functions E, F, G for local patches on a plane, a cylinder, a sphere and a saddle; satisfy yourself that they generalize the Euclidean distance function and Pythagoras's theorem. Find out more about Pythagoras [1].

Obviously, when we need more than one patch to define the surface (as for a sphere—why?, try wrapping a ball with paper!) then we want the change of patch maps to be diffeomorphisms on their overlaps. This allows us to define differentiability on a surface in terms of differentiability of components on local patches. The image of a patch together with the induced map from the surface is called a **chart**; the collection of charts used to define the surface is called an **atlas** and any given surface may have many different choices of atlases. Many of our constructions generalize further from dimension 2 to arbitrary dimension n—giving **n-manifolds** [5]. As is often the case in mathematics, the big step is from one to more than one—from two to many is usually straightforward, until the many becomes infinite.

Exercises on atlases

1. The unit 2-sphere $\mathbb{S}^2$ has an atlas consisting of two charts

$$\{(U_N, \phi_N),\ (U_S, \phi_S)\}$$

   where U_N consists of $\mathbb{S}^2$ with the north pole (n.p.) removed, U_S consists of $\mathbb{S}^2$ with the south pole (s.p.) removed, and the chart maps are stereographic projections. Thus, if $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centered at the origin then:

$$\phi_N : \mathbb{S}^2 \setminus \{n.p.\} \longrightarrow \mathbb{E}^2 : (x, y, z) \longmapsto \frac{1}{1+z}(x, y)$$

$$\phi_S : \mathbb{S}^2 \setminus \{s.p.\} \longrightarrow \mathbb{E}^2 : (x, y, z) \longmapsto \frac{1}{1-z}(x, y).$$

   What are the patches corresponding to these charts?

2. The same type of atlas works also for $\mathbb{S}^n$, which is an example of an n-dimensional generalization of surfaces.

3. $\mathbb{R}^n$ has an atlas consisting of just one chart, the identity map.

4. Find another atlas for $\mathbb{S}^2$ consisting of projections of six hemispheres onto three perpendicular planes through the origin.

5. Find atlases for the cylinder $\mathbb{S}^1 \times (0, 1)$, and for the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

6. Gray [6] gives a parametrization of, among many other surfaces, a Möbius strip:

$$\begin{aligned} \text{Möbius} \quad &: \quad (0, 2\pi) \times (-0.3, 0.3) \to \mathbb{E}^3 \\ &: \quad (u, v) \mapsto \{\cos u + v\cos(\frac{u}{2})\cos u, \sin u + v\cos(\frac{u}{2})\sin u, v\sin(\frac{u}{2})\} \end{aligned}$$

   Find an atlas for this surface.

7. The Klein bottle is an example of a non-orientable closed surface. It has several embeddings in 3-space but all involve self-intersections and in fact it requires 4 dimensions to allow an injective embedding. Here is one embedding in which the self-intersection is in the form of a figure-eight knot:

$$\begin{aligned} \text{Klein} \quad &: \quad [0, 2\pi] \times [0, 2\pi] \to \mathbb{E}^3 \\ &: \quad (u, v) \mapsto \{(2 + \cos\frac{u}{2}\sin t - \sin\frac{u}{2}\sin 2t)\cos u, \\ &\quad\quad (2 + \cos\frac{u}{2}\sin t - \sin\frac{u}{2}\sin 2t)\sin u, \\ &\quad\quad \sin\frac{u}{2}\sin t + \cos\frac{u}{2}\sin 2t\} \end{aligned}$$

6 Tangent Spaces

The derivative of a map $f : \mathbb{E}^m \to \mathbb{E}^n$ at $p \in \mathbb{E}^m$ is the limit of differences that is the best linear approximation to f, at p. Thus, we need vector spaces to define linearity for maps between surfaces and these are automatically present at each point of $\mathbb{E}^n$. At each point p of a surface M in $\mathbb{E}^n$ we construct a vector space T_pM, called the **tangent space to M at p**, from the tangent vectors to curves in M that pass through p.

$$T_pM = \{\alpha'(0) \in \mathbb{E}^n | \alpha \text{ is a curve in } M \text{ starting at } p \in M\}$$

A continuous choice of tangent vector on M is called a **tangent vector field** on M.
Exercises on tangent spaces

1. Construct $T_p\mathbb{E}^2$.

2. Construct a vector space structure on T_pM.

3. Differentiable maps between surfaces are called **surface maps** and induce tangent space maps; a surface map is a **local isometry** if it preserves norms.

4. Construct a tangent vector field on $\mathbb{S}^2$, and show that it must be zero somewhere [2, 4].

5. There is only one reasonable way to define the **derivative** $\nabla_y V$ of a vector field V on M in the tangent direction $y \in T_pM$:

$$\nabla_y V = \lim_{t\to 0} \frac{V \circ \alpha(t) - V \circ \alpha(0)}{t} \quad (8)$$

   where α is a curve in M beginning at p with $\alpha'(0) = y$.

6. Check that ∇ in (8) is linear in y and V, and find coordinate expressions for $\nabla_y V$ in terms of those for y and V.

7 Normal Vectors and Gauss Map

Where x_u, x_v from (4) are linearly independent they define a normal vector field $x_u \times x_v$.
For surfaces in $\mathbb{E}^3$, the **Gauss map** of the patch map

$$x : U \to \mathbb{E}^3$$

is defined at regular points by

$$G : U \to \mathbb{S}^2 : (u, v) \mapsto \frac{x_u \times x_v}{\|x_u \times x_v\|} \tag{9}$$

If it is possible to make a continuous assignation of a unit normal vector over the whole of a regular surface in $\mathbb{E}^3$, (eg when $x_u \times x_v$ is nowhere zero) then we say that the surface is **orientable**. On orientable surfaces the Gauss map on a patch extends to a continuous map on the whole surface.
Find out more about Gauss at webpage [1]

Exercises on orientability

1. For $\hat{n}$ a unit normal vector field, differentiate $\hat{n} \cdot x_u = \hat{n} \cdot x_v = 0$ with respect to u and with respect to v. We call the real functions

$$e = -\hat{n}_u \cdot x_u, \; f = -\hat{n}_v \cdot x_u \text{ and } g = -\hat{n}_v \cdot x_v \tag{10}$$

   the coefficients of the **second fundamental form** of x.

2. If a surface is orientable, then there are exactly two choices of the continuous unit normal vector field.

3. $\mathbb{S}^2$ is orientable but a Möbius strip is not.

8 Shape Operator and Curvature

Let $\hat{n}$ be a unit normal vector field defined in a neighbourhood of $p \in M$. Then the **shape operator** measures the vectorial rate of change of the unit normal vector field $\hat{n}$ as the linear map [8]

$$S : T_pM \to T_pM : y \mapsto -\nabla_y \hat{n} \tag{11}$$

The **normal curvature** of M at p in the direction $y \in T_pM$ is

$$k : T_pM \to \mathbb{R} : y \mapsto \frac{S(y) \cdot y}{\|y\|^2} \tag{12}$$

The normal curvature k induces a real-valued map on $\mathbb{S}^2$, which is compact, so k achieves its bounds; the extreme values are called **principal curvatures** k_1, k_2, and determine the **principal directions** in the surface. The shape operator S is a symmetric operator and so diagonalizable, with real eigenvalues which turn out to be the principal curvatures.
Recall from linear algebra that a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ has a 3×3 matrix representation and $|det(A)|$ is the volume (up to sign) of the image under A of a unit cube. The trace of A is the sum of the diagonal elements of A, which is actually also the sum of the eigenvalues (including multiplicities). These functions applied to the shape operator give measures of the curvature of surfaces.
The **Gaussian curvature** K and the **mean curvature** H of a surface $M \subset \mathbb{E}^3$ are defined by

$$K : M \to \mathbb{R} \quad : \quad p \mapsto det(S)_{|p} \tag{13}$$

$$H : M \to \mathbb{R} \quad : \quad p \mapsto \frac{1}{2} trace(S)_{|p} \tag{14}$$

M is **flat** if and only if K is the zero function and is a **minimal surface** if H is the zero function. See Osserman [9] for an easily readable account of minimal surfaces.

Gauss's **Theorium Egregium** (Remarkable Theorem) says that a local isometry preserves Gaussian curvature. It is remarkable because it demonstrates that Gaussian curvature is an intrinsic quality and not dependent on the particular embedding in $\mathbb{E}^3$. The problem of making flat maps of the world (or of wrapping a football) is due to the fact that a plane surface has zero curvature but the Earth has nonzero Gaussian curvature. Hence there is no local isometry between a flat map and the surface of the Earth—even if it was a perfect sphere.

Exercises on Gaussian and mean curvature

1. Investigate K and H for the surfaces: plane, cylinder, hemisphere and saddle.
2. Expressions for the Gaussian and mean curvatures in terms of the components of the first (Eq (7) and second (Eq (10) fundamental forms are

$$K = k_1 k_2 = \frac{eg - f^2}{EG - F^2} \quad \text{and} \tag{15}$$

$$H = \frac{1}{2}(k_1 + k_2) = \frac{eG - 2fF + gE}{2(EG - F^2)} \tag{16}$$

3. A helicoid is locally isometric to a catenoid.

References

[1] On-line mathematical materials:

Mathematicians: http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

Alfred Gray's Mathematica NoteBooks: http://library.wolfram.com/infocenter/Books/3759

Elementary Notes on:
Curves http://www.maths.manchester.ac.uk/ kd/curves/curves.pdf
Surfaces http://www.maths.manchester.ac.uk/ kd/curves/surfaces.pdf
Knots http://www.maths.manchester.ac.uk/ kd/curves/knots.pdf

LaTeX Tutorial:
http://www.maths.manchester.ac.uk/ kd/latextut/pdfbyex.htm

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