COMBUSTION OF A FINITE QUANTITY OF GAS RELEASED IN THE ATMOSPHERE

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Low Mach number and large activation energy asymptotic approximations are used to derive a model for the thermal behaviour of a finite planar, cylindrically or spherically symmetric fuel region released in an unconfined oxidant environment. A single one-step exothermic reaction is taken to occur. The induction stage has been studied in a previous paper [1] where it was shown that ignition occurs provided the fuel region exceeds a critical size. Below this size diffusive dilution starves the reaction too quickly for ignition to be possible.

In this paper combustion of the system following ignition is studied. The analysis is equally valid whether ignition arises through self-heating or whether it is brought about by some other means such as a spark. Deflagration flames are formed which quickly consume the locally lean species at any point. A criterion is identified for any diffusion flame to exist after the passage of deflagration flames. A non-linear moving boundary formulation is developed to describe the progress and eventual collapse of such diffusion flames as the remaining fuel is consumed. The linearised version of these equations gives an upper limit for the time in which all combustion ceases.

1. Introduction

We continue to study the behaviour of a quantity of gaseous fuel initially occupying a finite symmetric region of space within an unbounded oxidant environment at roughly the same temperature. At some moment the fuel is released and is then free to diffusively mix and react with the oxidant. Such a situation may practically arise as an accidental spillage of a combustible gaseous fuel in the atmosphere, or it may approximate to a breakaway eddy in a turbulent mixing stream of two initially un-mixed gaseous chemical reactants. In a previous paper [1] the progress towards self-ignition of this system was examined.

A related initial value problem was examined by Liñán and Crespo [2] in studying the mixing, ignition and combustion of two initially separate semi-infinite regions of fuel and oxidant. They observed three phases in the process: firstly, an induction phase in which the reactants mix diffusively and react slowly until at some point and time the feedback effect on the rate of reaction from the heat produced by the reaction causes thermal runaway to occur; next, deflagration flames consume as much of the lean species as is available at each point as they rapidly traverse the region in which fuel and oxidant have had the chance to mix significantly; finally, a diffusion flame is established.

In the case of a finite region of fuel the situation can be substantially different. With a small enough initial quantity of fuel, for instance, the dilution of the fuel through diffusion is found to happen too quickly for thermal runaway and combustion to occur. On the other hand finite regions above a certain size are found to ignite more quickly than the equivalent semi-infinite region would do. Results for this stage of the evolution have already been presented [1].

Diffusive mixing of the species may have diluted the fuel sufficiently to make it lean at all points. No diffusion flame will then remain after ignition and the passage of deflagration flames. Finally, any diffusion flame that does arise will necessarily have a limited lifespan since the finite quantity of fuel available must ultimately be consumed.

In this paper we firstly derive the system of equations used to model these processes (not having had enough space for this in [1]) and assume that ignition occurs at some position and time. The passage of deflagration flames accom-
panied by convective thermal expansion effects is then considered. Provided some unburnt fuel remains after deflagration a diffusion flame is formed. The progress of this flame is modelled as a nonlinear moving-boundary problem. Results for this are presented in the limit of low overall temperature-rise.

2. The Model

One can assemble a set of constant typical quantities with which to non-dimensionalise the system of equations governing the processes we are to examine here as follows, where the superscript * indicates the dimensional value of a quantity: it will be assumed for simplicity that all gases in the mixture have the same (constant) specific heat at constant pressure \( C_p \); temperatures will be measured in units of an absolute temperature \( T \); and it will also be convenient to identify a typical absolute pressure \( P \); with a constant mean molecular weight \( W \) it is then natural to measure densities in units of \( \rho \); thermal conductivity will be measured in units of \( \kappa \); so that a typical thermal diffusivity is given by \( \kappa = \frac{\kappa}{\rho C_p} \). We identify a time scale \( t \), which gives a rough measure of the induction interval between the start of the process of mixing and chemical reaction and the onset of ignition, so that it is now possible to construct a typical velocity \( u \), and a typical length scale \( r \), such that

\[
u = \frac{\kappa}{\rho C_p} \frac{t^2}{r^2},
\]

and a typical length scale \( r \), such that

\[
u = \frac{\kappa}{\rho C_p} \frac{t^2}{r^2}.
\]

These definitions make it clear that we anticipate a central role for the process of diffusion. With the assumption that Fick’s law will suffice to describe the diffusion of mass, and that there is therefore a single typical diffusion coefficient \( \kappa \), we define the Lewis number as \( \kappa / \kappa \). Since there must be a typical coefficient of dynamic viscosity \( \eta \) the Prandtl number is given by \( \nu = \eta / \kappa \).

Before setting down the collection of conservation and associated equations we must observe that local pressures do not in general change very dramatically from the average system value. At this juncture it is worth noting that variations from \( \rho \) are typically of the order of \( \rho u' \), although one should perhaps remember that if acoustic, or gas-dynamical, pressure changes were to play a significant part then these would best be measured in units of \( \rho a^2 \), where \( a^2 \) is the typical sound speed given by \( a^2 = \gamma \rho C_p / \rho \), and \( \gamma \) is the ratio of principal specific heats (most usefully, in general, calculated under conditions of frozen chemistry). A typical Mach number \( M \) is given by the ratio \( M = u / a \).

The rate of progress \( \mathcal{R} \) of the assumed single chemical reaction, \( vF + vX \rightarrow vP \) within the system where \( F, X \) and \( P \) are fuel, oxidant and product species, respectively, and where \( v_f, v_X \) and \( v_p \) are the respective stoichiometric integers, can be written in dimensional terms as follows:

\[
\mathcal{R} = \frac{(\rho^* C_p)^v (\rho^* C_X)^x \rho^* - 1 \times F^*(T^*) \exp \left\{ -E^*_A/(R^* T^*) \right\}}{F^*(T^*) - F^*(T^*)}
\]

where \( E^* \) is the molar activation energy of the reaction, and \( F^*(T^*) \) is a pre-exponential frequency factor with the dimensions of density to the power \( 1 - v_f - v_X \). If \( \alpha \) represents either \( F \) or \( X \) and by

\[
\eta = \frac{\rho W}{\nu P} \mathcal{R}^*
\]

for the product species \( P \), in units of mass per unit volume per unit time. Defining the dimensionless activation energy to be \( \Theta = E_A^*/(R^* T^*) \) it transpires that a useful form for the time \( t^* \) is

\[
t^* = \frac{\rho^* C_F}{\nu P W Q^*} \left( \frac{2}{\rho^* C_F} \right)^{v_f} \left( \frac{2}{\rho^* C_X} \right)^{v_X} \frac{\rho^*}{F^*(T^*)} \frac{e^{\Theta}}{\Theta}
\]

where \( C_F \) and \( C_X \) are the maximum initial values of \( C_F \) and \( C_X \), \( W \) is the molecular weight of the product species, and \( Q^* \) is the chemical energy per unit mass released by the reaction. With this definition a fuel region of size of order \( r^* \) can be expected to ignite spontaneously after an induction time of the order of \( t^* \). The equations that describe the system [3] can now be written in dimensionless form.

Species:

\[
\rho'(y_{at} + u \cdot \nabla y_a) - Le \nabla \cdot (\rho' \nabla y_a) = -\rho G'/(\Theta \xi_a)
\]

Energy:

\[
\rho'(T' + u \cdot \nabla T') - \nabla \cdot (\lambda' \nabla T') = \rho G'/\Theta + (\gamma - 1)M^2 \times \{p' + u \cdot \nabla p' + \nu \eta [S'] : [\nabla u']}\]
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Continuity:

\[ \rho' + \nabla \cdot (\rho' \mathbf{u}') = 0 \]  

Momentum:

\[ \rho'[\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{u}'] + \nabla p = \text{Pr} \nabla \cdot (\eta' \hat{\mathbf{S}}') \]  

Thermal Equation of State:

\[ \rho T' = 1 + \gamma M^2 \rho' \]  

where

\[ G' = (2\rho' y'_p)^x (2\rho' y'_p)^y F(T') \rho' e^{\theta(1-\gamma T')} \]  

and

\[ \hat{\mathbf{S}}' = [\nabla \mathbf{u}'] + [\nabla \mathbf{u}']^T - \frac{1}{2} (\nabla \cdot \mathbf{u}') \mathbf{I} \]  

\[ \mathbf{I} \] is the unit tensor. Also

\[ \zeta_o = (wpW_pQ^*C_o)/(\nu_o W_o \epsilon_p^* T_p). \]  

The new quantities that appear above are defined as follows

\[ \mathbf{I} = \beta^* / \beta^*, t = t^* / \beta^*, \mathbf{u} = \gamma / u^*, \]  

\[ T' = T^* / T_f, \rho' = \rho / \rho^*, \gamma = \gamma / \gamma^*, \mathbf{S}' = \mathbf{S}^* / \mathbf{S}_e, \eta = \eta / \eta^*, \]  

\[ \rho' = \rho u^* / \rho^* u^* \]  

and \( \nabla \) is the gradient operator in terms of the dimensionless coordinate \( r \). A useful alternative form of the continuity equation can be obtained by using the thermal equation of state (7) to eliminate \( \rho' + u' \cdot \nabla \rho' \), and hence to eliminate \( T' \) from the energy equation (4). This gives

\[ M^2 \rho' + \nabla \cdot (\rho' T' y' - \lambda \nabla T') = \rho' G'/\Theta \]  

\[ + (\gamma - 1) M^2 [y' \cdot \nabla \rho' + \text{Pr} \eta' \hat{\mathbf{S}}' : [\nabla \mathbf{u}']]. \]  

The constants \( M \) and \( \Theta \), appearing explicitly in these equations, provide scope for proceeding on the basis of an asymptotic analysis. For this purpose it may be noted that in many combustion situations the non-dimensional activation energy \( \Theta \) is large compared with unity, and the typical flow Mach number \( M \) is normally very small indeed. At any rate we will restrict attention here to situations in which the conditions, \( M \ll 1 \) and \( \Theta \gg 1 \), may be considered to hold good. Taking firstly the limit of small Mach number, the non-dimensional dependent variables defined in (11) may be assumed to behave in accordance with the asymptotic form

\[ f'(r,t;M) = f(r,t) + O(M^2) \]  

where \( f \) is any member of the set \( \{ y, y, \rho, T, \rho, \mathbf{u}, \lambda, \mathbf{S}, \eta \} \). As long as this is the case the asymptotic forms (unprimed) of the dependent variables will satisfy Eqs (3) to (12) with the primes and the terms containing \( M \) omitted.

The asymptotic form of Eq. (12), namely

\[ \nabla \cdot (\mathbf{u} - \lambda \nabla T) = G'(\Theta T) \]  

is of particular interest since it establishes a relationship between velocity and, in essence, thermal expansion effects associated with either thermal conduction or chemical heating. With \( \nabla \cdot \mathbf{u} \) so determined the asymptotic form of the momentum equation (6) — with the primes omitted — determines \( \nabla p \). There is a possible conflict here, since in general one should be free, for instance, to arbitrarily specify initial values for all of the variables \( y, y, \rho, T, T, \rho \) and \( \mathbf{u} \) and thus (at least at an initial instant) to violate the continuity and momentum asymptotic equations. Such a violation may be accounted for as an acoustic disturbance which the asymptotic form (13) is incapable of describing. In order to consider acoustic effects one should employ the definition of a family of acoustic time scales \( \tau(t;M,\tau_0) \) and (as indicated earlier) an acoustic pressure \( \hat{p} \), such that \( t = t_0 + \tau_0 \) and

\[ M \hat{p}'(r, t; M) = \hat{p}'(r, \tau t_0) + O(M) \]  

In terms of these, the appropriate asymptotic forms of the continuity and momentum equations become

\[ \hat{p}_t + \nabla \cdot (\mathbf{u} - \lambda \nabla T) = G'(\Theta T) \]  

and

\[ \mathbf{u}_t + T \nabla \hat{p} = 0. \]  

This pair of equations makes it clear that an acoustic propagation arises out of any disturbance of velocity from a solution of Eq. (14), or of pressure, from the condition \( \nabla \hat{p} = 0 \). It can be argued [4] that, being finite in extent, any acoustic disturbances set up at the time of release of the fuel region will tend to quickly propagate away from the region of interest, leaving the asymptotic forms (13) as a good approximation.

With conditions of symmetry and non-rotation Eq. (14) can be written in the form

\[ r^{-\eta} [\rho'(u - \lambda T')] = G'(\Theta T), \]
where $u$ and $r$ are now the radial velocity and position, and where $n$ takes the values: 0, for symmetry across a plane; 1, for cylindrical symmetry; and 2, for spherical symmetry. Because $u$ satisfies the symmetry condition $u(0,t) = 0$ Eq. (18) can be integrated to give

$$u = \lambda T_r + v/\Theta$$  \hspace{1cm} (19)$$

with

$$v = r^{-n} \int_0^r r^n G/T \, dr.$$  \hspace{1cm} (20)

Under these conditions the velocity can be seen to consist solely of thermal-conductive and chemical-heating induced expansion effects.

Using this result to eliminate $u$ from the convective terms of the appropriate symmetric forms of the species and energy equations gives

$$y_{ar} + \left[ \lambda T_r - \Theta (p^{(3)}) y_{ar} \right] t_r = - \Theta^{-1} \left( G - \Theta^{-1} v \right)$$  \hspace{1cm} (21)$$

and

$$T_t + \left[ \lambda T_r - \Theta (p^{(3)}) \right] T_r = - T \Theta^{-1} \left( G - \Theta^{-1} v \right).$$  \hspace{1cm} (22)$$

These equations show that a considerable simplification will result if $\lambda$, $p^{(3)}$, and $\Theta$ are modelled (not unrealistically) such that $\lambda = \rho^{(3)} = T$ and $\Theta = 1$. Further simplification follows if the Schvab-Zeldovitch formulation is adopted by defining the variables

$$Y_a = y_a + (T - 1)/T_a.$$  \hspace{1cm} (23)$$

The equations for $Y_a$ and $T$ can now be written in the form

$$Y_{ar} + v Y_{ar} /\Theta = T^2 r^{-n}(Y_{ar} r)$$  \hspace{1cm} (24)$$

and

$$T_t + v T_t /\Theta = G /\Theta = T^2 r^{-n}(Y_{ar} r),$$  \hspace{1cm} (25)$$

where

$$G = \left[ \frac{2}{T} \left( Y_{F} - \frac{T - 1}{\bar{\xi}_F} \right) \right]^{1/4} \times \left[ \frac{2}{T} \left( Y_{X} - \frac{T - 1}{\bar{\xi}_X} \right) \right]^{1/4} T F(T) e^{\Theta t - t / T}. $$  \hspace{1cm} (26)$$

A combination of the asymptotic forms of the continuity, energy and momentum equations can also be used to derive the following form of the momentum equation

$$T_p + \frac{1}{2} nu TT + \Theta^{-1}(u + uv)$$  \hspace{1cm} (27)$$

in which an obvious further simplification follows if the Prandtl number takes the value 3/4. With Eq. (19) this serves to uniquely determine the velocity $u$ and pressure gradient $p$, in terms of the species and temperature solutions of Eqs (24) to (26).

These equations have been used in [1] to model the release of a finite region of fuel. Accordingly, the initial conditions for $Y_F$, $Y_X$ and $T$ are expressed as follows:

$$Y(t) = \lim_{t \to 0} (Y_F, Y_X, T)$$

\begin{align*}
\text{if } |r| < a &= \left\{ \begin{array}{ll}
(1 + k / \bar{\xi}_F, 1 + k / \bar{\xi}_X, 1 + k / \Theta) & : |r| < a \\
(0, 1, 1) & : |r| \geq a.
\end{array} \right.
\end{align*}

Thus at the initial time, the size of the fuel region is determined by its radius $a$. Since $\Theta$ is large, the difference $h / \Theta$ between the initial temperatures of oxidant and fuel is correspondingly small. However, in view of the exponential dependence of the reaction-rate term $G$ on changes in temperature of order $\Theta^{-1}$ this small initial temperature difference is significant, especially during the induction stage of the reaction as seen in [1].

For the purpose of the analysis here it will be assumed that an ignition time $\tilde{t}$ and position $\tilde{r}$ exist, in a small neighbourhood of which an ignition process gives rise to the production of deflagration flames. Where ignition is brought about by self-heating the quantities $r_i$ and $t_i$ calculated in [1] can be considered to provide first-order estimates of the ignition position and ignition time in the limit as $\Theta \to \infty$.

3. Deflagration Stage

Before the ignition time $\tilde{t}$ the temperature growth is of the order of $\Theta^{-1}$ about the background temperature of unity, apart from an exceptionally short time in a small neighbourhood of $\tilde{t}$ as ignition occurs [5,6]. Until this happens $Y_F$ and $Y_X$ can thus be found to have the asymptotic solutions
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\[ Y_F = x/2 + O(\Theta^{-1}) \]

and

\[ Y_X = 1 - x/2 + O(\Theta^{-1}) \] (29)

where \( x(r,t) \) satisfies the diffusion equation

\[ x_t = x_{rr} + n x/r \] (30)

and, initially, \( x = 2 \) for \( |r| < a \) and \( x = 0 \) for \( |r| > a \). This gives [7]:

\[
x = \begin{cases} 
\text{erf}\left(\frac{r+a}{2t^{1/2}}\right) - \text{erf}\left(\frac{r-a}{2t^{1/2}}\right) & : n = 0 \\
2\sum_{m=0}^\infty \left[ \frac{r^{2m}}{m!(4t)^m} \sum_{l=0}^{m} \frac{a^{2l}}{l!(4t)^l} \right] & : n = 1 \\
\text{erf}\left(\frac{r+a}{2t^{1/2}}\right) - \text{erf}\left(\frac{r-a}{2t^{1/2}}\right) + 2t^{1/2} \left[ \text{exp}\left(\frac{-(r+a)^2}{4t}\right) - \text{exp}\left(\frac{-(r-a)^2}{4t}\right) \right] & : n = 2
\end{cases}
\] (31)

If \( r > 0 \) then two flames can be considered to emerge from \( r = \tilde{r} \) following the paths \( r = r_i(t) \) and \( r = r_o(t) \), with \( r_i \) decreasing to zero and \( r_o \) increasing. If \( \tilde{r} = 0 \), or once \( r_i \) has collapsed to zero, then only \( r_o \) need be considered (\( r_i \) should be set to zero in the equations below). Allowing for volumetric expansion at a point by a factor \( T_b \), since a flame raises the temperature from asymptotically near unity to \( T_b \) as it passes, a convected coordinate \( \tilde{t} \) can be defined such that

\[
x = \frac{\text{erf}\left(\frac{r+a}{2t^{1/2}}\right) - \text{erf}\left(\frac{r-a}{2t^{1/2}}\right)}{\text{erf}\left(\frac{\tilde{r}+a}{2\tilde{t}^{1/2}}\right) - \text{erf}\left(\frac{\tilde{r}-a}{2\tilde{t}^{1/2}}\right)} + 2t^{1/2} \left[ \text{exp}\left(\frac{-(\tilde{r}+a)^2}{4\tilde{t}}\right) - \text{exp}\left(\frac{-(\tilde{r}-a)^2}{4\tilde{t}}\right) \right].
\] (32)

The domain of integration \( D \) is empty for \( r < r_i \); \([\tilde{r}_i, \tilde{r}]\) for \( r_i < r < r_o \); and \([\tilde{r}, \tilde{r}_o]\) for \( r > r_o \) where \( \tilde{r}_i \) and \( \tilde{r}_o \) are flame positions in terms of \( \tilde{r} \). Using Eq. (26) the “burnt” temperature \( T_b \), at which the reaction ceases because the locally lean species is completely consumed, can be defined as

\[ T_b(\tilde{r}) = 1 + \min\{\xi_F Y_F, \xi_X Y_X\} \]

\[ - \left( 1 + \frac{\xi_X}{\xi_F} x(\tilde{r}, \tilde{t}) \right) : x(\tilde{r}, \tilde{t}) > \Omega \]

\[ 1 + \frac{\xi_F}{\xi_X} x(\tilde{r}, \tilde{t}) : x(\tilde{r}, \tilde{t}) \leq \Omega \] (33)

where

\[ \Omega = 2(\xi_X / (\xi_F + \xi_X)) = 2/[1 + C_F v_X W_X / (C_X v_F W_F)]. \]

Using results obtained in the appendix, where the flame-speed relative to the upstream (unburnt) gases is derived, the propagation of the flames is such that

\[ \frac{dr_i}{dt} = -S_i \Theta^{-1} v_L / 2 e^{\xi_X (\Theta - 1)/T_b} \] (34)

(or zero if \( r_i = 0 \)) and

\[ \frac{dr_o}{dt} = -\frac{dr_i}{dt} (T_i - 1) \left( \frac{T_i}{r_i} \right)^s \]

\[ + T_b S_i \Theta^{-1} v_L / 2 e^{\xi_X (\Theta - 1)/T_b} \] (35)

with \( v_L \) and \( S \) as defined in the appendix, and subscripts \( i \) and \( o \) denoting evaluation at \( r_i \) and \( r_o \). Clearly, in terms of the length scale \( r \) and time \( t \), these flames propagate through the system in an exponentially short time.

4. Diffusion Flame Stage

Following the passage of deflagration flames the species parameters and temperature are left with the values:

\( Y_F = \hat{x}/2 + O(\Theta^{-1}) \), \( Y_X = 1 + \hat{x}/2 + O(\Theta^{-1}) \)

and

\[ T = O(\Theta^{-1}) + \left\{ \begin{array}{ll}
1 + \hat{x}/(1 - \hat{x}/2) : \hat{x} > \Omega \\
1 + \xi_F \hat{x}/2 : \hat{x} \leq \Omega
\end{array} \right. \] (36)

where

\[ \lim_{t \to t_o} \hat{x}(r, t) = x(\tilde{r}, \tilde{t}) \] (37)

with \( \hat{t} \) given by Eq. (32) for the case \( r_i = 0 \) and \( r_o = \infty \). Provided a region exists in which the fuel species is rich at the time of ignition, then some unburnt fuel remains after the locally lean species is consumed by deflagration flames. It follows that a diffusion flame is created at the boundary where the remaining fuel and oxidant meet.

A condition for a diffusion flame to form is therefore that \( \xi_F Y_F > \xi_X Y_X \) in some region at \( t = \tilde{t} \). To first order this requires that \( x(0, t) > \Omega \) since \( x \) is greatest at \( r = 0 \). Alternatively, since \( x(0, t) \) decreases with time, ignition must occur before a critical time \( t_d = t_{do} + O(\Theta^{-1}) \) which, to first order, satisfies \( x(0, t_{do}) = \Omega \). Values of \( t_{do} \) as a function of \( \Omega \) are plotted in Fig. 1. Assuming \( t < t_d \), a diffusion flame is formed at \( r = r_f(t) = r_{do} + O(\Theta^{-1}) \) where, initially, \( \hat{x}(r_{do}, \tilde{t}) = \Omega \).
Fig. 1. Time-limits for the generation of diffusion flames as functions of $\Omega$, where $\Omega = \frac{2}{1 + C_{x0}W_{1} / (C_{x0}W_{1} T_{f})}$; Solid lines $t_{00}/a^2$; Dotted lines $a^2/t_{00}$.

Modelling the flame as a Burke-Schumann flame sheet with $y_{x} = 0$ for $r < r_{f}$ and $y_{f} = 0$ for $r > r_{f}$, the asymptotic forms (36) persist outside the flame-sheet with $\delta(r,t)$ satisfying

$$
\delta = \left\{ \begin{array}{ll}
(1 + \frac{\delta}{2})^{2}(\delta_{f} + \frac{n\delta}{r}) : \delta > \Omega \\
(1 + \frac{\delta}{2})^{2}(\delta_{f} + \frac{n\delta}{r} - \beta \frac{\delta}{r}) : \delta \leq \Omega.
\end{array} \right. \tag{38}
$$

The function $\beta(r_{f},t)$ arises through evaluating $v/\theta$ (Eq. (20)) which can only be done through considering the “inner” flame region.

Denoting $Y_{a}(r,t) = 1 + \zeta_{f}Y_{f}(r,t) = 1 + \zeta_{a}Y_{a}(r,t)$ by $Y_{a}$ and $T$, respectively, the flame region is highlighted by rescaling such that

$$
r = r_{f} + \mu \delta, \quad Y_{a} \sim Y_{a0} + \mu a_{a}, \quad \text{and} \quad T \sim T_{f} - \mu \Psi \tag{39}
$$

with

$$
\mu^{1 + \frac{\alpha}{F}} = \Theta_{e}^{-\theta(1/T_{f})} \times (\zeta_{F}T_{f}/2)^{v_{P}}(\zeta_{x}T_{f}/2)^{v_{X}}T_{f}/F(T_{f}). \tag{40}
$$

It follows that

$$
\omega_{af} = 0 \tag{41}
$$

and

$$
\psi_{f} = (\zeta_{F}a_{F} + \Psi)^{v_{F}}(\zeta_{x}a_{x} + \Psi)^{v_{x}}. \tag{42}
$$

Matching of $Y_{f}$ and $Y_{x}$ between the regions shows, from (41), that $\delta$ and $\delta_{x}$ must be continuous across the flame-sheet. Using Eq. (20), $v/\theta$ can be evaluated in the inner region to give

$$
v/\theta \sim T_{f} \int_{-\infty}^{t} \psi_{f} d\delta = T_{f}[\psi_{f}(\delta) - \psi_{f}(-\infty)]. \tag{43}
$$

Fig. 2. Diffusion-flame paths for planar symmetry and small overall temperature rise.

Matching of $T$ between the regions gives $\Psi_{f}(-\infty) = \zeta_{f}\delta_{f}(r_{f},t)/2$ and $\Psi_{x}(x) = -\zeta_{x}\delta_{x}(r_{x},t)/2$, so that in the outer region $v/\theta$ becomes zero for $r < r_{f}$ and

$$
v/\theta \sim \frac{1}{2}(\zeta_{F} + \zeta_{x}) T_{f} \delta_{f}(r_{f},t) (r_{f}/r)^{n} \tag{44}
$$

for $r > r_{f}$. To first order, the flame path is $r = r_{f}$, where $\delta(r_{f},t) = \Omega$. This serves to define $\beta$ of Eq. (38) as follows:

$$
\beta = -\frac{1}{2}(\zeta_{F} + \zeta_{x}) \delta_{f}(r_{f},t) (r_{f}/r)^{n}. \tag{45}
$$

Provided $\delta(0,t) > \Omega$, so that a flame exists, Eq. (38) now presents a nonlinear moving boundary problem. Two different equations govern the evolution of $\delta$ for $|r| < r_{f}$ ($\delta > \Omega$) where diffusion dominates and $|r| > r_{a}$ ($\delta < \Omega$) where diffusion and convection operate. The necessary two conditions to be satisfied at the flame-boundary have already been deduced:

$$
\delta(r_{f},t) = \delta(r_{f}^{+},t) \quad \text{and} \quad \delta(r_{a},t) = \delta(r_{a}^{+},t). \tag{46}
$$

since $\delta$ and $\delta_{x}$ must be continuous across the flame-sheet. With initial conditions provided by
Eq. (37) $\xi$ is thus uniquely determined, giving $Y_f$, $Y_X$ and $T$ to first-order from Eqs (36).

After a finite time (bounded above by $t_0$) the moving boundary disappears as the flame collapses to the origin. At this point all of the fuel is consumed. No value for $r$, exists (or it may be considered equal to zero) and only the second part of Eq. (38), with $\beta = 0$, governs the subsequent behavior of $r$.

Solutions in the limit of small overall temperature rise ($\zeta_f$ and $\zeta_x$ being small) are particularly simple. Eq. (38) reduces to Eq. (30) and $r$ becomes equal to $x$ which is known analytically. Since volumetric expansion effects also become negligible, the results after deflagration are the same regardless of the actual time of ignition $t$. Some examples of flame-paths calculated under these conditions are shown in Figs. 2, 3 and 4.

5. Conclusions

A set of equations with which to model low Mach number combustion processes under symmetric conditions has been derived. Taking the case of a finite region of fuel released into an oxidant atmosphere, the various stages of combustion have been studied using large activation energy asymptotic methods.

An earlier paper [1] showed that self-ignition arises provided the size of fuel region exceeds a critical level. In this paper it has been shown that ignition must occur before a critical time (at which diffusive dilution makes the mixture fuel-lean everywhere) if deflagration flames are to leave a diffusion flame in their wake. This diffusion flame ultimately disappears as the remaining fuel is consumed. The passage of deflagration flames is described using quasi-steady flame speeds and the diffusion-flame stage is found to be modelled by a nonlinear moving-boundary problem.

REFERENCES


Appendix: Quasi-Steady Flame Speed

The passage of a deflagration flame can be modelled in the pre-heat and reaction zones using

\[ T \sim T_p(z) \text{ and } T \sim T_b - \Theta^{-1} T_b^2 \ T/ \Theta(z) \]  

respectivey, where \( z = \kappa T - k^2 Vt \) with

\[
k^2 = \left( \frac{2T_b}{\xi L} \right)^{v_L} \left[ \frac{2}{\xi R T_b} (\xi R Y_R - \xi L Y_L) \right]^{v_R} \times \frac{F(T_b) e^{\Theta(1/T_f)}}{T_b^2} \Theta^{2+v_L}. \]  

(A2)

Here, \( L \) and \( R \) are used to denote the locally lean and rich species respectively and \( T_b \) is written for the burnt temperature, \( 1 + \xi L Y_L \). The unburnt temperature is taken to be asymptotically close to unity. It is assumed that spatial and temporal variations are slow enough for \( T_b \) etc. to be taken as constant for order one values of \( z \).

In the pre-heat zone, \( VT_b^2 + T_p^2 T_p^u = 0 \), which solves to give

\[
Vz = \ln \left[ (T_b - 1)/(T_p - 1) \right] + T_b - T_p \]  

satisfying the conditions \( T_p(0) = T_b \) and \( T_p(\infty) = 1 \). In the reaction zone, \( T_r' = T_r^{3, \theta \exp(-T_f)} \), which solves to give \( T_r' \) in terms of \( T_f' \),

\[
T_f'/2 = 2 \int_0^{T_f} e^{-y} dy. \]  

(A4)

This gives \( T_f' = 0 \) for \( T = T_b \) and shows \( T_f'^2 \) approaching a constant value of \( 2(\nu L + 1) \) (or \( 2vL \) if \( v_L \) is an integer) as \( T_f' \to \infty \). Matching of the temperature gradient between the two zones leads to the result for the flame-speed:

\[
\frac{\partial T}{\partial z} \bigg|_z = kV = 5 \Theta^{-1} v_L/2 \nu L \Theta(1/T_f) \]  

(A5)

where

\[
S^2 = \left( \frac{2T_b}{\xi L} \right)^{v_L} \left[ \frac{2}{\xi R T_b} (\xi R Y_R - \xi L Y_L) \right]^{v_R} \times \frac{T_b^2 F(T_b)}{(T_b - 1)^2} 2(\nu L - 1). \]  

(A6)

COMMENTS

T. Hirano, University of Tokyo, Japan. In your theory, it seems that the pressure field is assumed to be uniform. In the situation that you are dealing with, the pressure disturbance must exist and must greatly affect the phenomena. Would you comment on the pressure disturbance effects?

Author's Reply. Pressure gradients are absolutely necessary for velocity changes to occur. But with small Mach-number flows, the required pressure gradients are also generally small and a good approximation for calculating velocities is to use the rate of thermal expansion at constant pressure. Knowing the velocity field the actual small deviation of pressure from the background constant value can then be calculated [using equation (27) in the paper]. The overall error in this procedure turns out to be only of the order of the square of the Mach number.

T. David, University of Leeds, U.K. Could you please clarify the assumption of the "annihilation" or collapse of the deflagration waves propagating into the domain bounded by the diffusion flame sheet.

Author's Reply. By their nature, deflagration flames consume all of the locally-lean species in their path. Thus, once they have converged on the centre of symmetry, there is no mixture of reactants left to sustain deflagration.