Introducing Curves

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These notes supplement the lectures and provide practise exercises. We begin with some material you will have met before, perhaps in other forms, to set some terminology and notation. Further details on unfamiliar topics may be found in, for example Cohn [3] for algebra, Dodson and Poston [5] for linear algebra, topology and differential geometry, Gray [6] for curves, surfaces and calculations using the computer algebra package *Mathematica*, and Wolfram [10] for *Mathematica* itself. Several on-line hypertext documents are available to support this course [1].

Introduction

This document briefly summarizes definitions and hints at proofs of principal results for a first course on curves. It is intended as an *aide memoire*—a companion to lectures, tutorials and computer lab classes, with exercises and proofs to be completed by the student. Exercises include the statements to be verified—mathematics needs to be done, not just read!

The prereqisites here are: elementary knowledge of Euclidean geometry and the definition of $\mathbb{R}^n$, familiarity with vector and scalar product, norms and basic linear algebra, fundamental theorem of calculus, inverse function theorem, implicit function theorem and a little vector calculus.

Where possible, we encourage use of computer algebra software to experiment with the mathematics, to perform tedious analytic calculations and to plot graphs of functions that arise in the studies. For this purpose, we shall make use of Gray's book [6]—which contains all of the theory we need for curves and surfaces—and we use the computational packages he provides free in the form of Mathematica NoteBooks [1].

For further study of more general differential geometry and its applications to relativity and space-time geometry, see Dodson and Poston [5]. For an introduction to algebraic topology see Armstrong [2] and for more advanced topics and their applications in analysis, geometry and physics, see Dodson and Parker [4]. The abovementioned books contain substantial bibliography lists for further reference.

The document you are reading was created with LaTeX and a LaTeX tutorial is available at [1].

1 Sets and maps

A **function** or **map** from a set X to a set Y is a set of ordered pairs from X and Y (pairs like (x, y) are the coordinates in the graph of the function) satisfying the **uniqueness of image** property:

for all $x \in X$, there exists a **unique** $y \in Y$ that is related to the given x

Then we usually write $y = f(x)$ or just $y = fx$, and $f : X \to Y : x \mapsto f(x)$.

A map $f : X \to Y$ may have any or none of the following properties:

injectivity (1 to 1)	$f(x) = f(y)$ implies $x = y$
surjectivity (onto)	$\mathrm{im}\ f = Y$; denoted $f : X \twoheadrightarrow Y$
bijectivity (both)	injectivity and surjectivity

We shall use sometimes the following common abbreviations:

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural, integer, rational, real, complex numbers.
$x \in V$	x is a member of set V.
$x \notin V$	x is not a member of set V.
$\exists\, x \in V$	There exists at least one member x in V.
$\forall\, x \in V$	For all members of V.
$W \subseteq V$	W is a subset of set V: so $(\forall\, x \in W)\ x \in V$.
$\{x \in V \mid p(x)\}$	The set of members of V satisfying property p.
$\emptyset$	The empty set.
$f : V \to W$	f is a map or function from V to W.
$f : x \mapsto f(x)$	f sends a typical element x to $f(x)$.
dom f	Domain of f: the set $\{x \mid \exists\, f(x)\}$.
im f	Image of f: the set $\{f(x) \mid x \in \text{dom } f\}$.
fU for $U \subseteq \text{dom } f$	Image of U by f: the set $\{f(x) \mid x \in U\}$.
$f^{\leftarrow}M$ for $M \subseteq \text{im } f$	Inverse image of M by f: the set $\{x \mid f(x) \in M\}$.
1_X	Identity map on x: the map given by $1_X(x) = x$ for all $x \in X$.
$U \cap V$	Intersection of U and V: the set $\{x \mid x \in U$ and $x \in V\}$.
$U \cup V$	Union of U and V: the set $\{x \mid x \in U$ or $x \in V$ or both$\}$.
$V \setminus U$	Complement of U in V: the set $\{x \in V \mid x \notin U\}$.
$f \circ g$	Composite of maps: apply g then f.
$\sum_{i=1}^{n} x_i$	Sum $x_1 + x_2 + \cdots + x_n$.
$\prod_{i=1}^{n} x_i$	Product $x_1 x_2 \cdots x_n$.
$\Rightarrow$	Implies, then.
$\Leftrightarrow$	Implies both ways, if and only if.
$a \times b$	Vector cross product of two vectors.
$a \cdot b$	Scalar product of two vectors.
$\|\|a\|\|$	Norm, $\sqrt{a \cdot a}$, of a vector.

2 Euclidean Space $\mathbb{E}^n$

We distinguish between $\mathbb{R}^n$ the real **vector space** of n-tuples of real numbers, and $\mathbb{E}^n$ the **affine point space** of n-tuples of real numbers with difference map:

$$\text{difference} : \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}^n : (p, q) \mapsto \overrightarrow{pq} = (q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n)$$

Thus, we view $\mathbb{E}^n$ as the set of points, together with the standard Euclidean (Pythagorean) distance structure and angles, and $\mathbb{R}^n$ as providing the vectors of directed differences between points. Not all books make this distinction so you need to be prepared to encounter the unstated identification $\mathbb{E}^n = \mathbb{R}^n$.

The derivative of a map $f : \mathbb{E}^m \to \mathbb{E}^n$ at $p \in \mathbb{E}^m$ is the limit of differences that is the best linear approximation to f, at p. Thus, we need vector spaces to define linearity for maps between Euclidean spaces and suitable vector spaces are automatically present at each point of $\mathbb{E}^n$. At each point p in $\mathbb{E}^n$ we construct a vector space $T_p\mathbb{E}^n$, called the **tangent space to $\mathbb{E}^n$ at** p, from the directed difference vectors to lines in $\mathbb{E}^n$ that pass through p.

$$T_p\mathbb{E}^n = \{\alpha'(0) \in \mathbb{E}^n | \alpha \text{ is a line in } \mathbb{E}^n \text{ starting at } p \in \mathbb{E}^n\} = \{\overrightarrow{pq} | q \in \mathbb{E}^n\}$$

where α' is the n-tuple of derivatives of the coordinates of α.

Technically, we collect all of the $T_p\mathbb{E}^n$ together in one large product space:

$$T\mathbb{E}^n \cong \mathbb{E}^n \times \mathbb{R}^n$$

called the **tangent bundle** to $\mathbb{E}^n$, which comes equipped with a natural projection map onto its first component to keep track of the points to which tangent vectors are attached [5].

Curves A (parametrized) **curve** in $\mathbb{E}^n$ is a continuous map

$$\alpha : (a,b) \to \mathbb{E}^n : t \mapsto (\alpha_1(t), \alpha_2, \ldots, \alpha_n(t)) \tag{1}$$

Note that α is a map and its image, path or trace, $\alpha((a,b))$, is a subset of $\mathbb{E}^n$; we keep these concepts distinct. For our purposes, we shall suppose that our curves are **differentiable**, in the sense that the component real functions possess derivatives of all orders. The **velocity** of α is the map

$$\alpha' : (a,b) \to T\mathbb{E}^n : t \mapsto (\alpha_1'(t), \alpha_2'(t), \ldots, \alpha_n'(t)) \tag{2}$$

and its **acceleration** is $\alpha'' = (\alpha')'$. We are particularly interested in **regular** curves which are differentiable and have nowhere zero velocity; then we usually choose the parameter set to make α' a unit vector for all t. The **length** of the curve (1) is the definite integral

$$\text{Length}[\alpha] = \int_{(a,b)} \|\alpha'(t)\| \tag{3}$$

and it is independent of reparametrization.

Exercises on parametrization of curves

Some obvious things should be checked and proofs supplied.

1. Investigate reparametrizations of the curve (1) in the form of diffeomorphisms of intervals

   $$f : (c,d) \to (a,b) \tag{4}$$

   for which $\alpha \circ f = \beta$ defines the same image as α. In particular, there are only two kinds of these; what happens to the velocity components under reparametrizations?

2. There exists a unit speed reparametrization of every regular curve.

3. Unit speed curves are parametrized by arc length.

3 Euclidean Space $\mathbb{E}^3$

This is the space of our normal experience and we distinguish between $\mathbb{R}^3$, the **vector space** or **linear space** of triples of real numbers, and Euclidean 3-space $\mathbb{E}^3$, the **point space** of triples of real numbers. Intuitively, we can think of a vector in $\mathbb{R}^3$ as an arrow corresponding to the *directed line* in $\mathbb{E}^3$ from one point (the blunt end of the vector arrow) to another point (the sharp end of the vector arrow).

In this course we shall be concerned only with three dimensional $\mathbb{E}^3$ but the basic definitions of points, difference vectors and distances are the same for all $\mathbb{E}^n$ with $n = 1, 2, 3, \ldots$; of course, in dimensions higher than 3, the extra directions will arise from other features than ordinary space—such as time, temperature, pressure etc. The important fact to hang onto is that $\mathbb{E}^3$ consists of points represented by coordinates $p = (p_1, p_2, p_3)$ while the directed difference between a pair of such points p, q is a vector $\overrightarrow{pq}$ with components $(q_1 - p_1, q_2 - p_2, q_3 - p_3)$. In modern mathematics, it is customary to omit the overbar when writing vectors and this will be our usual practice; we identify vectors with their sets of components and points with their sets of coordinates.

The space $\mathbb{E}^3$ has one particularly important feature: the availability of the vector cross product on $\mathbb{R}^3$, which simplifies many geometrical proofs. Given two vectors in $\mathbb{R}^3$, their cross product is perpendicular to both and if the original vectors were unit vectors then so is their cross product. We shall use this device to construct normal vectors to curves and to surfaces; in the case of a circle and a sphere, the radial coordinate vector is also a unit normal vector.

> **Our main interest in this course is to develop the geometry of curves and surfaces in $\mathbb{E}^3$. The basic ideas are very simple: a curve is a continuous image of an interval and a surface is a continuous image of a product of intervals; in each case the intervals may be open or closed or neither.** The rate of change of direction of unit normal vectors with respect to arc length gives curvature information.

Difference vectors and distances

The **difference map** gives the vector arrow from one point to another and is defined by

$$\text{difference} : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}^3 : (p, q) \mapsto v = \overrightarrow{pq} = (q_1 - p_1, q_2 - p_2, q_3 - p_3).$$

The **distance map** takes non-negative real values and is defined by

$$\text{distance} : \mathbb{E}^3 \times \mathbb{E}^3 \to [0, \infty) : (p, q) \mapsto ||\overrightarrow{pq}||$$

here, $||\ ||$ denotes the operation of taking the **norm** or absolute value of the vector, defined by

$$||(q_1 - p_1, q_2 - p_2, q_3 - p_3)|| = +\sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$$

Then we can view $\mathbb{E}^3$ as the set of points representing ordinary space, together with the standard Euclidean angles and Pythagorean distances and $\mathbb{R}^3$ provides the vectors of **directed differences** between points. Not all books make this distinction so you need to be prepared to encounter the unstated identification $\mathbb{E}^3 = \mathbb{R}^3$. Often, we use the coordinates (x, y, z) for points in $\mathbb{E}^3$ and denote by $\mathbb{E}^2$ the set of points in $\mathbb{E}^3$ with $z = 0$ and then we abbreviate $(x, y, 0)$ to (x, y).

The **standard unit sphere** $\mathbb{S}^n$ in a Euclidean n-space is the set of points unit distance from the origin; we shall often use $\mathbb{S}^1$ in $\mathbb{E}^2$ and $\mathbb{S}^2$ in $\mathbb{E}^3$. A parametric equation for the unit 2-sphere $\mathbb{S}^2$ in $\mathbb{E}^3$ is given by

$$g : [0, 2\pi] \times [-\pi/2, \pi/2] :\to \mathbb{E}^3 : (u, v) \mapsto (\cos v \cos u, \cos v \sin u, \sin v).$$

4 Group actions

In algebra, geometry and topology we often exploit the fact that important structures arise from families of morphisms that are indexed by a group. For example, rotations in the plane about the origin are indexed by the unimodular group of complex numbers; we say that this group **acts on the plane** and the **orbit** of a point at distance r from the origin is the circle of radius r.

We use in geometry the groups that act on subsets of $\mathbb{E}^n$ while preserving Euclidean distances and angles; these are **groups of isometries** of $\mathbb{E}^n$. They form subgroups of matrix groups. The set of $n \times n$ nonsingular real matrices forms a group $GL(n, \mathbb{R})$, often just written $GL(n)$, the **general linear group**, under matrix multiplication. So does $O(n)$, the subset consisting of **orthogonal** matrices, and its subset $SO(n)$ consisting of those with determinant $+1$.

The **Euclidean group** $E(n)$ consists of all isometries of Euclidean n-space $\mathbb{E}^n$. Isometries can always be written as an ordered pair from $O(n) \times \mathbb{R}^n$ with action on $\mathbb{E}^n$ given by

$$(O(n) \times \mathbb{R}^n) \times \mathbb{E}^n \longrightarrow \mathbb{E}^n : ((\alpha, u), x) \longmapsto \alpha(x) + u$$

and composition

$$(\alpha, u)(\beta, v) = (\alpha\beta, \alpha(v) + u)\,.$$

Thus, topologically $E(n)$ is the product $O(n) \times \mathbb{R}^n$ but algebraically it is not the product group. It is called a **semidirect product** of $O(n)$ and $\mathbb{R}^n$.

Definitions

A group G is said to **act** on a set (for example, a group, vector space, manifold, topological space) X **on the left** if there is a map (for example, homomorphism, linear, smooth, continuous)

$$\alpha : G \times X \longrightarrow X : (g, x) \longmapsto \alpha_g(x)$$

such that $\alpha_{g*h}(x) = \alpha_g(\alpha_h(x))$ and $\alpha_e(x) = x$ for all $x \in X$. Normally, we shall want each $\alpha_g : X \to X$ to be an isomorphism in the category for X; in this case, an **action** is the same as a representation of G *in* the automorphism group of X, or a representation *on* X. We sometimes abbreviate the notation to $g \cdot x$, especially when α is fixed for the duration of a discussion. There

is a dual theory of actions on the right; we have to keep the concepts separate because every group acts on itself by its group operation, but it may be different on the right from on the left.
The **orbit** of $x \in X$ under the action α of G is the set

$$G \cdot x = \{\alpha_g(x) \mid g \in G\}\,.$$

It is easy to show that the orbits partition X, so they define an equivalence relation on X:

$$x \sim y \iff \exists\, g \in G \text{ with } \alpha_g(x) = y\,.$$

The quotient object (set, space, *etc.*) is called the **orbit space** and denoted by X/G.
The **stabilizer** or **isotropy subgroup** of x is defined to be the set

$$\mathrm{stab}_G(x) = \{g \in G \mid \alpha_g(x) = x\},$$

and it is always a subgroup of G.
The action is called **transitive** if for all $x, y \in X$ we can find $g \in G$ such that

$$\alpha_g(x) = y \qquad (\text{so also } \alpha_{g^{-1}}(y) = x)\,,$$

free if the only α_g with a fixed point has $g = e$ (the identity of G),
and **effective** if

$$\alpha_g(x) = x \quad (\forall\, x \in X) \implies g = e.$$

Note that an action being transitive is equivalent to it having exactly one orbit, or to its orbit space being a singleton.
The situations of most practical interest are when:

- X is a subset of Euclidean space, a group or vector space—especially $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{R}^n$;
- G, X are **topological groups**, so each has a topology with respect to which its binary operation and the taking of inverses is continuous;
- G is a topological group and X is a topological space;
- G is a **Lie group**, so G has a differentiable structure with respect to which its binary operation and the taking of inverses is smooth, and X is a smooth manifold. Here smooth means all derivatives of all orders exist and are continuous. Important examples of Lie groups are $\mathbb{R}^n$, $GL(n)$ and $\mathbb{S}^1$, where the differentiability arises from that of the underlying real functions.

Exercises on group actions

1. $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$.
2. The **symmetric group** S_n of permutations of n objects is not abelian for $n > 2$.
3. Find a group G consisting of four, 2×2 real matrices such that G acts on the plane $\mathbb{E}^2$. For the case $n = 2$ find discrete subgroups $G < E(2)$ such that $\mathbb{R}^n/G$ is: (i) the cylinder; (ii) the torus.
4. The general linear group $GL(n; \mathbb{R})$ is not abelian if $n > 1$.
5. Prove that $GL(2)$ has a subgroup consisting of rotations in a plane

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \ \theta \in \mathbb{R} \right\}.$$

   This is actually $SO(2)$, the **special orthogonal group** of 2×2 real matrices.

6. Find an isomorphism

$$f : SO(2) \to \{z \in \mathbb{C}|\ |z| = 1\}$$

and give its inverse.

7. Prove that, for all elements a in group G, the map

$$c_a : G \to G : x \mapsto a^{-1}xa$$

is an automorphism; find the inverse of c_a.

8. The group $SO(2)$ of rotations in a plane acts on a sphere $\mathbb{S}^2$ as rotations of angles of longitude. The orbits are circles of latitude and the quotient space by this action is the interval $[-1, 1]$. The action is neither transitive nor free, but it is effective.

9. Prove that $SO(2)$ defines a left action on $\mathbb{E}^2$ by

$$\rho : SO(2) \times \mathbb{E}^2 \to \mathbb{E}^2 : (A, p) \mapsto L_A p$$

where $L_A p$ denotes matrix multiplication of the coordinate column vector p by the matrix A. To establish this you need to show that the map ρ is well-defined and that it satisfies two rules for all $p \in \mathbb{E}^2$ and all $A, B \in SO(2)$, namely

Product $L_A(L_B p) = L_{AB} p$

Identity $L_I p = p$

[In fact, the whole of the general linear group $GL(2)$ acts on $\mathbb{E}^2$.]

10. Prove that the action ρ is effective but neither free nor transitive. Find the orbits under this action of the points on the x-axis of $\mathbb{E}^2$.

11. Prove that the action ρ preserves the scalar product; that is, for all $p, q \in \mathbb{E}^2$ and all $A \in SO(2)$,

$$L_A p \cdot L_A q = p \cdot q$$

Hence deduce that the action preserves Euclidean angles, lengths and areas.

12. Show that

$$L_J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO(2)$$

and find the image under L_J of the unit square in the upper right quadrant of $\mathbb{E}^2$. [Hint: Check the edge vectors.] Find an element $K \in GL(2)$ with $K \notin SO(2)$ and $\det K = -1$. This defines a linear map L_K; compare its effect on the unit square with the image found for L_J.

13. It is clear that $GL(3)$, which acts on $\mathbb{E}^3$, has a subgroup $SO(3)$, consisting of 3×3 real matrices having determinant $+1$. Find three distinct subgroups of $SO(3)$, consisting of rotations around the three coordinate axes, respectively, by finding three group homomorphisms $SO(2) \to SO(3)$ with trivial kernels.

14. Use the subgroups of $SO(3)$ found in the previous exercise, and the parametric equation for the equator of $\mathbb{S}^2$, to show how any other great circle on $\mathbb{S}^2$ can be found by appropriate combinations of rotations of the equator.

15. Find two matrices, R_1 and R_2 from $SO(3)$ which represent, respectively, rotation by $\pi/3$ about the y-axis and rotation by $\pi/4$ about the z-axis; each rotation must be in a right-hand-screw sense in the positive direction of its axis. Find the product matrix $R_1 R_2$ and show that its transpose is its inverse.

5 Curves in $\mathbb{E}^3$

Given a point $p \in \mathbb{E}^3$ and a vector $v \in \mathbb{R}^3$, there is always a unique point $q \in \mathbb{E}^3$ such that $\overrightarrow{pq} = v$; intuitively, q is at the point of the arrow v when its tail is at p. Then we can write $q = p + v$ and the **line segment** from p (in the direction v) to q is given in parametric form by

$$L : [0,1] \to \mathbb{E}^3 : t \mapsto p + tv$$

or equivalently

$$L : [0,1] \to \mathbb{E}^3 : t \mapsto (p_1, p_2, p_3) + t(v_1, v_2, v_3)$$

and we say that the **tangent vector**, or **velocity vector** of this line at $t \in [0,1]$ is v. Thus, we can write this as the derivative, which is the limit of differences:

$$D_t L = \lim_{h \to 0} \frac{L(t+h) - L(t)}{h} = \frac{dL}{dt} = v$$

because L is a linear function of t, since p and v are constant.

As we shall see, a curve may have a nonlinear dependence on its parameter and a velocity vector that varies in magnitude and direction, so curves are natural generalisations of line segments.

Tangent vector, speed and acceleration vector

A **curve** in $\mathbb{E}^3$ with parameter t satisfying $a \leq t \leq b$ is a continuous map

$$\alpha : [a,b] \to \mathbb{E}^3 : t \mapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \tag{5}$$

Note that α is a map and its image, (also called track or path), $\alpha([a,b])$, is a subset of $\mathbb{E}^3$; we keep these concepts distinct. The curve starts at the point $\alpha(a)$ and ends at the point $\alpha(b)$. Sometimes, one or both endpoints of the curve are absent; so in general the domain of a curve may be an interval of any kind.

For our purposes, we shall suppose that our curves are **differentiable**, in the sense that the components, $\alpha_1, \alpha_2, \alpha_3$, are real functions of t possessing derivatives of all orders—so no corners like those in the graph of $|x|$. The **tangent vector** or **velocity** of α is the vector valued map $D_t\alpha = \alpha'$ which in components is given by

$$\alpha' : [a,b] \to \mathbb{R}^3 : t \mapsto (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)) \tag{6}$$

and its **speed** is the absolute value of the velocity vector. The **acceleration** of α is the vector $\alpha'' = (\alpha')'$, given by the derivative of the velocity. Observe that the velocity and acceleration vectors are attached to the curve and change as the parameter moves the point of attachment.

We are particularly interested in **regular** curves which are differentiable and have nowhere zero velocity (they are always going somewhere, not stopped); then we make calculations easier if we choose the parameter set $0 < s < L$ to make α' a unit vector for all s, and L is actually the total length of the curve.

The **length** of the curve (1) is defined as the integral of the speed over the domain $[a,b]$

$$\text{Length}[\alpha] = \int_{[a,b]} \|\alpha'(t)\| = \int_a^b \sqrt{\alpha_1'(t)^2 + \alpha_2'(t)^2 + \alpha_3'(t)^2}\; dt \tag{7}$$

and it is independent of reparametrization. Unit speed curves are parametrized by arc length because then $\|\alpha'(t)\| = 1$ for all t. In general, it is difficult to calculate analytically the arc length of a given curve—because of the presence of the square root of sums of functions in the integrand; the same is true for other calculations for curves and surfaces but computer algebra software can help[1].

[1] On this course we use the computer algebra package *Mathematica* [10] to perform calculus and create graphics for curves and surfaces; Gray [6] provides the necessary *Mathematica* input to plot and study virtually all named curves and surfaces and perform analytic calculations on them—including the solution of geodesic equations on the surfaces and construction of curves with prescribed curvature and torsion. The necessary files can be found via the web server [1] An important source of curves is from solutions of ordinary differential equations. Gray et al. [7] provided a definitive text on this subject, together with associated *Mathematica* functions to solve ordinary differential equations analytically and numerically, and plot families of solutions.

6 Plane Curves

A **plane curve** is a curve that lies in some plane in $\mathbb{E}^3$. If the curve lies in the $z = 0$ plane, then we may write the curve with just two components in the form

$$\alpha : [a, b] \to \mathbb{E}^2 : t \mapsto (\alpha_1(t), \alpha_2(t)) \tag{8}$$

In general, of course, whether a curve lies in a plane is not obvious from its equation; we shall construct in the section on space curves a function called **torsion** that measures departure from planarity for general curves in $\mathbb{E}^3$.

On regular plane curves, we can measure the curvature as the rate of change of the direction of a unit tangent vector with arc length. We call this the **signed curvature** $\kappa 2$, of α, defined by

$$\kappa 2[\alpha](t) = \frac{\alpha''(t) \cdot J\alpha'(t)}{\|\alpha'(t)\|^3}, \tag{9}$$

in which J is the linear operator

$$J : \mathbb{R}^2 \to \mathbb{R}^2 : (p, q) \mapsto (-q, p), \quad \text{so always } J\alpha' \cdot \alpha' = 0. \tag{10}$$

At points where the curvature $\kappa 2$ is nonzero, $1/\kappa 2[\alpha]$ is called the **radius of curvature** of α.

The rate at which the angular direction Θ of a regular plane curve changes can be calculated by differentiating its normalized velocity vector and we find that this coincides with the curvature for unit speed curves. Denote the speed by $v(t) = ||\alpha'(t)|| > 0$, then we deduce:

$$\begin{aligned}
\frac{\alpha'(t)}{v(t)} &= (\cos(\Theta(t), \sin(\Theta(t)) & (11)\\
\frac{v(t)\alpha''(t) + v'(t)\alpha'(t)}{v(t)^2} &= \Theta'(t)(-\sin(\Theta(t), \cos(\Theta(t)) & (12)\\
J\alpha'(t) &= v(t)(-\sin(\Theta(t), \cos(\Theta(t)) & (13)\\
\kappa 2(t) &= \frac{\Theta'(t)}{v(t)}, \quad \text{using (10) and (9).} & (14)
\end{aligned}$$

So, we have proved the following for the case of constant $v = 1$ in (14):

Theorem 6.1 (Curvature of Plane Curves) *A regular unit speed plane curve has curvature $\kappa 2$ given by the rate of change with arc length of the angular direction of its tangent vector.*

In fact, $\kappa 2$ gives a complete classification of regular plane curves, up to a Euclidean motion:

Theorem 6.2 (Fundamental Theorem of Plane Curves) *Two regular plane curves defined on the same interval with the same curvature $\kappa 2$, can be transformed into one another by application of a translation and an orthogonal transformation.*

Gray [6] proves this classification theorem and studies applications in detail, giving many examples and a *Mathematica* algorithm for drawing a curve in $\mathbb{E}^2$ with specified curvature.

Exercises on plane curves

1. Show that a parametric equation for the line segment from $(3, 5)$ to $(6, 1) \in \mathbb{E}^2$ is given by

   $$\ell : [0, 1] \to \mathbb{E}^2 : t \mapsto (3, 5) + t(3, -4).$$

   Note that the line segment is the **image** of the map ℓ. What is the tangent vector of ℓ? Show that it has zero acceleration and zero curvature.

2. Find another parametric equation for the line segment in the previous example, but having a tangent vector with half the magnitude of that used for ℓ.

3. Clearly, there is a well-defined continuous curve given by

$$h : [-1, 1] \to \mathbb{E}^2 : t \mapsto (t, |t|). \tag{15}$$

What happens to its tangent vector at $t = 0$? Show that the curve has a well-defined length.

4. Show that for all $\theta \in [0, 2\pi]$ the matrix

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

when applied to the coordinates of a curve (viewed as a column vector) rotates the curve through angle θ in the plane, that is, round the z-axis. Find a suitable θ value that rotates the curve in the previous question through 30^o; give an explicit equation for the rotated curve.

5. Why does the rotation matrix leave the length of a curve unaltered? What does the rotation matrix do to the curvature of a curve?

6. Can a reflection in the line $y = x$ be a rotation?

7. Find θ for the rotation matrix corresponding to the linear operator

$$J : \mathbb{R}^2 \to \mathbb{R}^2 : (p, q) \mapsto (-q, p)$$

and show that J gives an anticlockwise rotation of $\pi/2$ in $\mathbb{R}^2$.

8. Apply a rotation

9. Express the equation for the parabola $y = 3x^2 + 2$ in parametric form and find its velocity, speed and acceleration. Show that the same image can be obtained by another curve with velocity vector the negative of that of the first one.

10. Verify that a curve defining the unit circle in $\mathbb{S}^1 \subset \mathbb{E}^2$ with centre at the origin O given by the set

$$\mathbb{S}^1 = \{p \in \mathbb{E}^2 | \ \mathrm{dist}(p, O) = 1\}$$

has a parametric equation given by

$$f : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto (\cos t, \sin t).$$

Find its velocity, speed, acceleration, curvature and length. Show that the same image can be obtained by another curve with velocity vector the negative of that of the first one.

11. For a unit speed plane curve α, show that the acceleration is related to the curvature by $\alpha'' = \kappa 2[\alpha] J\alpha'$.

12. Find an expression for $\kappa 2[\alpha](t)$ in terms of its component functions.

13. Investigate the hyperbola

$$f : [-1, 1] \to \mathbb{E}^2 : t \mapsto (\cosh t, \sinh t).$$

14. Plot the limaçon (French name for slug—why?) given by

$$l : [0, 2\pi] \to \mathbb{E}^2 : t \mapsto ((2\cos t + 1)\cos t, (2\cos t + 1)\sin t)$$

and show that the curve passes twice through the origin in different directions, which emphasises why we have to specify the parameter value and not the point on the curve when we require the velocity vector.

15. Find a parametric equation for the equator of the sphere $\mathbb{S}^2$, and for a perpendicular circle of longitude.

Implicitly defined plane curves
We know that some curves are defined implicitly, like the unit circle,

$$x^2 + y^2 - 1 = 0 \tag{16}$$

However, for $f(x, y) = 0$ to define a parametrized curve near some point (x_0, y_0) where f is zero, it is sufficient for f to have at least one of its partial derivatives nonzero there.
Exercises on implicit curves

1. Use the implicit function theorem to prove this assertion.
2. Investigate the sets of zeros of the following function and a slightly perturbed version

$$f(x, y) = x^3 + y^3 - 3xy \tag{17}$$
$$f^*(x, y) = x^3 + y^3 - 3xy - 0.01 \tag{18}$$

for which Gray [6] gives graphs on pages 59 and 60.

Evolutes and involutes of plane curves
The loci of centres of circles (called **osculating circles**), that are tangent to the plane curve α in (1) and have radii equal to the radii of curvature at the points of tangency, is a new plane curve called the **evolute** of α. So, the evolute of a circle is its centre point. Explicitly,

$$\text{evolute}[\alpha] = \alpha + \frac{1}{\kappa 2}\frac{J\alpha'}{\|\alpha'\|} \tag{19}$$
$$= \alpha + \frac{\|\alpha'\|^2 J\alpha'}{\alpha'' \cdot J\alpha'} \tag{20}$$

The **involute**, starting at $c \in (a, b)$, of the plane curve α in (1) is the plane curve

$$\text{involute}[\alpha, c] = \alpha + (c - s)\frac{\alpha'}{\|\alpha'\|} \tag{21}$$

where s is the arc length function of α.
Exercises on evolutes and involutes

1. The formula for evolute is independent of reparametrization of α.
2. The evolute of an involute of α is again α.

7 Space Curves

In 3-space we take advantage of the usual vector algebra operations available on $\mathbb{R}^3$ to study the curvature (departure from linearity) and torsion (departure from planarity) of curves in space. Since we are interested in curves with nonzero speed everywhere, we can always reparametrize to have unit speed; then the parameter coincides with arc length along the curve, often denoted by s.
Curvature, torsion and the Frenet-Serret equations
The **curvature** of the unit speed space curve

$$\beta : [a, b] \to \mathbb{E}^3 \tag{22}$$

is the norm of its acceleration

$$\kappa[\beta](s) = \|\ddot{\beta}(s)\| \quad \text{where } \dot{\beta} = \frac{d\beta}{ds} \text{ and } s \text{ is arc length} \tag{23}$$

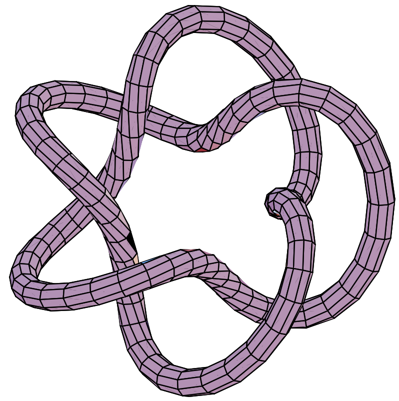

Figure 1: **A torus knot: This tube is a thickened embedding of a circle that has been mapped onto the surface of a torus; note the twisting of the ruling lines, showing high torsion when there is rapid departure from a plane.**

It is easy to show that the velocity vector $\dot{\beta}$ is perpendicular to the acceleration vector $\ddot{\beta}$ by differentiating $(\dot{\beta} \cdot \dot{\beta}) = 1$. So if we take their cross product we get a vector perpendicular to both; we have only three dimensions and so the derivative of the new vector must be expressible in terms of the others. In this way, three, mutually perpendicular unit vectors $\{T, N, B\}$ arise at each point: $T = \dot{\beta}$, $N = \dot{T}/\kappa$ and $B = T \times N$. These vector functions along the curve β with curvature κ are controlled by the famous
Frenet-Serret equations for unit-speed curves:

$$\dot{T} = \kappa N \qquad \text{recall that we have } \kappa > 0 \tag{24}$$
$$\dot{N} = -\kappa T + \tau B \tag{25}$$
$$\dot{B} = -\tau N \tag{26}$$

Here, N is the **principal normal**, B is the **binormal** and τ is the **torsion**. $\{T, N, B\}$ is called the **Frenet frame field** along β, and consists of three mutually perpendicular unit vectors—a triad that moves along the curve with T pointing always forward.
For a regular curve α with arbitrary speed $\sqrt{\alpha' \cdot \alpha'} = ||\alpha'|| = v > 0$, we have the
Frenet-Serret equations for arbitrary-speed curves:

$$T' = v\kappa N \qquad \text{recall that we have } \kappa > 0 \tag{27}$$
$$N' = -v\kappa T + v\tau B \tag{28}$$
$$B' = -v\tau N. \tag{29}$$

Exercises on Frenet-Serret equations
Here, β is the unit speed curve in equation (22).

1. Show that the helix
$$\gamma : [0, 10] \to \mathbb{E}^3 : s \mapsto (2\cos(\frac{s}{\sqrt{5}}), 2\sin(\frac{s}{\sqrt{5}}), \frac{s}{\sqrt{5}}) \tag{30}$$
is a unit speed curve and has constant curvature and torsion.

2. Why do we always have $\kappa[\beta] \geq 0$?

3. For all s, $\ddot{\beta}(s) \cdot \dot{\beta}(s) = 0$; so the acceleration is always perpendicular to the acceleration along unit-speed curves. What about $\alpha'(t) \cdot \alpha''(t)$ on arbitrary speed curves?

4. Derive the **Frenet-Serret equations for an arbitrary-speed regular curve** α and show

that the following hold for a curve α with speed $\sqrt{\alpha' \cdot \alpha'} = ||\alpha'|| = v > 0$:

$$T = \alpha'/v, \qquad N = B \times T, \qquad B = \frac{\alpha' \times \alpha''}{||\alpha' \times \alpha''||} \tag{31}$$

$$\kappa = \frac{||\alpha' \times \alpha''||}{v^3} \tag{32}$$

$$\tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{||\alpha' \times \alpha''||^2}. \tag{33}$$

5. Viviani's curve: is the intersection of the cylinder $(x-a)^2 + y^2 = a^2$ and the sphere $x^2 + y^2 + z^2 = 4a^2$ and has parametric equation:

$$\alpha : [0, 4\pi] \to \mathbb{E}^3 : t \mapsto a(1 + \cos t, \sin t, 2 \sin \frac{t}{2}).$$

   An animated Frenet-Serret frame graphic for this curve is given at:
   http://www.maths.manchester.ac.uk/ kd/latextut/pdfbyex.htm

   Show that it has curvature and torsion given by

$$\kappa(t) = \frac{\sqrt{13 + 3 \cos t}}{a\,(3 + \cos t)^{\frac{3}{2}}} \quad \text{and} \quad \tau(t) = \frac{6 \cos \frac{t}{2}}{a(13 + 3 \cos t)}.$$

6. Investigate the following curves for $n = 0, 1, 2, 3$

$$\gamma : [0, 2\pi\sqrt{6}] \to \mathbb{E}^3 : s \mapsto (\sqrt{6}\cos(\frac{s}{\sqrt{6}}), \sqrt{\frac{3}{2}}\sin(\frac{s}{\sqrt{6}}), \frac{\sqrt{3}}{2}\sin(\frac{ns}{\sqrt{6}})) \tag{34}$$

7. Show that for all $\theta \in [0, 2\pi]$ the matrix

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   when applied to the coordinates of a curve in $\mathbb{E}^3$ rotates the curve through angle θ in the (x, y)-plane, that is, round the z-axis. Find a matrix $R_y(\theta)$ representing rotation round the y-axis and hence obtain explicitly the result of rotating the curves in the previous question by 60^o round the y-axis.

8. On plane curves, $\tau = 0$ everywhere and we use the signed curvature $\kappa 2$, defined by

$$\kappa 2[\alpha](t) = \frac{\alpha''(t) \cdot J\alpha'(t)}{\|\alpha'(t)\|^3}, \qquad \text{where } J : \mathbb{R}^2 \to \mathbb{R}^2 : (p, q) \mapsto (-q, p). \tag{35}$$

   Show how $\kappa 2$ is related to κ for a general planar curve in $\mathbb{E}^3$, not necessarily in the $z = 0$ plane.

9. Give an equation of a regular curve in $\mathbb{E}^2$ and then apply a rotation out of the $z = 0$ plane. Show that for this rotated curve the torsion is zero and find the radius of curvature $1/\kappa 2$.

10. **(i)** Find two matrices, R_y and R_z from $SO(3)$ which represent, respectively, rotation by $\pi/3$ about the y-axis and rotation by $\pi/4$ about the z-axis; each rotation must be in a right-hand-screw sense in the positive direction of its axis. Find the product matrix $R_y R_z$ and show that its transpose is its inverse.
    (ii) By considering $(R_y R_z)^{-1}$, or otherwise, show that the curve

$$\gamma : [0, \infty) \to \mathbb{E}^3 : t \mapsto (\frac{1}{\sqrt{2}}\cosh t/2 + \frac{t}{2\sqrt{2}}, -\sqrt{2}\cosh t/2 + \frac{t}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}\cosh t/2 - \frac{\sqrt{3}t}{2\sqrt{2}})$$

    lies in a plane and find its curvature function and arc length function.

11. Vertical projection from $\mathbb{E}^3$ onto its xy-plane is given by the map

$$\pi : \mathbb{E}^3 \to \mathbb{E}^3 : (x, y, z) \mapsto (x, y, 0).$$

A unit speed curve $\beta : [0, L] \to \mathbb{E}^3$ lies above the xy-plane and has vertical projection

$$\pi \circ \beta : [0, L] \to \mathbb{E}^3 : s \mapsto (\frac{s}{2} \cos(\log s/2), \frac{s}{2} \sin(\log s/2), 0).$$

Find explicitly a suitable β and for it compute the Frenet-Serret frame, curvature and torsion.

12. Investigate a selection of named curves from Gray [6, 1].

Classification of curves in $\mathbb{E}^3$

Regular space curves with nonzero curvature are classified by their curvature and torsion, up to a Euclidean transformation (translation plus reflection and/or rotation):

Theorem 7.1 (Fundamental Theorem of Space Curves) *Two space curves defined on the same interval with the same torsion and nonzero curvature can be transformed into one another by application of a translation and a Euclidean transformation.*

Gray [6] proves this classification theorem and studies applications in detail, giving many examples and a *Mathematica* algorithm for drawing a curve in $\mathbb{E}^3$ with specified curvature and torsion.

White's Theorem: $Lk = Tw + Wr$

For a smooth simple curve α with arc length function s and arbitrary speed $s' = v = ||\alpha'||$, we have the Frenet orthonormal frame T, N, B and the Frenet-Serret equations:

$$\begin{aligned} T' &= v\kappa N \qquad \text{recall that we have } \kappa > 0 & (36)\\ N' &= -v\kappa T + v\tau B & (37)\\ B' &= -v\tau N. & (38) \end{aligned}$$

Let U be a unit normal vector field along α. The **Twist** of U along α is

$$Tw(\alpha, U) = \frac{1}{2\pi} \int_\alpha T \times U \cdot \dot{U}\, ds \tag{39}$$

Denote by $U^\perp = T \times U$. Then $Tw(\alpha, U)$ is precisely the net rotation of the frame $(T, U, U^\perp)$ along α in the direction T.

Express the frame $(T, U, U^\perp)$ in the form

$$\begin{aligned} U &= \cos\phi\, N + \sin\phi\, B & (40)\\ U^\perp &= -\sin\phi\, N + \cos\phi\, B & (41) \end{aligned}$$

By orthonormality, $U^\perp \cdot dU = B \cdot dN + d\phi$ and integrating we obtain

$$\begin{aligned} Tw(\alpha, U) &= \frac{1}{2\pi} \int_\alpha B \cdot dN + \frac{1}{2\pi} \int_\alpha d\phi & (42)\\ &= tor(\alpha) + \Phi(\alpha, U). & (43) \end{aligned}$$

So in fact we have decomposed $Tw(\alpha, U)$ into the total torsion of α, that is the integral of torsion, plus the winding number of U round α.

Consider a pair of smooth simple disjoint curves α, μ with arc length functions $s_1,\ s_2$, defined on the same interval. Put $w(t) = \mu(t) - \alpha(t)$, we have a well-defined normal field along α given by $V = w - (w \cdot T)T$ and, since V is never zero, we have a unit normal field $U = V/||V||$. Hence, $Tw(\alpha, U)$ represents also the **total twist of curve μ about curve α**.

$$Tw(\alpha, \mu) = tor(\alpha) + \Phi(\alpha, \mu). \tag{44}$$

Now, we can extend α to a smooth map $\alpha^\dagger : \mathbb{B}^2 \to \mathbb{E}^3$ on the whole of the unit disk $\mathbb{B}^2$. We define the **Link Number** $Lk(\alpha, \mu)$ of α, μ to be the number of times μ intersects $\alpha^\dagger$ in the direction of its unit normal. It turns out that $Lk(\alpha, \mu)$ is independent of the choice of extension of α and is computable by

$$Lk(\alpha, \mu) = \frac{1}{4\pi} \int_{\alpha \times \mu} \frac{\partial U}{\partial s_1} \times \frac{\partial U}{\partial s_2} \cdot U \; ds_1 \; ds_2 \tag{45}$$

where again $w(t) = \mu(t) - \alpha(t)$, $V = w - (w \cdot T)T$ and $U = V/||V||$.

The **Writhe** of the smooth closed simple curve α is

$$Wr(\alpha) = \frac{1}{4\pi} \int_{\alpha \times \alpha} \frac{\partial U}{\partial s_1} \times \frac{\partial U}{\partial s_2} \cdot U \; ds_1 \; ds_2. \tag{46}$$

White's Theorem states that

$$Lk(\alpha, \mu) = Tw(\alpha, \mu) + Wr(\alpha). \tag{47}$$

Examples

1. In the case that α is planar, $\tau = 0$ so $tor(\alpha) = 0$, and since $\frac{\partial U}{\partial s_1}$, $\frac{\partial U}{\partial s_2}$ and U all lie in a plane, $Wr(\alpha) = 0$. Then, correctly, we have $Lk(\alpha, \mu) = \Phi(\alpha, \mu)$.

2. If α lies on the surface of $\mathbb{S}^2$, then the choice $U(t) = \alpha(t)$ and hence $\mu(t) = 2\alpha(t)$ leads to $Lk(\alpha, \mu) = 0$. Also, $U \cdot T = 0$ and $T = \frac{dU}{ds_1}$ so $Tw(\alpha, \mu) = 0$ and therefore $Wr(\alpha) = 0$.

White's theorem has applications in molecular biology, where the two curves α, μ correspond to the edges of DNA ribbons and supercoiling corresponds to writhe. Enzymes exist that alter the constituents in the equation $Lk(\alpha, \mu) = Tw(\alpha, \mu) + Wr(\alpha)$ in order to compactify the molecule as protection against a virus or locally expand it to allow replication.

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Mathematicians: http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/

Alfred Gray's Mathematica NoteBooks: http://library.wolfram.com/infocenter/Books/3759

Elementary Notes on:
Curves http://www.maths.manchester.ac.uk/ kd/curves/curves.pdf
Surfaces http://www.maths.manchester.ac.uk/ kd/curves/surfaces.pdf
Knots http://www.maths.manchester.ac.uk/ kd/curves/knots.pdf

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