Homotopy Decompositions and the Homotopy Exponent Problem

Case for Support

Part 1. Previous Research and Track Record.

Previous Research. It is standard in mathematics to investigate an object by decomposing it into smaller pieces, investigate the pieces individually and then reassemble that information to gain insight into the original object. *Homotopy decompositions* play a prominent role in determining the homotopy theoretical properties of topological spaces. Theoretically, any space X could be decomposed into a product of indecomposable factors. A thorough understanding of them and the way they are assembled would lead towards an understanding of the homotopy type of all spaces.

Much of the work on decompositions has been motivated by attempts to calculate upper bounds on homotopy exponents. The *p-homotopy exponent* of a space X is p^t if t is the minimal power of p which annihilates the p-torsion in the homotopy groups of X. James [13] was the first one to consider decompositions of loop suspensions. He did not decompose $\Omega\Sigma X$ unstably, instead he showed that $\Sigma\Omega\Sigma X \simeq \bigvee_{i=1}^{\infty} \Sigma X^{(n)}$, where $X^{(n)}$ is the n-fold smash product of X with itself. James applied this stable decomposition to the case $X = S^m$ to show that the homotopy exponent of S^{2n+1} is bounded above by 2^{2n}. At the same time, using similar methods, Toda [19] proved that the odd The research environment at the University of Manchseter is very attractive. Manchester is recognised in the UK and internationally for its traditionally high expertise in the area of Topology and its applications in Algebra and Geometry. Being one of the largest mathematical departments in the UK, the MIMS offers expertise in almost every mathematical discipline; including Modular Representation Theory relevant to the project. Prof Ray, whose research interests overlap significantly with the subject of the proposal, and Prof Buchstaber, a world leading expert in the homotopy theory, make Manchester the best place for realising the project. The expertise of Prof Eccles, Symondes in homotopy theory, and Profs Premet, Prest in group representation theory will also be of significant value.primary homotopy exponent of S^{2n+1} is bounded above by p^{2n}. A giant leap forward came with the seminal work of Cohen, Moore and Neisendorfer [2, 3, 4] in the late 1970's. They developed new methods based on mod-p homotopy theory (p is an odd prime) and differential Lie algebras to decompose $\Omega\Sigma P^{m-1}(p^r)$ into irreducible pieces, where $P^{m-1}(p^r)$ is a mod-p^r Moore space. Using this decomposition as the key datum, Cohen, Moore and Neisendorfer obtained the remarkable result that the homotopy exponent of S^{2n+1} is exactly p^n. Neisendorfer [15] took this a step further and showed that the homotopy exponent of $P^m(p^r)$ is p^{r+1}. This result offered the first example confirming a major exponent conjecture posed by Barratt in the 1950's. The *Barratt conjecture* can be formulated as follows. Let $f\colon \Sigma^2 X \to \Sigma^2 X$ be a map of order p^r in $[\Sigma^2 X, \Sigma^2 X]$. Then $p^{r+1}\pi_*(\Sigma X) = 0$. The Barratt conjecture is still open.

The problem of calculating the homotopy exponent of $P^m(2)$ motivated Selick and Wu [16] to develop recently a powerful new decomposition method for studying loop suspensions. They found a functorial decomposition of $\Omega\Sigma X$, and obtained its minimal functorial retract. Wu [21] used these techniques to calculate the homotopy groups of $P^m(2)$ in low dimensions. The exponent problem, however, remains elusive. The experts in the field believe that the Selick and Wu approach has a wealth of application, but the method so far has been little used.

In 1931 Hopf defined what is nowadays known as the Hopf invariant in order to study maps between spheres of different dimensions which cannot be distinguished homologically. Ever since, Hopf invariants and their relations with Whitehead products have been widely studied (cf. [5, 22]). Nevertheless, many important problems related to them are still open. This motivated Wu and the PI [11, 12] to study the groups of the homotopy classes of maps from the James construction to loop spaces. The approach taken is completely new. It introduces simplicial methods in the subject and has a strong algebraic backstage. Natural linear and coalgebra transformations of tensor algebras were studied by connecting them to the representations of certain combinatorial groups. By applying this set up, Wu and the PI [12] calculated the generalized k-th Hopf invariant of the n-fold Whitehead product when k does not divide n and the second Hopf invariant of the 4-fold Whitehead product. The second application of their method gives a strong result estimating the rate of growth of the order of the map $J(f)$ in the Barratt conjecture. This programme was supported by EPSRC overseas travel grant EP/D033144/1 and its realisation was overall assessed as "Tending to Outstanding".

The PI's Track Record. Grbić's research interests focus on modern homotopy theory and its applications. She is active in the rapidly developing area of decompositions of loop spaces, which is currently, the strongest approach for attacking the homotopy exponent problem.

As a PhD student, the PI [6] developed a recognition method to identify certain universal spaces in the category of homotopy associative, homotopy commutative H-spaces, an analogue of the James construction in the category of associative H-spaces. The method is based on linking homotopy decompositions of loop spaces together with the Lie algebra structure appearing in the homology of appropriate spaces. As an application the PI [6] used universal spaces to derive an explicit formula for the differential d_1 in the EHP spectral sequence in a certain range for calculating the unstable homotopy groups of spheres. Universal spaces have proved to be a useful tool for finding different homotopy properties of spaces, such as homotopy exponents. Generalising Cohen, Moore and Neisendorfer's work [2, 3, 4], the PI [7] reproved Neisendorfer's result on the homotopy exponent of $S^{2n+1}\{p\}$, the homotopy fibre of the degree p map on S^{2n+1}, and gave upper and lower bounds for exponents of universal spaces of two-cell complexes. The methods used to approach these problems are of significant use for the proposed project. In 2003 the PI led the school "Classical homotopy theory and beyond" at the Belgrade Mathematical Institute. There, she made a parallel between the work of Cohen, Moore, Neisendorfer [2, 3] and Gray with recent work of Theriault and herself on decomposition of loop spaces. The PI [11, 12] continued research on loop decompositions and their applications with Wu at the National University of Singapore. The research concerned a fundamental homotopy exponent problem, the Barratt conjecture. The approach taken was new, guided by an idea to develop an algebraic obstruction theory using the Selick-Wu [16] loop decomposition programme. This point of view directed the PI to write this proposal.

Grbić has also shown that homotopy theory can be applied to produce solutions to a broad range of problems. She has been working to establish links between toric topology and various configuration spaces. In particular, she has calculated the unstable homotopy type of the complement of a coordinate subspace arrangement, in collaboration withTheriault [8, 9]. This work can be seen as preparation for thePI to approach the proposed connection between decompositions of double loop spaces and configuration spaces. More recently, in an ongoing project with Theriault, the PI [10] is using Lie group decompositions [17] and the concept of universal spaces to study multiplicative self-maps of low rank Lie groups. Some of the milestones of this proposal are based on that work. The PI's research is currently supported by the *Nuffield Foundation grant* NAL/00967/G.

Collaborative Work. The PI has participated in many conferences and research visits during which she has established connections with mathematicians of different interests. In particular, the PI has been involved in several research activities with Profs Cohen and Wu; gave a talk at the conference in honour of Prof Cohen, and successfully realised a 7-month project with Wu. Working on String theory, the PI has collaborated with Prof Terzić. Since her PhD studies, the PI has had a thriving collaboration with Dr Theriault. Related to this proposal, she also established research contacts with Buchstaber, Neumann and Ray working on homotopy theory, configuration spaces and toric topology.

Expertise available. The School of Mathematics at the University of Manchester is one of the largest in the UK, and is well-known to the international community for its long-standing expertise in topology, algebra, and geometry. It holds weekly seminars and regular meetings in these disciplines, which often attract external funding and international speakers, and is planning a major conference at which Cohen has agreed to speak during 2008. The School also hosts a stream of visitors in related fields, and about 10 of its PhD students have been engaged in relevant projects since 2000. Current work of Buchstaber, Eccles, and particularly Ray, involves major homotopy theoretical overlap with the proposal, and the algebraic expertise of Premet, Prest, and especially Symonds in representation theory, will also be of significant value. Four other staff members have related interests.

Part 2. Description of the Proposed Research

The overall objectives of this programme are developing new decomposition methods for loop spaces and some fundamental spaces (including Moore spaces, classical Lie groups and Stiefel manifolds), and studying the homotopy characteristics of topological spaces using all recently developed decomposition machineries. Two types of homotopy properties are focused on: H-structure preserved by decompositions and homotopy exponents. Following the nature of the overall objectives, the project is organised in two parts: Homotopy Decompositions and the Homotopy Exponent Problem.

I HOMOTOPY DECOMPOSITIONS

Background. *(i) Loop Space Decompositions.* A general approach in understanding a mathematical object is to decompose it into simpler objects, then to study those objects and in the end to assemble all the information in order to obtain some properties of the initial object. Therefore in classifying any mathematical structure, it is helpful to analyse the irreducible or indecomposable components first.

A space X is an *H-space* if it has a continuous multiplication which satisfies a unital property. By the *H-structure* on X, we mean whether the multiplication is homotopy associative, homotopy commutative, satisfies higher homotopy associativities, or higher commutativities. In the context of a homotopy decomposition $X \simeq Y \times Z$, if X is an H-space, then so are Y and Z as they retract off X. The equivalence, however, may be just on the level of spaces, not preserving H-structure. Therefore it is important to know to what extent the equivalence carry on the initial H-structure.

As already mentioned in the section on Previous Research, the first decomposition of loop suspensions was done stably, that is, James [13] showed that after suspending once, $\Omega\Sigma X$ breaks into $\bigvee_{i=1}^{\infty} \Sigma X^{(n)}$. Using a completely new set of tools (mod-p homotopy theory and differential Lie algebras), Cohen, Moore and Neisendorfer [2, 3] considered the case of $\Omega\Sigma X$, where X is a mod-p^r Moore space $P^{m-1}(p^r)$ and p is an odd prime. The PI generalised their approach introducing a notion of universal spaces in the category of homotopy associative and homotopy commutative H-spaces. This method led to a decomposition of the loop suspension of certain two-cell complexes [7]. Further on, the PI [6] attained a decomposition of the loop suspension of a three-cell complex which allowed for improved calculations in the EHP-spectral sequence. A major breakthrough in decomposing loop suspensions was achieved by Selick and Wu [16]. They consider functorial decompositions of $\Omega\Sigma X$ in the case where X is a p-local spaces. The problem reduces to that of finding a natural coalgebra decomposition of tensor algebras. Their first main result is the construction of a functor $A^{\min}(X)$ which gives the smallest natural retract of $\Omega\Sigma X$ whose mod p homology contains $\widetilde{H}_*(X; Z/pZ)$. While, by construction, $A^{\min}(X)$ is the smallest natural nontrivial retract of $\Omega\Sigma X$, it is possible that $A^{\min}(X)$ could decompose further for particular spaces X. Some work along these lines has already been done by the PI [7].

(ii) Finite H-spaces. Another use of Selick and Wu's decomposition method was to give a different construction of certain low rank finite H-spaces first constructed by Cooke, Harper and Zabrodsky [1]. The original construction obtained the finite H-space directly, by explicitly constructing multiplication. The new construction obtained the finite H-space indirectly, by retracting it off a loop space. Theriault [18] and later on together with the PI [10] developed this further by using the detailed input from the loop space decompositions to prove that a subclass of these finite H-spaces are homotopy associative, homotopy commutative and universal for a certain subspace.

These finite H-spaces are related to another important class of finite H-spaces which arise as factors in decompositions of the p-localisation of classical Lie groups. Focusing on $SU(n)$, Mimura, Nishida and Toda [14] showed that there is a decomposition $SU(n) \simeq \prod_{i=1}^{p-1} B_i(n)$ for indecomposable spaces $B_i(n)$. Each $B_i(n)$ is an H-space as it retracts off the H-space $SU(n)$. However, the homotopy equivalence is on the level of spaces, not H-spaces, so that the H-structure of the factors remains a mystery. Knowing this, and the extent to which the homotopy equivalence is multiplicative, would be useful. Given the ubiquity of Lie groups through mathematics, such information would be also of use elsewhere.

Programme and Methodology. (i) Exploring Selick and Wu's Decompositions. In this part of the project we propose to study a very important homotopy invariant of certain indecomposable factors, namely, the *homology of the atomic retract* containing the bottom cell of the loop suspension of a Hopf invariant one complex.

Although this problem is interesting in its own right, its solution would provide an important link between algebraic topology and representation theory. It turns out that the problem of calculating the homology of the bottom piece of the loop suspension of an arbitrary complex is equivalent to the fundamental problem in the modular theory of the symmetric group, namely, determining possible decompositions of the identity. Mathematicians working in representation theory have tried hard to solve problems related to the fundamental problem of the representation theory of the symmetric groups in the modular case (when the characteristic of the field divides the order of the group) for over a hundred years, after the rational representation theory of symmetric groups was done by Young diagrams. In spite of much effort, the modular representation theory of the symmetric groups remains, for the most part, a mystery.

Programme: This part of the proposal is based on work of Selick and Wu [16] on natural coalgebra decompositions of tensor algebras and loop suspensions. They consider functorial decompositions of $\Omega\Sigma X$ in the case where X is a p-local space. The problem reduces to that of finding a natural coalgebra decomposition of tensor algebras. The first main result is the construction of a functor $A^{\min}(X)$ which gives the smallest natural retract of $\Omega\Sigma X$ whose mod p homology contains $\widetilde{H}_*(X; Z/pZ)$. Equivalently, their results give the construction of the algebraic functor $A^{\min}(\ )$ which recognises the minimal natural coalgebra retract of the tensor algebra functor $T(\ )$. The results tie in with the representation theory of the symmetric group and in particular produce the maximal projective submodule of the important Σ_n-module $Lie(n)$, which arises in the homology of various interesting spaces. While, by construction, $A^{\min}(X)$ is the smallest natural nontrivial retract of $\Omega\Sigma X$, it is possible that $A^{\min}(X)$ could decompose further for particular spaces X. Some work along these lines has already been done. The PI [7] worked on decompositions of $\Omega\Sigma X$ where X is a p-local two-cell suspension such that the bottom cell is in an arbitrary even dimension, while the top cell is in an arbitrary odd dimension. It was shown that the smallest retract F of $\Omega\Sigma X$ which contains the bottom cell is a nontrivial retract of $A^{\min}(X)$. In this case the mod p homology of F is isomorphic as a Hopf algebra to the free commutative algebra generated by $\widetilde{H}_*(X)$. Moreover, the PI calculated upper and lower bounds for the exponent of the homotopy groups of F and showed that F is the universal space of X in the category of homotopy associative, homotopy commutative H-spaces. There are several more examples of decompositions of the loop suspension of a p-local two-cell complex, as listed in [16]. Yet nothing is known about the atomic piece containing the bottom cell of decompositions of the loop suspension of a p-local two-cell complex having nontrivial Steenrod power $\mathcal{P}^1$. The mod 2 case as well has not yet been considered.

Milestones:

- **obtain the decompositions of loop suspensions of 2-local two-cell complexes such as $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$, and $\mathbb{K}P^2$, namely, the Hopf invariant one complexes.**
- **calculate the homology of the atomic retracts of the above decompositions that contain the bottom cell.**

Methodology: We want to produce decompositions

$$X^{(n)} \simeq A(X) \vee B(X) \vee \ldots$$

where A and B are functors on a co-H space X. The key idea is to apply the modular representation theory of the symmetric group. First let Σ_n act on $X^{(n)}$ by permuting coordinates. Then for each $\sigma \in \Sigma_n$, we obtain a map $\sigma\colon X^{(n)} \longrightarrow X^{(n)}$. Note that $\sigma_*\colon \widetilde{H}_*(X^{(n)}) = \widetilde{H}_*(X)^{\otimes n} \longrightarrow \widetilde{H}_*(X)^{\otimes n} = \widetilde{H}_*(X^{(n)})$ is a graded permutation of coordinates, where the homology is taken over the fields $\mathbb{Z}/p$ or $\mathbb{Q}$. Next, for $\sigma + \tau \in \mathbb{Z}(\Sigma_n)$, because $X^{(n)}$ is a co-H space, we can define a map

$$\sigma + \tau\colon X^{(n)} \longrightarrow X^{(n)}$$

such that in homology, $(\sigma + \tau)_* = \sigma_* + \tau_*$. In general, for any element $\alpha \in \mathbb{Z}_{(p)}(\Sigma_n)$ we can define a map $\alpha\colon X^{(n)} \longrightarrow X^{(n)}$ with the desired homological properties.

Now suppose that

$$1 = \sum_{i=1}^{q} e_i$$

is an orthogonal decomposition of the identity in $\mathbb{Z}_{(p)}(\Sigma_n)$ in terms of primitive idempotents. Then we have a homotopy decomposition given by

$$X^{(n)} \longrightarrow \bigvee_{i=1}^{q} X^{(n)} \xrightarrow{\bigvee_{i=1}^{q}\tilde{e}_i} \bigvee_{i=1}^{q} \operatorname*{hocolim}_{e_i} X^{(n)}$$

with

$$\widetilde{H}_*(\operatorname*{hocolim}_{e_i} X^{(n)}) = \mathrm{Im}(e_{i*}\colon (\widetilde{H}_*(X))^{\otimes n} \longrightarrow (\widetilde{H}_*(X))^{\otimes n}).$$

By using modular representation theory (the primitive idempotents correspond to Young diagrams), we will produce functorial decompositions of self-smash products of two-cell suspensions. The homology of each factor can be explicitly determined. Then we take X to be a Hopf invariant one complex. (Note that $\mathbb{R}P^2$ is not a co-H space but there are some techniques available (cf. [21]) for solving this problem.) By using Steenrod operations, we intend to show that each factor is indecomposable. In other words, we will give a complete homotopy decomposition of self smashes of Hopf invariant one complexes. It should be emphasised that this decomposition comes from functorial decompositions of self smashes of general spaces.

We will then proceed to calculate the homology of the atomic piece containing the bottom cell of $\Omega\Sigma X$. The methodology of the programme is justified by recent partial results of Selick and Wu when X is an n-dimensional mod 2 Moore space, that is, $X = \Sigma^{n-2}\mathbb{R}P^2$.

(ii) Finite H-spaces.

Programme: In this part of the proposal we propose to analyse the homotopy theory of classical Lie groups and Stiefel manifolds, motivated by the calculation of upper bounds on their homotopy exponents, particularly in low rank (cf. [17, 18]). Our main goal will be to analyse the H-structure of the factors appearing in the decompositions of Lie groups and looped Stiefel manifolds, and determine to what extent the decomposition preserves H-structure. We will proceed by studying multiplicative p-local self-maps of low rank Lie groups by making use of universal spaces appearing in the favourable decomposition.

Methodology: The methods to be used are a mixture of standard and new decomposition techniques. The standard techniques are based on mod-p homotopy theory and differential graded Lie algebras. There are several new tools. First, Selick and Wu's decompositions of loop spaces. Second, the use of universal spaces to extract exponent and H-structure information.

As before, we focus on $SU(n)$ for ease of exposition. We consider the decomposition $SU(n) \simeq \prod_{i=1}^{p-1} B_i(n)$ given by Selick and Wu which holds in low rank, when $n \leq (p-1)(p-2)$. Its advantage over Mimura, Nishida and Toda's decomposition is the existence of homotopy fibration sequences

$$\Omega\Sigma A_i \xrightarrow{*} B_i(n) \longrightarrow R_i \xrightarrow{\lambda_i} \Sigma A_i$$

where (i) $H_*(B_i(n)) \cong \Lambda(\tilde{H}_*(A_i))$; (ii) $\Omega\Sigma A_i \simeq B_i(n) \times \Omega R_i$ and (iii) R_i is a wedge of spaces and λ_i is a wedge sum of Whitehead products . Here A_i is a wedge of summand of $\Sigma\mathbb{C}P^{n-1}$. The detailed input in the construction of $B_i(n)$ allows for a detailed analysis of its homotopy properties. Note that $B_i(n)$ is an H-space because it is a retract of the H-space $SU(n)$, but the decomposition in not necessary multiplicative.

Milestones:

(a) **determine the extent to which the homotopy equivalence $SU(n) \simeq \prod_{i=1}^{p-1} B_i(n)$ is multiplicative;**

(b) **use the universal property of $B_i(n)$ to obtain upper bounds on its homotopy exponent and consequently on $SU(n)$;**

(c) **use the previous properties to help determine the group of self-maps $[SU(n), SU(n)]$.**

There is a clear strategy for approaching the Milestones. The decomposition methods in [6, 7] used to study finite H-spaces establish the universal property of $B_i(n)$, facilitating the calculations in (b) and (c). It remains to be determined exactly what the statement about the H-equivalence should be in part (a). The equivalence rarely happens to be multiplicative integrally or at odd primes. Therefor our approach might shed light on obstructions to homotopy commutativity of $SU(n)$.

Similar questions can be asked of Stiefel manifolds, with a modification. A Stiefel manifold V is not usually an H-space, so it may be necessary to loop before decomposing. But V appears cohomologically to be a product of spaces similar to the $B_i(n)$'s, so it may be the case that ΩV decomposes as a product of loop spaces. In low rank, the new decomposition methods give this a chance of being proved.

Milestones:

- **in low rank, decompose ΩV as a product of loop spaces;**
- **determine whether a decomposition can be chosen to be an H-map;**
- **find upper bounds on the homotopy exponents of the factors and hence on V.**

The success of this decomposition method in giving new information about H-spaces suggests more is possible. It tends to be difficult to construct and analyse finite H-spaces. The new methods may offer a better tool with which to make progress.

Milestones:

- **use the new decomposition method as a tool to construct more finite H-spaces;**
- **analyse the H-structure of these H-spaces and calculate their homotopy exponent.**

II THE HOMOTOPY EXPONENT PROBLEM

Background. *Exponents.* The second strand of our research proposal is concerned with the exponent problem in homotopy theory. In general, two types of exponents of a given space X can be considered. Let p be a prime. The mod p homotopy exponent of a space X is p^r if that is the least power of p which annihilates the p-torsion component of $\pi_*(X)$. Denote the mod p homotopy exponent of X by $\exp(X) = p^r$. If X is a simply connected CW complex of finite type, then each $\pi_n(X)$ has a bounded exponent. Not much is known for a "global" exponent. For instance $\pi_*(S^2 \vee S^2)$ does not have a bounded exponent. On the other hand, one can check that the homotopy groups of the mapping space from a simply connected finite torsion space to a space has a bounded exponent. A stronger notion is that of a multiplicative (or H-) exponent. If Y is an H-space, then the p-th power map is given by the composite $p\colon Y \xrightarrow{\Delta} Y^{\times p} \xrightarrow{\mu} Y$, where μ is the multiplication on Y. The multiplicative exponent of Y is p^r if that is the least power of p such that $p^r\colon Y \longrightarrow Y$ is null homotopic, while $p^{r-1}\colon Y \longrightarrow Y$ is essential. We say that Y has no multiplicative exponent if the p^r-th power map on Y is essential for all $r \in \mathbb{N}$.

One of the longstanding conjectures in homotopy theory, dating from the 1950's, and a major exponent problem is the *Barratt Conjecture.* Let $f\colon \Sigma^2 X \longrightarrow Y$ be a map of order p^r in $[\Sigma^2 X, Y]$. Then

$$p^{r+1}\,\mathrm{Im}\left(f_*\colon \pi_*(\Sigma^2 X) \longrightarrow \pi_*(Y)\right) = 0.$$

In particular, if the identity map on $\Sigma^2 X$ is of order p^r in $[\Sigma^2 X, \Sigma^2 X]$, then the mod p exponent of $\Sigma^2 X$ is p^{r+1}, that is,

$$p^{r+1}\pi_*(\Sigma^2 X) = 0.$$

The late J. F. Adams, attributed with founding stable homotopy theory as a coherent discipline by people like Denis Sullivan, implies in his writing that it was a lack of progress in solving basic questions like the Barratt conjecture that caused him to move away from unstable homotopy theory. New developments now suggest that such questions can be tackled in a coherent manner. Today, homotopy theorists have a more general point of view on the Barratt conjecture. A *strong version of the Barratt Conjecture* is concerned with the multiplicative exponent of $\Sigma^2 X$ and can be stated as follows. Let $f\colon \Sigma^2 X \longrightarrow Y$ be a map of order p^r in $[\Sigma^2 X, Y]$. Then

$$\Omega^2 f\colon \Omega^2\Sigma^2 X \longrightarrow \Omega^2 Y$$

has order bounded by p^{r+1} in $[\Omega^2\Sigma^2 X, \Omega^2 Y]$.

It was proved by Neisendorfer [15] that the Barratt Conjecture holds for the Moore space $P^n(p^r)$ with $p > 2$. It should be pointed out that it is still unknown whether or not $\pi_*(P^n(2))$ has bounded exponent.

Programm and Methodology. *Programme:* How is the Barratt Conjecture related to our project? We know that the strong form of the Barratt Conjecture is equivalent to

$$p^{r+1} \cdot [\Omega A^{\min}(X), \Omega A^{\min}(X)] = 0$$

for any suspension X with the trivial degree p^r map.

We propose to use the information on $A^{\min}$ obtained in the first part of our research project to make progress on the Barratt Conjecture.

Milestone:

- **calculate the homotopy groups of a mod 2 Moore space in certain range;**
- **find an upper bound on the homotopy exponent of a mod 2 Moore space.**

Methodology: There is another way of approaching the Barratt Conjecture. Wu and the PI [11, 12] brought simplicial methods into the study of homotopy exponent problems. They established a link between topology and algebra by associating combinatorial groups to homotopy exponent problems via natural coalgebra transformations of tensor algebras. In such a way geometrical problems become purely algebraic, allowing a systematic analysis of the obstruction to the exponent problem using algebraic machinery. In Wu and the PI's work, the growth of the exponent was controlled by displaying the obstructions to the exponent problem in combinatorial groups known as the Cohen groups.

Our next step is to study the quotient groups of natural coalgebra transformations of tensor algebras that admit representations into self functorial maps of double loop spaces. Extra information should be obtained by looking at the composite

$$\text{Cohen group} \longrightarrow [\Omega\Sigma^2, \Omega\Sigma^2] \xrightarrow{\Omega} [\Omega^2\Sigma^2, \Omega^2\Sigma^2].$$

The furthest right group is abelian and the Cohen group is not abelian. Therefore the composite must factor through the abelianisation of the Cohen group. Wu [20] recognised that shuffled relations can be factorised out of the Cohen group without changing the composite. Nowadays, one way of studying double loop spaces is to investigate related configuration spaces. We want to study the evaluation map from double loop spaces to single loop spaces using configuration space models, hoping to kill off the obstructions to the exponent problem.

Milestones:

- **find a suitable notion of the evaluation map from double loop suspension spaces to single loop suspension spaces using configuration space models;**
- **find more relations such that the composite factors through further quotients;**
- **ideally, find "secondary relations" in the Cohen group which, when quotiented out, would resolve the Barratt conjecture.**

Further Research. The simplicial methods permit a development of the project in many directions. For example, it can be related to the *Adams spectral sequence*. Consider the Milnor free monoid construction $F(X)$ on X. It is known that $F[X] \simeq \Omega\Sigma X$. The Cohen group consists of certain self maps $J(X) \longrightarrow F[X]$. We know that the James construction $J(X)$ is the free monoid generated by X and that $J(X) \simeq F[X]$ if X is path-connected. So the Cohen maps should be extendable to self maps of $F[X]$. Algebraically, we know that the Cohen maps

$$\Omega\Sigma X \longrightarrow \Omega\Sigma X$$

preserve the augmentation ideal filtration in homology. The aim is to show that all Cohen maps

$$F[X] \longrightarrow F[X]$$

preserve induced descending central series (although they are not group homomorphisms). Recall that the Adams spectral sequence is constructed from descending central series. Therefore if we can extend the Cohen maps $J(X) \longrightarrow F[X]$ to $F[X] \longrightarrow F[X]$ and the extended maps preserve the filtration of descending central series, then all of our natural coalgebra decompositions would induce a "natural decomposition of Adams spectral sequences".

Timeliness and Novelty. The functorial decomposition of loop spaces is a new subject in Algebraic Topology. It provides not just a meeting point between different problems in Homotopy Theory but also connects it with Representation Theory. We propose a novel research programme which focuses on the topological, simplicial and combinatorial aspects of the theory. There are many directions in which the new objects and methods could be investigated. The project represents new and original approaches to fundamental questions in Algebraic Topology.

Relevance to Beneficiaries. The beneficiaries of this project are mathematicians working in Algebraic Topology, Algebra and Representation Theory. The work on Moore spaces, Lie groups and Stiefel manifolds would be of immediate interest particularly to members of the long standing Japanese School of Homotopy (especially in Kyoto, Kyushu, Okayama and Osaka), to research teams in North America (such as in Chicago, Rochester, Toronto), Singapore and individuals working in Europe (such as in Aberdeen, Glasgow, Leicester, Manchester, Barcelona, Lausanne). The results will have an important impact on the representation theory of symmetric groups and will be of interest to experts especially in Aberdeen, Cambridge, Leicester, Manchester, Oxford and Singapore. The research assistant working on this project will benefit from developing skills and knowledge in her/his area of speciality and from working on a subject which currently receives a lot of attention and exposure.

Dissemination and Exploitation. The content and results of the research will be disseminate through internationally acknowledged channels for Pure Mathematics, including: publication of papers in appropriate scholarly journals of international repute; presentations at seminars, workshops and conferences within the UK and overseas; and distribution of preprints via electronic archives and personal webpages. The PI has already been invited for month-long visits to the University of Toronto and the National University of Singapore in the academic year 2007/2008. During those visits the PI will work on the solutions to the problems posed in this proposal and present advances made up to that point. Manchester is one of the nodes in the English consortium which has recently been awarded an EPSRC Taught Course Centre grant. The PI hopes to be able to use this advance media of communication to deliver a series of lectures aimed at postgraduate students based on the results of the project.

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