Introduction to Matrices for Engineers*

C.T.J. Dodson, School of Mathematics, Manchester University

1 What is a Matrix?

A *matrix* is a rectangular array of *elements*, usually numbers, e.g.

$$\begin{pmatrix} 1 & 3 & 0 \\ -2 & 8 & 2 \\ 4 & 0 & -1 \\ \frac{1}{2} & 0 & 117 \end{pmatrix}$$

The above matrix is a (4×3)–matrix, i.e. it has three columns and four rows.

1.1 Why use Matrices?

We use matrices in mathematics and engineering because often we need to deal with several variables at once—eg the coordinates of a point in the plane are written (x, y) or in space as (x, y, z) and these are often written as column matrices in the form:

$$\begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix}$$

It turns out that many operations that are needed to be performed on coordinates of points are **linear operations** and so can be organized in terms of rectangular arrays of numbers, matrices. Then we find that matrices themselves can under certain conditions be added, subtracted and multiplied so that there arises a whole new set of algebraic rules for their manipulation.

In general, an $(n \times m)$–matrix A looks like:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,m-1} & a_{2,m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,m-1} & a_{3,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

Here, the entries are denoted a_{ij}; e.g. in Example 4.1.1 we have $a_{11} = 1$, $a_{23} = 2$, etc. Capital letters are usually used for the matrix itself.

1.2 Dimension

In the above matrix A, the numbers n and m are called the *dimensions* of A.

*Based on original lecture notes on matrices by Lewis Pirnie

1.3 Addition

It is possible to add two matrices together, but *only if they have the same dimensions.* In which case we simply add the corresponding entries:

$$\begin{pmatrix} 1 & 3 & 0 \\ -2 & 8 & 2 \\ 4 & 0 & -1 \\ \frac{1}{2} & 0 & 117 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 1 \\ 0 & -8 & -3 \\ 5 & 1 & -2 \\ \frac{1}{2} & 1 & -50 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 1 \\ -2 & 0 & -1 \\ 9 & 1 & -3 \\ 1 & 1 & 67 \end{pmatrix}$$

If two matrices don't have the same size (dimensions) then they can't be added, or we say the sum is 'not defined'.

1.4 Example

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 6 & -\frac{1}{4} \\ 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 7 & \frac{7}{4} \\ 1 & 0 \\ 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 3 & -2 \end{pmatrix} \quad \text{is undefined}$$

2 Multiplying Matrices

2.1 Why should we want to?

We can motivate the process by looking at rotations of points in the plane. Consider the point P on the x-axis at $x = x, y = 0$, so it has coordinates

$$\begin{pmatrix} \text{x} \\ 0 \end{pmatrix}$$

Now, let P' be the point obtained by rotating P round the origin in an anticlockwise direction through angle θ, keeping the distance from the origin (which is actually the square root of the sum of squares of the coordinates) constant. What are the coordinates of the point P'?

It is not difficult trigonometry to work out that these are:

$$\begin{pmatrix} x\cos\theta \\ x\sin\theta \end{pmatrix}$$

Next consider the point Q on the y-axis with coordinates

$$\begin{pmatrix} 0 \\ \text{y} \end{pmatrix}$$

and rotate this point round the origin in an anticlockwise direction through angle θ, keeping the distance from the origin constant. We find that the new coordinates are:

$$\begin{pmatrix} -y\sin\theta \\ y\cos\theta \end{pmatrix}$$

Now the payoff, it is actually a 2×2 matrix that does the rotating here because the operation is linear on the coordinate components. We can express the rotation of *any* point with coordinates (x, y) as the following matrix equation:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \text{x} \\ \text{y} \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Can you guess what will be the matrix for rotation in a clockwise direction?

2.2 Exercise on Trigonometry

1. Three common right angled triangles can be used to obtain the following values for the trigonometric functions:

$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$
$\sin\frac{\pi}{6} = \frac{1}{2}$	$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\cos\frac{\pi}{3} = \frac{1}{2}$
$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$	$\tan\frac{\pi}{4} = 1$	$\tan\frac{\pi}{3} = \sqrt{3}$

2. Obtain the rotation matrices explicitly for rotations of $\theta = \pm 30°,\ \pm 45°,\ \pm 60°,\ \pm 90°,\ \pm 180°$.

2.3 Rules

When multiplying matrices, keep the following in mind: lay the first row of the first matrix on top of the first column of the second matrix; only if they are both of the same size can you proceed.

The rule for multiplying is: go across the first matrix, and down the second matrix, multiplying the corresponding entries, and adding the products. This new number goes in the new matrix in position of the row of the first matrix, and the column of the second matrix. For example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 1.2+2.1 & 1.0+2.5 & 1.3+2.7 \\ 3.2+4.1 & 3.0+4.5 & 3.3+4.7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 10 & 17 \\ 10 & 5 & 37 \end{pmatrix}$$

Symbolically, if we have the matrices A and B as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3,m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,m} \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1,q} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2,q} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3,q} \\ \dots & \dots & \dots & \dots & \dots \\ b_{p-1,1} & b_{p-1,2} & b_{p-1,3} & b_{p-1,q-1} & b_{p-1,q} \\ b_{p,1} & b_{p,2} & b_{p,3} & \dots & b_{p,q} \end{pmatrix}$$

then the product AB is given by:

$$\begin{pmatrix} \sum_{i=1}^m a_{1i}b_{i1} & \sum_{i=1}^m a_{1i}b_{i2} & \dots & \sum_{i=1}^m a_{1i}b_{iq} \\ \sum_{i=1}^m a_{2i}b_{i1} & \sum_{i=1}^m a_{2i}b_{i2} & \dots & \sum_{i=1}^m a_{2i}b_{iq} \\ \dots & \dots & \dots & \dots \\ \sum_{i=1}^m a_{ni}b_{i2} & \sum_{i=1}^m a_{ni}b_{i2} & \dots & \sum_{i=1}^m a_{ni}b_{iq} \end{pmatrix}$$

where $\sum_{i=1}^m a_{1i}b_{i1}$ stands for $a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \ldots + a_{1n}b_{n1}$, etc.

Note that we must have $m = p$, i.e. the number os columns in the first matrix must equal the number of rows of the second; otherwise, we say the product is *undefined*.

2.4 Example

We multiply the following matrices:

(i) $\begin{pmatrix} -1 & 2 & 0 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & -1 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} -1.1+2.0+0.7 & -1.3+2.-1+0.5 \\ 4.1+1.0+-2.7 & 4.3+1.-1+-2.5 \end{pmatrix} = \begin{pmatrix} -1 & -5 \\ -10 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ -8 & \frac{1}{2} \\ 2 & 1 \end{pmatrix}$ is undefined

(iii) $\begin{pmatrix} 4 & -\frac{1}{2} \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 4.2+-\frac{1}{2}.-1 & 4.2+-\frac{1}{2}.4 \\ 2.2+1.-1 & 2.2+1.4 \end{pmatrix} = \begin{pmatrix} \frac{17}{2} & 6 \\ 3 & 8 \end{pmatrix}$

(iv) $\begin{pmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 & 1 \\ 3 & 7 & 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1.0+1.3 & -1.-1+1.7 & -1.2+1.0 & -1.1+1.-1 \\ 0.0+2.3 & 0.-1+2.7 & 0.2+2.0 & 0.1+2.-1 \\ 1.0+4.3 & 1.-1+4.7 & 1.2+4.0 & 1.1+4.-1 \end{pmatrix} = \begin{pmatrix} 3 & 8 & -2 & -2 \\ 6 & 14 & 0 & -2 \\ 12 & 27 & 2 & -3 \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1.2+0.5 \\ 3.2+1.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$

(vi) $\begin{pmatrix} 1 & 3 & 0 \\ -2 & 8 & 2 \end{pmatrix} \begin{pmatrix} 4 & 0 & 1 & 6 \\ 0 & -8 & -3 & 0 \\ 5 & 1 & -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1.4+3.0+0.5 & 1.0+3.-8+0.1 & 1.1+3.-3+0.-2 & 1.6+3.0+0.1 \\ -2.4+8.0+2.5 & -2.0+8.-8+2.1 & -2.1+8.-3+2.-2 & -2.6+8.0+2.1 \end{pmatrix}$

$= \begin{pmatrix} 4 & -24 & -8 & 6 \\ 2 & -62 & -30 & -10 \end{pmatrix}$

We see from the examples:

(i)	Product of	(2×3)–matrix with a	(3×2)–matrix is a	(2×2)–matrix.
(iii)	Product of	(2×2)–matrix with a	(2×2)–matrix is a	(2×2)–matrix.
(iv)	Product of	(3×2)–matrix with a	(2×4)–matrix is a	(3×4)–matrix.
(v)	Product of	(2×2)–matrix with a	(2×1)–matrix is a	(2×1)–matrix.
(vi)	Product of	(2×3)–matrix with a	(3×4)–matrix is a	(2×4)–matrix.

Note that the two middle numbers must be the same if the product is defined; and then the dimensions of the answer is just the two outer numbers. Thus, the product of an $(n \times m)$–matrix with a $(m \times q)$–matrix is an $(n \times q)$–matrix.

3 Scalar Multiplication

There is another type of multiplication involving matrices called *scalar multiplication*. This means just multiplying each entry of the matrix by a number. For example:

$$3 \begin{pmatrix} -1 & 2 & 0 \\ 4 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 6 & 0 \\ 12 & 3 & -6 \end{pmatrix}$$

3.1 Rules

There are some rules which matrix addition and multiplication obey:

Associative	$(A+B)+C = A+(B+C)$
Commutative	$A+B = B+A$

Distributive	$A(B+C) = AB + AC$
Distributive	$(A+B)C = AC + BC$
Associative	$(AB)C = A(BC)$
Non–Commutative	$AB \neq BA$ (usually)
Moving Constants	$A(\lambda B) = \lambda(AB)$

(Assuming that the sums and products are defined in all cases.)

3.2 Example

$$\text{Let } A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} \text{-1} & 4 \\ 1 & \text{-2} \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Consider the following:

$$AB = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \text{-1} & 4 \\ 1 & \text{-2} \end{pmatrix} = \begin{pmatrix} 1.-1+0.1 & 1.4+0.-2 \\ 3.-1+2.1 & 3.4+2.-2 \end{pmatrix} = \begin{pmatrix} \text{-1} & 4 \\ \text{-1} & 8 \end{pmatrix}$$

$$BA = \begin{pmatrix} \text{-1} & 4 \\ 1 & \text{-2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1.1+4.3 & -1.0+4.2 \\ 1.1+-2.3 & 1.0+-2.2 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ \text{-5} & \text{-4} \end{pmatrix}$$

Now, $AB \neq BA$, and we see that two matrices are *not* the same if they are multiplied the other way around. Also consider:

$$AC = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

$$BC = \begin{pmatrix} \text{-1} & 4 \\ 1 & \text{-2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ \text{-4} \end{pmatrix}$$

$$AB + BC = \begin{pmatrix} 2 \\ 12 \end{pmatrix} + \begin{pmatrix} 10 \\ \text{-4} \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} \text{-1} & 4 \\ 1 & \text{-2} \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

$$(A+B)C = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

and we notice that $(A+B)C = AC + BC$ as required.

4 Transpose

Another operation on matrices is the *transpose*. This just reverses the rows and columns, or equivalently, reflects the matrix along the leading diagonal. The transpose of A is normally written A^t thus

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,m-1} & a_{2,m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,m-1} & a_{3,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,m-1} & a_{n,m} \end{pmatrix},$$

$$A^t = \left(\begin{array}{cccccc} a_{11} & a_{21} & a_{31} & \cdots & a_{m-1,1} & a_{m,1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m-1,2} & a_{m,2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m-1,3} & a_{m,3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n-1} & a_{2,n-1} & a_{3,n-1} & \cdots & a_{m-1,n-1} & a_{m,n-1} \\ a_{1,n} & a_{2,n} & a_{3,n} & \cdots & a_{m-1,n} & a_{m,n} \end{array} \right)$$

Note that the transpose of a $(n \times m)$–matrix is a $(m \times n)$–matrix.

4.1 Example

As an example of the transpose:

$$A = \left(\begin{array}{ccc} 2 & 4 & -1 \\ 0 & 3 & 5 \end{array} \right), A^t = \left(\begin{array}{cc} 2 & 0 \\ 4 & 3 \\ -1 & 5 \end{array} \right)$$

5 Square Matrices

Given a number n, $(n \times n)$–matrices have very special properties. Note that if we have two $(n \times n)$–matrices, the product is defined and will also be an $(n \times n)$–matrix.

5.1 The Identity Matrix

There also exists a special matrix, known as the identity, $I_{n \times n}$:

$$I_{2\times 2} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), I_{3\times 3} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), I_{4\times 4} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \text{etc.}$$

which has 1's on the main diagonal, and 0's everywhere else. This matrix has the property that, given any $(n \times n)$–matrix A:

$$AI_{n\times n} = A \text{ and } I_{n\times n}A = A$$

i.e. multiplying by the identity on either side doesn't change the matrix. This is similar to the property of 1 when multiplying numbers. We usually abbreviate $I_{n\times n}$ to just I when it's obvious what n is.

5.2 Example

$$\text{Let } A = \left(\begin{array}{cc} 1 & 7 \\ 3 & 2 \end{array} \right), \quad I = I_{2\times 2} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$

Now, we check the properties of the identity:

$$AI = \left(\begin{array}{cc} 1 & 7 \\ 3 & 2 \end{array} \right) \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc} 1.1 + 7.0 & 1.0 + 7.1 \\ 3.1 + 2.0 & 3.0 + 2.1 \end{array} \right) = \left(\begin{array}{cc} 1 & 7 \\ 3 & 2 \end{array} \right)$$

$$IA = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \left(\begin{array}{cc} 1 & 7 \\ 3 & 2 \end{array} \right) = \left(\begin{array}{cc} 1.1 + 0.3 & 1.7 + 0.2 \\ 0.1 + 1.3 & 0.7 + 1.2 \end{array} \right) = \left(\begin{array}{cc} 1 & 7 \\ 3 & 2 \end{array} \right)$$

as required.

6 Determinants

One of the most important properties of square matrices is the *determinant*. This is a number obtained from the entries.

6.1 Determinant of a (2×2)–Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, the determinant of A, denoted $\det A$ or $|A|$ is given by $ad - bc$.

6.2 Example

$$\det \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} = 2.5 - 3.1 = 10 - 3 = 7$$

$$\det \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} = (-1)(-6) - 2.3 = 6 - 6 = 0$$

Before we go on to larger matrices, we need to define *minors*.

6.3 Minors

Let A be the $(n \times n)$–matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,m-1} & a_{2,m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,m-1} & a_{3,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

Then, the minor m_{ij}, for each i, j, is the determinant of the $(n-1 \times n-1)$–matrix obtained by deleting the i^{th} row and the j^{th} column. For example, in the above notation:

$$m_{11} = \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2,m-1} & a_{2,m} \\ a_{32} & a_{33} & \cdots & a_{3,m-1} & a_{3,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,2} & a_{n,3} & \cdots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

$$m_{21} = \det \begin{pmatrix} a_{12} & a_{13} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{32} & a_{33} & \cdots & a_{3,m-1} & a_{3,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,m-1} & a_{n-1,m} \\ a_{n,2} & a_{n,3} & \cdots & a_{n,m-1} & a_{n,m} \end{pmatrix}$$

6.4 Example

We compute all the minors of $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -5 & 0 & -2 \end{pmatrix}$

$$m_{11} = \begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} = -8 \qquad m_{12} = \begin{vmatrix} 0 & 3 \\ -5 & -2 \end{vmatrix} = 15 \qquad m_{13} = \begin{vmatrix} 0 & 4 \\ -5 & 0 \end{vmatrix} = 20$$

$$m_{21} = \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = -2 \qquad m_{22} = \begin{vmatrix} 2 & -1 \\ -5 & -2 \end{vmatrix} = -9 \qquad m_{23} = \begin{vmatrix} 2 & 1 \\ -5 & 0 \end{vmatrix} = 5$$

$$m_{31} = \begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix} = 7 \qquad m_{32} = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 6 \qquad m_{33} = \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8$$

6.5 Minors and Cofactors

The numbers called 'cofactors' are almost the same as minors, except some have a minus sign in accordance with the following pattern:

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The best way to remember this is as an 'alternating' or 'chessboard' pattern. The cofactors from the previous example are:

$$\begin{array}{lll} c_{11} = m_{11} = -8 & c_{12} = -m_{12} = -15 & c_{13} = m_{13} = 20 \\ c_{21} = -m_{21} = 2 & c_{22} = m_{22} = -9 & c_{23} = -m_{23} = -5 \\ c_{31} = m_{31} = 7 & c_{32} = -m_{32} = -6 & c_{33} = m_{33} = 8 \end{array}$$

7 Determinant of a (3×3)–Matrix

In order to calculate the determinant of a (3×3)–matrix, choose *any* row or column. Then, multiply each entry by its corresponding cofactor, and add the three products. This gives the determinant.

7.1 Example

Letting $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -5 & 0 & -2 \end{pmatrix}$ as before, we compute the determinant using the top row:

$$\det A = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 2.(-8) + 1.(-15) + (-1).20 = -16 - 15 - 20 = -51$$

Suppose, we use the second column instead:

$$\det A = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} = 1.(-15) + 4.(-9) + 0.(-6) = -15 - 36 - 0 = -51$$

It doesn't matter which row or column is used, but the top row is normal. Note that it is not necessary to work out *all* the minors (or cofactors), just three.

7.2 Example

Let $B = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. We compute the determinant of B:

$$\det \begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(1-0) - 0(-2-0) + 4(-4-3) = 1 - 28 = -27$$

8 Determinant of an $(n \times n)$–Matrix

The procedure for larger matrices is exactly the same as for a (3×3)–matrix: choose a row or column, multiply the entry by the corresponding cofactor, and add them up. But of course each minor is itself the determinant of an $(n-1 \times n-1)$–matrix, so for example, in a (4×4) determinant, it is first necessary to do four (3×3) determinants — quite a lot of work!

9 Inverses

Let A be an $n \times n$ matrix, and let I be the $n \times n$ identity matrix. Sometimes, there exists a matrix A^{-1} (called the *inverse* of A) with the property:

$$A\,A^{-1} = I = A^{-1}A$$

In this section, we demonstrate a method for finding inverses.

9.1 Inverse of a 2×2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, the inverse of A, A^{-1} is given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

To check, we multiply:

$$A^{-1}A = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{pmatrix}$$
$$= \frac{1}{ad-bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In a similar fashion we could show that $A\,A^{-1} = I$.
Of course, the inverse could also be written

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note, that if $\det A = 0$, then we have a division by zero, which we can't do. In this situation there is no inverse of A.

9.2 Inverse of 3×3 (and higher) Matrices

Recall the definition of a *minor* from Section 4.3.3: given an $(n \times n)$–matrix A, the minor m_{ij} is the determinant of the $(n-1 \times n-1)$–matrix obtained by omitting the i^{th} row and the j^{th} column.

9.3 Example

Let $A = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. We calculate the minors:

$$m_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \qquad m_{12} = \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = -2 \qquad m_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} = -7$$
$$m_{21} = \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} = -8 \qquad m_{22} = \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} = -11 \qquad m_{23} = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2$$
$$m_{31} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4 \qquad m_{32} = \begin{vmatrix} 1 & 4 \\ -2 & 0 \end{vmatrix} = 8 \qquad m_{33} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$$

Recall also the pattern of + and − signs from which we obtain the cofactors:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Now, we put the minors into a matrix and change their signs according to the pattern to get the matrix of cofactors:

$$\begin{pmatrix} 1 & 2 & -7 \\ 8 & -11 & -2 \\ -4 & -8 & 1 \end{pmatrix}$$

The next stage is take the transpose:

$$\begin{pmatrix} 1 & 8 & -4 \\ 2 & -11 & -8 \\ -7 & -2 & 1 \end{pmatrix}$$

and finally we must divide by the determinant, which is −27, from Example 4.3.8:

$$A^{-1} = \frac{1}{-27}\begin{pmatrix} 1 & 8 & -4 \\ 2 & -11 & -8 \\ -7 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{27} & -\frac{8}{27} & \frac{4}{27} \\ -\frac{2}{27} & \frac{11}{27} & \frac{8}{27} \\ \frac{7}{27} & \frac{2}{27} & -\frac{1}{27} \end{pmatrix}$$

This shows how to calculate the inverse of a (3×3)–matrix. We check the result:

$$A^{-1}A = -\frac{1}{27}\begin{pmatrix} 1 & 8 & -4 \\ 2 & -11 & -8 \\ -7 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1-16-12 & 0+8-8 & 4+0-4 \\ 2+22-24 & 0-11-16 & 8+0-8 \\ -7+4+3 & 0-2+2 & -28+0+1 \end{pmatrix}$$

$$= -\frac{1}{27}\begin{pmatrix} -27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as required.

The same procedure works for $(n \times n)$–matrices.

(I) Work out the minors.
(II) Put in the − signs to form the cofactors.
(III) Take the transpose.
(IV) Divide by the determinant.

Furthermore, an $n \times n$ matrix has an inverse if and only if the determinant is not zero. So, it's a good idea to calculate the determinant *first*, just to check whether the rest of the procedure is necessary.

10 Linear Systems

We discuss one very important application of finding inverses of matrices.

10.1 Simultaneous Equations

Often, when solving problems in mathematics, we need to solve simultaneous equations, e.g. something like:

$$\begin{array}{cccccc} 2x & + & y & = & 3 \\ 5x & + & 3y & = & 7 \end{array}$$

from which we would obtain $x = 2$ and $y = -1$. The process we have used up until now is a little messy: we combine the equations to try and eliminate one of the unknown variables. There is a more systematic way using matrices. We can write the equations in a slightly different way:

$$\begin{pmatrix} 2x + y \\ 5x + 3y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Now we can check that the first matrix is equal to the product:

$$\begin{pmatrix} 2x + y \\ 5x + 3y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and so altogether we have a matrix equation:

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

The next stage is to use the inverse of the (2×2)–matrix, so let's calculate that now.

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \text{ then } A^{-1} = \frac{1}{2.3 - 1.5} \begin{pmatrix} 3 & \text{-1} \\ \text{-5} & 2 \end{pmatrix} = \begin{pmatrix} 3 & \text{-1} \\ \text{-5} & 2 \end{pmatrix}.$$

Now, we take the matrix equation above, and multiply by A^{-1}

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & \text{-1} \\ \text{-5} & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & \text{-1} \\ \text{-5} & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Then, doing the multiplication:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.2 + -1.7 \\ -5.3 + 2.7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

and so $x = 2$ and $y = -1$, as required. So, provided we can work out the inverse of the matrix of coefficients, we can solve simultaneous equations.

11 Larger Systems

The same thing works with 3 equations in x, y and z. Suppose we have

$$\begin{array}{lllllll} x & + & 2y & + & 2z & = & -1 \\ & & 3y & - & 2z & = & 2 \\ 2x & - & y & + & 8z & = & 7 \end{array}$$

Then, the matrix form is

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & \text{-2} \\ 2 & \text{-1} & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \text{-1} \\ 2 \\ 7 \end{pmatrix}$$

Now, we denote the 3×3 matrix by A, and calculate the inverse of A. The minors are as follows:

$$m_{11} = \begin{vmatrix} 3 & -2 \\ -1 & 8 \end{vmatrix} = 22 \qquad m_{12} = \begin{vmatrix} 0 & -2 \\ 2 & 8 \end{vmatrix} = 4 \qquad m_{13} = \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} = -6$$

$$m_{21} = \begin{vmatrix} 2 & 2 \\ -1 & 8 \end{vmatrix} = 18 \qquad m_{22} = \begin{vmatrix} 1 & 2 \\ 2 & 8 \end{vmatrix} = 4 \qquad m_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$m_{31} = \begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} = -10 \qquad m_{32} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2 \qquad m_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

Recall the chessboard pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

So we have the following matrix of cofactors:

$$\begin{pmatrix} 22 & -4 & -6 \\ -18 & 4 & 5 \\ -10 & 2 & 3 \end{pmatrix}$$

We can calculate the determinant by taking any row or column, and multiplying the original matrix entry by its corresponding cofactor, and then adding — let's choose the top row:

$$\det A = 1.22 + 2. - 4 + 2. - 6 = 22 - 8 - 12 = 2$$

and so the determinant is 2, and so we will be able to find the inverse. From the matrix of cofactors, we take the transpose, and then divide by the determinant to get A^{-1}:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 22 & -18 & -10 \\ -4 & 4 & 2 \\ -6 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -9 & -5 \\ -2 & 2 & 1 \\ -3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

Now, we return to solving the simultaneous equations, where we had:

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix}$$

Multiplying both sides on the left by A^{-1}, we have:

$$\begin{pmatrix} 11 & -9 & -5 \\ -2 & 2 & 1 \\ -3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & -2 \\ 2 & -1 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 & -9 & -5 \\ -2 & 2 & 1 \\ -3 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix}$$

and we know that $A^{-1}A = I$, so

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11. - 1 + -9.2 + -5.7 \\ -2. - 1 + 2.2 + 1.7 \\ -3. - 1 + \frac{5}{2}.2 + \frac{3}{2}.7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -64 \\ 13 \\ \frac{37}{2} \end{pmatrix}$$

and so we get $x = -64$, $y = 13$ and $z = \frac{37}{2}$. It's a very good idea to check calculations like this!

$$\begin{aligned} x + 2y + 2z &= -64 + 2(13) + 2(\tfrac{37}{2}) = -64 + 26 + 37 &&= -1 \\ 3y - 2z &= 3(13) - 2(\tfrac{37}{2}) = 39 - 37 &&= 2 \\ 2x - y + 8z &= 2(-64) - 13 + 8(\tfrac{37}{2}) = -128 - 13 + 148 &&= 7 \end{aligned}$$

as required.

12 Appendix

12.1 Scientific Wordprocessing with LaTeX

This pdf document with its hyperlinks was created using LaTeX which is the standard (free) mathematical wordprocessing package; more information can be found via the webpage [1].

12.2 Computer Algebra Methods

The computer algebra package Mathematica [2] can be used to manipulate and invert matrices. Similarly, Maple and Matlab also can be used for working with matrices.

References

[1] On-line mathematical materials:

Mathematicians:
http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

Alfred Gray's Mathematica NoteBooks on differential geometry:
http://library.wolfram.com/infocenter/Books/3759

Elementary Notes on:
Curves http://www.maths.manchester.ac.uk/ kd/curves/curves.pdf
Surfaces http://www.maths.manchester.ac.uk/ kd/curves/surfaces.pdf
Knots http://www.maths.manchester.ac.uk/ kd/curves/knots.pdf

LaTeX Tutorial:
http://www.maths.manchester.ac.uk/ kd/latextut/pdfbyex.htm

[2] S. Wolfram. **The Mathematica Book** Cambridge University Press, Cambridge 1996.