Modeling a class of stochastic porous media

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Abstract

This note extends E.H. Lloyd’s model of pore structure in random fibre networks, to a large class of stochastic fibre networks containing the random model as a special case. The key to the generalization is the substitution of a family of gamma distributions for the negative exponential family used for inter-crossing distances on fibres. This allows closed expressions to be obtained for the variance and mean of the equivalent pore size distributions in a planar array of line elements representing fibres. The analytical details have been made available in a Mathematica notebook, via the World Wide Web. The model has application in modeling the forming of non-woven textiles and paper from fibre suspensions, and in modeling their void structures and transmission of fluids.

Keywords: Stochastic, Porous media, Fibre network, Pore statistics

1 Introduction

In a planar network of random lines, the mean number of sides per polygon is four, so Corte and Lloyd [1, 2] estimated the pore size distribution as the product of two negative exponential distributions, which are known to give a good approximation to the inter-crossing lengths in a random network [3]. Here we repeat this analysis using the gamma distribution as a generalisation of inter-crossing length distributions for more general stochastic fibre networks; then the negative exponential distribution is a special case. We used the computer algebra package Mathematica for the calculus and graphics; the code for our calculations is available from the authors or via the World Wide Web [4].

In applications to non-woven textiles and paper, a random arrangement of fibres is a common target structure. However, in commercial production such a degree of dispersion is hard to achieve because of a tendency for the fibres to clump together, or ‘flocculate’. So the random case becomes an ‘upper bound on uniformity’. The new models provide a family of commercially realizable structures.

2 Pore size distributions

The gamma distribution has a probability density function given by:

\[ f(x) = \frac{b^k}{\Gamma(k)} x^{k-1} e^{-bx}, \]

with mean, \( \bar{x} = \frac{k}{b} \) and variance \( \text{Var}(x) = \frac{k}{b^2} \). The negative exponential distribution is a gamma distribution with \( k = 1 \). Thus, \( k \) and \( b \) are the parameters which represent the departure from the random case, for which \( k = 1 \) and \( b_{\text{rand}} = 1/\bar{x} \). For clumped or ‘flocculated’ stochastic fibre networks, we expect \( \frac{k}{b^2} > \frac{1}{b_{\text{rand}}^2} \) and to increase with increasing fibre clumping. For disperse stochastic fibre networks, we expect \( b^2 \gg k \) such that \( \frac{k}{b^2} < \frac{1}{b_{\text{rand}}^2} \) and to decrease with increasing uniformity.

We consider the product of two independent identical gamma distributions \( f(x) \) and \( f(y) \) such that \( xy = a \) where \( a \) is the area of a rectangular pore. The probability density of \( a \) will be given by:

\[ p(a) = \int_0^\infty \frac{1}{x} f(x) f\left(\frac{a}{x}\right) dx, \]

Evaluation of the integral in equation 2 gives us,

\[ p(a) = \frac{2a^{k-1} b^2 K_0(z)}{\Gamma(k)^2}, \quad \text{where} \quad z = 2b\sqrt{a} \]
and $K_0(z)$ is the zeroth order modified Bessel function of the second kind. The distribution given by equation (3) has mean, $\bar{a} = k^2 b^2$ and variance $\text{Var}(a) = \frac{k^2(1+2k)}{b^2}$. Following Corte and Lloyd, we define an equivalent pore radius $r$ which is given by $a = \pi r^2$. The probability of finding an equivalent pore radius $r_1 \leq r \leq r_2$, is given by:

$$\int_{\pi r_1^2}^{\pi r_2^2} p(a) \, da = \int_{r_1}^{r_2} p(\pi r^2) \, 2\pi r \, dr .$$

(4)

So the probability density function for equivalent pore radii is:

$$q(r) = 2 \pi r p(\pi r^2) ,$$

(5)

which gives us:

$$q(r) = \frac{4 b^{2k} \pi^k r^{2k-1} K_0(z)}{\Gamma(k)^2} , \quad \text{where } z = 2br\sqrt{\frac{r}{\pi}},$$

(6)

and $\int_0^\infty q(r) \, dr = 1$. The mean and variance of $q(r)$ are given by:

$$\bar{r} = \frac{\Gamma(k + \frac{1}{2})^2}{b \sqrt{\pi} \Gamma(k)^2}$$

(7)

and

$$\text{Var}(r) = \frac{k^2 \Gamma(k)^4 - \Gamma(k + \frac{1}{2})^4}{b^2 \pi^2 \Gamma(k)^4} = \bar{r}^2 \left( \frac{k^2 \Gamma(k)^4}{\Gamma(k + \frac{1}{2})^4} - 1 \right) .$$

(8)

For a random network, $k = 1$ and the distribution of pore radii has mean, $\bar{r} = \frac{\sqrt{\pi}}{4b}$, and variance, $\text{Var}(r) = \frac{1}{b^2} \left( \frac{1}{\pi} - \frac{1}{16} \right)$ in agreement with Corte and Lloyd [1, 2].

Corte and Lloyd used a multiplanar model of paper with layers of capillaries of distributed radii. Then for a fluid flow proportional to $r^\kappa$ in a capillary of radius $r$, the mean effective radius averaged over $m$ layers is

$$r_{\text{eff}}(m, \kappa) = \left( \sum_{i=1}^{m} r_i^{-\kappa} \right)^{1/\kappa} .$$

(9)

Typical flow regimes that may be used are: laminar or Poiseuille $\kappa = 4$; molecular or Knudsen $\kappa = 3$; turbulent $\kappa = 2$; capillary $\kappa = \frac{1}{2}$. The same procedure may be used for the new family of pore radii distributions.

The derivation of pore size distribution for random networks allows a relatively simple determination of the relationship between the mean pore size and the standard deviation. This relationship is linear in the theoretical case, and approximately linear for experimental measurements [1, 2]. A property of the gamma distribution, and of the new distribution $q(r)$, is that a given value of the mean may be associated with an infinite number of variances. Some plots are provided in [3], together with results of comparisons with measured pore size distributions in paper, which will be reported in detail elsewhere.

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References


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