

HOMOTOPY THEORY AND THE COMPLEMENT OF A COORDINATE SUBSPACE ARRANGEMENT

JELENA GRBIĆ

ABSTRACT. The main aim of this paper is to underline the role of Homotopy Theory in Toric Topology. In particular, the homotopy type of the complement of a coordinate subspace arrangement is studied by considering the connection between its topological and combinatorial structures. A family of arrangements for which the complement is homotopy equivalent to a wedge of spheres is described. One consequence is an application in commutative algebra: certain local rings are proved to be Golod. On other hand in homotopy theory a better understanding of higher Whitehead and Samelson products is proposed.

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2000 *Mathematics Subject Classification.* Primary 13F55, 55P15, Secondary 52C35.

Key words and phrases. coordinate subspace arrangements, homotopy type, Golod rings, toric topology, cube lemma.

1. INTRODUCTION

In this paper we elaborate on the natural role of Homotopy Theory in Toric Topology by studying the homotopy type of the complement of a (complex) coordinate subspace arrangement. The complement of a coordinate subspace arrangement can be, up to homotopy, identified with the moment angle complex, one of the central objects in Toric Topology.

This paper is based on the results on the homotopy theory of the complement of a complex coordinate spaces written by Stephen Theriault and the author in [GT, GT2] and the talk given by author at the International Conference on Toric Topology at Osaka City University, 2006.

We begin the paper with a preliminary section on Homotopy Theory. The more advance information on Homotopy Decompositions are given in Section 3. With these first two section we intend to recall a necessary homotopy background for tackling the problem of determining the homotopy type of the complement of a coordinate arrangement. In Section 4 we introduce main definitions and constructions of Toric Topology running through several related mathematical disciplines: Combinatorics, Algebra, Topology. Sections 5 through 7 illustrate on several examples the main method used to determine the homotopy type of the complement of an subspace arrangement. The last section states some applications in Topology and Algebra.

2. HOMOTOPY THEORY

Throughout this section and the remainder of the paper, we work in the category of based, connected topological spaces and continuous maps. Recall that for pointed topological spaces X and Y , the (*reduced*) *join* of X and Y is the space

$$X * Y = X \times I \times Y / \sim$$

where $(x, 0, y) \sim (x', 0, y)$, $(x, 1, y) \sim (x, 1, y')$, and $(*, t, *) \sim (*, t', *)$ for all $x, x' \in X$, $y, y' \in Y$, and $t, t' \in I$.

The quotient map $X * Y \longrightarrow \Sigma X \wedge Y$ is a natural homotopy equivalence.

Lemma 2.1. *The homotopy pushout of the projection maps $\pi_1: X \times Y \longrightarrow X$ and $\pi_2: X \times Y \longrightarrow Y$ is homotopy equivalent to $X * Y$.*

Proof. To establish the homotopy type of the homotopy pushout Q of the projection maps π_1 and π_2 we embed the homotopy pushout diagram in the diagram of homotopy cofibration sequences

$$\begin{array}{ccccccc} X \times Y & \xrightarrow{\pi_1} & X & \longrightarrow & C_X & \longrightarrow & \Sigma(X \times Y) \\ \downarrow \pi_2 & & \downarrow & & \parallel & & \downarrow \\ Y & \longrightarrow & Q & \longrightarrow & C_X & \longrightarrow & \Sigma Y \\ \downarrow & & \downarrow & & & & \\ C_Y & \xlongequal{\quad} & C_Y & & & & \end{array}$$

After suspending the topological space $X \times Y$ once, there is a homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. From the top row cofibration sequence it follows that $C_X \simeq \Sigma Y \vee \Sigma(X \wedge Y)$. Now considering the cofibration sequence $Y \longrightarrow Q \longrightarrow \Sigma Y \vee \Sigma(X \wedge Y)$, it follows that $Q \simeq \Sigma X \wedge Y \simeq X * Y$. \square

2.1. Samelson and Whitehead Products. Let G be a group-like space, that is, an H -space with a homotopy multiplication and a homotopy inverse. Let $[\cdot, \cdot]: G \times G \rightarrow G$ be a commutator defined by $[x, y] = xyx^{-1}y^{-1}$ for all $x, y \in G$. Its restriction to $G \vee G$ is trivial, so it factors through a map $[\cdot, \cdot]: G \wedge G \rightarrow G$. Using that the left hand map in the cofibration $G \vee G \rightarrow G \times G \rightarrow G \wedge G$ has a left homotopy inverse when suspended, we show that the factorisation is unique up to homotopy.

Definition 2.2. Let $f: X \rightarrow G$ and $g: Y \rightarrow G$. The composite

$$\langle f, g \rangle: X \wedge Y \xrightarrow{f \wedge g} G \wedge G \xrightarrow{[\cdot, \cdot]} G$$

which is defined pointwise by sending $\langle x, y \rangle$ to $f(x)g(y)f(x)^{-1}g(y)^{-1}$, is called the *external Samelson product* of f and g .

Remark 2.3. The homotopy class of $\langle f, g \rangle$ is uniquely determined by those of f and g .

Definition 2.4. Suppose X is any space and maps $f: \Sigma A \rightarrow X$ and $g: \Sigma B \rightarrow X$ are given. Let f' and g' be the adjoints of f and g respectively. The *external Whitehead product*

$$[f, g]: \Sigma A \wedge B \rightarrow X$$

is the adjoint of the external Samelson product $\langle f', g' \rangle$.

Example 2.5. Let ι_m and ι_n be the canonical inclusion of S^m and S^n into $S^m \vee S^n$. The space formed by attaching an $m+n$ -cell to $S^m \vee S^n$ by means of the Whitehead product $[\iota_m, \iota_n]: S^{m+n-1} \rightarrow S^m \vee S^n$ is $S^m \times S^n$.

External Whitehead products are natural with respect to maps $\psi: X \rightarrow Z$. Suppose in addition that X is a suspension, $X = \Sigma Y$. Then the external Whitehead product $[1_{\Sigma Y}, 1_{\Sigma Y}]: \Sigma Y \wedge Y \rightarrow \Sigma Y$ is defined and it is natural with respect to suspensions, $\Sigma\phi: \Sigma Z \rightarrow \Sigma Y$.

One particular case of an external Whitehead product is of substantial interest to us. Let X and Z be spaces. Let i_X and i_Z be the inclusions of X and Z into the wedge $X \vee Z$. Consider the external Samelson product $\langle \Omega i_X, \Omega i_Z \rangle$. Adjoining gives an external Whitehead product $[\zeta_X, \zeta_Z]$, where $\zeta_X = i_X \circ ev$, $\zeta_Z = i_Z \circ ev$ and ev is the canonical evaluation map $ev: \Sigma \Omega Y \rightarrow Y$.

Lemma 2.6. *The homotopy fibre of the inclusion*

$$X \vee Y \xrightarrow{i} X \times Y$$

is $\Omega X * \Omega Y$.

Proof. Consider the diagram

$$(1) \quad \begin{array}{ccccc} Y & \xrightarrow{\quad} & X \vee Y & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & * & \downarrow & & X \\ & \downarrow & \downarrow & & \downarrow \\ X \times Y & \xlongequal{\quad} & X \times Y & \xlongequal{\quad} & X \times Y \\ & \searrow & \downarrow & \searrow & \\ & & X \times Y & \xlongequal{\quad} & X \times Y \end{array}$$

and extend the vertical maps into fibrations, denoting by F the homotopy fibre of the inclusion $i: X \vee Y \rightarrow X \times Y$. As the top side of diagram (1) is a pushout, the newly obtained diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & F \\ \pi_1 \uparrow & & \uparrow \\ \Omega X \times \Omega Y & \xrightarrow{\pi_2} & \Omega Y \end{array}$$

is a pushout as well. Now by Lemma 2.1, $F \simeq \Omega X * \Omega Y$. \square

Lemma 2.7. *There is a homotopy fibration*

$$\Sigma \Omega X \wedge \Omega Z \xrightarrow{[\zeta_X, \zeta_Z]} X \vee Z \xrightarrow{i} X \times Z,$$

where i is the inclusion.

Proof. See [Selick, Theorem 7.7.4a]. It is important to notice that the map from the fibre into the wedge can be realised as the external generalised Whitehead product on X and Z . \square

3. HOMOTOPY DECOMPOSITIONS

The purpose of this section is to identify the homotopy type of several pushouts. We begin by stating Mather's Cube Lemma [M], which relates homotopy pullbacks and homotopy pushouts in a cubical diagram.

Lemma 3.1. *Suppose there is a homotopy commutative diagram*

$$\begin{array}{ccccc} E & \xrightarrow{\quad} & F & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & G & \downarrow & & H \\ & \downarrow & \downarrow & & \downarrow \\ A & \xrightarrow{\quad} & B & & \\ & \searrow & \downarrow & \searrow & \\ & & C & \xrightarrow{\quad} & D. \end{array}$$

Suppose the bottom face $A - B - C - D$ is a homotopy pushout and the sides $E - G - A - C$ and $E - F - A - B$ are homotopy pullbacks.

- (a) If the top face $E - F - G - H$ is also a homotopy pushout then the sides $G - H - C - D$ and $F - H - B - D$ are homotopy pullbacks.
- (b) If the sides $G - H - C - D$ and $F - H - B - D$ are also homotopy pullbacks then the top face $E - F - G - H$ is a homotopy pushout.

□

We next set some further notation. Let $*$ denote the basepoint. For spaces X and Y , let $X \rtimes Y = (X \times Y)/(* \times Y)$, $X \wedge Y = (X \rtimes Y)/(X \times *)$, and $X * Y = \Sigma X \wedge Y$. Fix spaces X_1 and X_2 and let $1 \leq j \leq 2$. Let $\pi_j: X_1 \times X_2 \rightarrow X_j$ be the projection onto the j^{th} factor and let $i_j: X_j \rightarrow X_1 \times X_2$ be the inclusion into the j^{th} factor. Let $q_j: X_1 \vee X_2 \rightarrow X_j$ be the pinch map onto the j^{th} wedge summand. Unless otherwise specified, we adopt the Milnor-Moore notation of denoting the identity map on a space X by X . Denote the map which sends all points to the basepoint by $*$.

Turning to the homotopy types of pushouts, we begin with a Lemma which was proved in [GT]. For completeness, we recall the proof.

Lemma 3.2. *Let A, B and C be spaces. Define Q as the homotopy pushout*

$$\begin{array}{ccc} A \times B & \xrightarrow{* \times B} & C \times B \\ \downarrow \pi_1 & & \downarrow \\ A & \longrightarrow & Q. \end{array}$$

Then $Q \simeq (A * B) \vee (C \rtimes B)$. □

Proof. Consider the diagram of iterated homotopy pushouts

$$\begin{array}{ccccc} A \times B & \xrightarrow{\pi_2} & B & \xrightarrow{i_2} & C \times B \\ \downarrow \pi_1 & & \downarrow * & & \downarrow s \\ A & \xrightarrow{*} & A * B & \xrightarrow{t} & \overline{Q} \end{array}$$

where π_2, i_2 are the projection and inclusion respectively. Here, it is well known that the left square is a homotopy pushout (see Lemma 2.1), and the right homotopy pushout defines \overline{Q} . Note that $i_2 \circ \pi_2 \simeq * \times B$. The outer rectangle in an iterated homotopy pushout diagram is itself a homotopy pushout, so $\overline{Q} \simeq Q$. The right pushout then shows that the homotopy cofibre of $C \times B \rightarrow Q$ is $\Sigma B \vee (A * B)$. Thus t has a left homotopy inverse. Further, $s \circ i_2 \simeq *$ so pinching out B in the right pushout gives a homotopy cofibration $C \rtimes B \rightarrow Q \xrightarrow{r} A * B$ with $r \circ t$ homotopic to the identity map. □

Lemma 3.3. *Let A, B, C and D be spaces. Define Q as the homotopy pushout*

$$\begin{array}{ccc} A \times B & \xrightarrow{* \times B} & C \times B \\ \downarrow A \times * & & \downarrow \\ A \times D & \longrightarrow & Q. \end{array}$$

Then $Q \simeq (A * B) \vee (C \rtimes B) \vee (A \rtimes D)$.

Proof. Let Q_1 be the homotopy pushout of the maps $A \times D \longrightarrow Q$ and $A \times D \xrightarrow{\pi_1} A$. Then there is a diagram of iterated homotopy pushouts

$$\begin{array}{ccc} A \times B & \xrightarrow{* \times B} & C \times B \\ \downarrow A \times * & & \downarrow \\ A \times D & \longrightarrow & Q \\ \downarrow \pi_1 & & \downarrow \\ A & \longrightarrow & Q_1. \end{array}$$

Observe that the outer rectangle is also a homotopy pushout, so by Lemma 3.2 we have $Q_1 \simeq (A * B) \vee (C \times B)$. Further, the outer rectangle shows that the map $A \longrightarrow Q_1$ is null homotopic. Since $A \times B \xrightarrow{A \times *}$ is homotopic to the composite $A \times B \xrightarrow{\pi_1} A \xrightarrow{i_1} A \times D$, there is an iterated homotopy pushout diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{* \times B} & C \times B \\ \downarrow \pi_1 & & \downarrow \\ A & \longrightarrow & Q_1 \\ \downarrow i_1 & & \downarrow \\ A \times D & \longrightarrow & Q. \end{array}$$

Since $A \longrightarrow Q_1$ is null homotopic, we can pinch out A in the lower pushout to obtain a homotopy pushout

$$\begin{array}{ccc} * & \longrightarrow & Q_1 \\ \downarrow & & \downarrow \\ A \times D & \longrightarrow & Q. \end{array}$$

Hence $Q \simeq Q_1 \vee (A \times D) \simeq (A * B) \vee (C \times D) \vee (A \times D)$. \square

Lemma 3.4. *Let A, B and C be spaces. Define Q as the homotopy pushout*

$$\begin{array}{ccc} A \times (B \vee C) & \xrightarrow{\pi_2} & B \vee C \\ \downarrow A \times q_2 & & \downarrow \\ A \times C & \longrightarrow & Q. \end{array}$$

*Then $Q \simeq (A * B) \vee C$. Further, the composite $B \vee C \longrightarrow Q \xrightarrow{\simeq} (A * B) \vee C$ is homotopic to $* \vee C$.*

Proof. First consider the homotopy pushout

$$\begin{array}{ccc} B & \longrightarrow & B \vee C \\ \downarrow & & \downarrow q_2 \\ * & \longrightarrow & C. \end{array}$$

In general, if M is the homotopy pushout of maps $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ then an easy application of the Cube Lemma (Lemma 3.1) shows that $N \times M$ is the homotopy pushout of $N \times X \xrightarrow{N \times f} N \times Y$ and $N \times X \xrightarrow{N \times g} N \times Z$. In our case, taking the product with A gives a homotopy pushout

$$\begin{array}{ccc} A \times B & \longrightarrow & A \times (B \vee C) \\ \downarrow \pi_1 & & \downarrow A \times q_2 \\ A & \xrightarrow{i_1} & A \times C. \end{array}$$

Now consider the diagram of iterated homotopy pushouts

$$\begin{array}{ccccccc} A \times B & \longrightarrow & A \times (B \vee C) & \xrightarrow{\pi_2} & B \vee C & \xrightarrow{q_1} & B \\ \downarrow \pi_1 & & \downarrow A \times q_2 & & \downarrow & & \downarrow \\ A & \xrightarrow{i_1} & A \times C & \longrightarrow & Q & \longrightarrow & Q' \end{array}$$

where the right pushout defines Q' . Because the squares are all homotopy pushouts so is the outermost rectangle. Thus, as the top row is homotopic to the projection π_2 , we see that $Q' \simeq A * B$. The right pushout then implies that there is a homotopy cofibration $C \longrightarrow Q \longrightarrow Q' \simeq A * B$.

On the other hand, the composite $A \times B \longrightarrow A \times (B \vee C) \xrightarrow{\pi_2} B \vee C$ is homotopic to the composite $A \times B \xrightarrow{\pi_2} B \xrightarrow{j_1} B \vee C$, where j_1 is the inclusion. Thus there is an iterated homotopy pushout diagram

$$\begin{array}{ccccc} A \times B & \xrightarrow{\pi_2} & B & \xrightarrow{j_1} & B \vee C \\ \downarrow \pi_1 & & \downarrow & & \downarrow \\ A & \longrightarrow & A * B & \longrightarrow & Q. \end{array}$$

As $q_1 \circ j_1$ is homotopic to the identity map on B , the composite $A * B \longrightarrow Q \longrightarrow Q' \simeq A * B$ is homotopic to the identity map. Hence the homotopy cofibration $C \longrightarrow Q \longrightarrow A * B$ splits as $Q \simeq (A * B) \vee C$.

Further, this decomposition of Q implies that the restriction of $B \vee C \longrightarrow Q$ corresponds to the inclusion $C \longrightarrow (A * B) \vee C$. The right square in the previous diagram shows that the restriction of the map $B \vee C \longrightarrow Q$ to B is null homotopic as this restriction factors through the map $B \longrightarrow A * B$ which is null homotopic. Thus the composite $B \vee C \longrightarrow Q \simeq (A * B) \vee C$ is homotopic to $* \vee C$. \square

Lemma 3.5. *Let A, B, C and D be spaces. Define Q as the homotopy pushout*

$$\begin{array}{ccc} A \times (B \vee C) & \xrightarrow{\pi_2} & B \vee C \\ \downarrow A \times (* \vee C) & & \downarrow \\ A \times (D \vee C) & \longrightarrow & Q. \end{array}$$

*Then $Q \simeq (A * B) \vee (A \times D) \vee C$. Further, letting $M = (A * B) \vee (A \times D)$, the composite $B \vee C \longrightarrow Q \xrightarrow{\simeq} M \vee C$ is homotopic to $* \vee C$.*

Proof. Observe that the map $* \vee C$ is homotopic to the composite $B \vee C \xrightarrow{q} C \xrightarrow{i} D \vee C$ where q is the pinch map and i is the inclusion. Then there is a diagram of iterated homotopy pushouts

$$(2) \quad \begin{array}{ccccc} A \times (B \vee C) & \xrightarrow{A \times q} & A \times C & \xrightarrow{A \times i} & A \times (D \vee C) \\ \downarrow \pi_2 & & \downarrow f & & \downarrow g \\ B \vee C & \longrightarrow & Q' & \longrightarrow & Q \end{array}$$

which defines the space Q' and the maps f and g . By Lemma 3.4, $Q' \simeq (A * B) \vee C$. We will show that there is a homotopy cofibration $Q' \rightarrow Q \rightarrow A \times D$ and the second map has a right homotopy inverse. If so then $Q \simeq Q' \vee (A \times D) \simeq (A * B) \vee (A \times D) \vee C$, and the additional statement identifying the composite $B \vee C \rightarrow Q \xrightarrow{\simeq} M \vee C$ as $* \vee C$ follows from Lemma 3.4, proving the Lemma.

Consider the homotopy cofibration $C \xrightarrow{i} D \vee C \rightarrow D$. Regard D as the homotopy pushout of $C \xrightarrow{i} D \vee C$ and $C \rightarrow *$. Then taking the product with A gives a homotopy pushout

$$\begin{array}{ccc} A \times C & \xrightarrow{A \times i} & A \times (D \vee C) \\ \downarrow \pi_1 & & \downarrow \\ A & \xrightarrow{i_1} & A \times D \end{array}$$

where i_1 is the inclusion. The homotopy cofiber of i_1 , and therefore of $A \times i$, is $A \times D$. Thus the right homotopy pushout in (2) shows that there is a homotopy cofibration $Q' \rightarrow Q \rightarrow A \times D$.

Next, the projection in the left square of (2) implies that the restrictions of f and g to A are null homotopic. So A can be pinched out to give a homotopy pushout diagram

$$\begin{array}{ccccc} A \times C & \xrightarrow{A \times i} & A \times (D \vee C) & \longrightarrow & A \times D \\ \downarrow & & \downarrow & & \parallel \\ Q' & \longrightarrow & Q & \longrightarrow & A \times D. \end{array}$$

The inclusion $A \times D \rightarrow A \times (D \vee C)$ induces an inclusion $A \times D \rightarrow A \times (D \vee C)$ which is a right homotopy inverse of $A \times (D \vee C) \rightarrow A \times D$. Thus the composite $A \times D \rightarrow A \times (D \vee C) \rightarrow Q$ is a right homotopy inverse of $Q \rightarrow A \times (D \vee C)$. This completes the proof. \square

Lemma 3.6. *Suppose there is a homotopy pushout*

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & D \\ \downarrow * \times B & & \downarrow \\ C \times B & \xrightarrow{g} & E \end{array}$$

where the restriction of f to B is null homotopic. Then g factors through a map $g' : C \times B \rightarrow E$ and g' has a left homotopy inverse.

Proof. As the restriction of f to B is null homotopic, the homotopy commutativity of the diagram in the statement of the Lemma implies that the restriction of g to B is also null homotopic. Pinching B out on the left side results in a homotopy pushout

$$\begin{array}{ccc} A \rtimes B & \xrightarrow{f'} & D \\ \downarrow * \rtimes B & & \downarrow \\ C \rtimes B & \xrightarrow{g'} & E \\ \downarrow & & \downarrow \\ Y & \xlongequal{\quad} & Y \end{array}$$

for maps f' and g' . Since $* \rtimes B$ is null homotopic, we have $Y \simeq (C \rtimes B) \vee \Sigma(A \rtimes B)$, implying that g' has a left homotopy inverse. \square

4. TORIC TOPOLOGY - MAIN DEFINITIONS AND CONSTRUCTIONS

In this section we introduce main Toric Topology objects of our interest. The motivation for their definitions we find in rich connection of Toric Topology with Combinatorics and Algebra. We start with Combinatorics by introducing a key combinatorial object, an abstract simplicial complex.

4.1. Combinatorics. Let $V = \{v_1, \dots, v_n\}$ be a set of vertices. We identify V with the index set $[n]$.

Definition 4.1. An (*abstract*) *simplicial complex* K on a set of vertices V is a finite set

$$K = \{\sigma_1, \dots, \sigma_s \mid \sigma_i \subset V\}$$

which is closed under formation of subsets and the empty set \emptyset belongs to K .

An element σ of a simplicial complex K is called *simplex*. The dimension of a simplex is its cardinality minus 1. The dimension of a simplicial complex K is the dimension of a maximal simplex, that is,

$$\dim(K) = \max_{\sigma \in K} \{\dim(\sigma)\}.$$

To such defined combinatorial object one wants to associate an algebraic widget which will reflect combinatorial structure of a simplicial complex.

4.2. Algebra. Let R be a commutative ring with unit and enhance each v_i from the set V with topological grading, that is, the degree of each v_i is 2. With $R[V]$ we denote the polynomial ring on V over R . Now to each subset σ of the index set $[n]$ associate the sigma power of v , that is,

$$v^\sigma := \prod_{i \in \sigma} v_i$$

namely the monomial

$$v^\sigma = v_{i_1} \dots v_{i_r} \quad \text{for } \sigma = \{i_1, \dots, i_r\}.$$

The main algebraic object of this paper associated to a simplicial complex K is the Stanley-Reisner algebra, also called the face ring of K .

Definition 4.2. Let K be a simplicial complex on the set of vertices V . The *Stanley-Reisner algebra*, or the *face ring* of K is defined as the quotient ring of the polynomial ring $R[V]$ with the ideal generated by sigma powers of v whenever sigma doesn't belong to K , that is,

$$R[K] := R[v_1, \dots, v_n]/(v^\sigma : \sigma \notin K).$$

4.3. Topological models for the algebraic objects. Having such a nice algebraic ring as Stanley-Reisner ring, one can wonder whether there are different “topological models” which realise the given algebraic object $R[K]$.

4.3.1. The Davis-Januszkiewicz space.

Definition 4.3. Let K be a simplicial complex. The it Davis-Januszkiewicz space $DJ(K)$ is a topological realisation of the Stanley-Reisner ring $R[K]$, specifically, the cohomology ring of $DJ(K)$ is the Stanley-Reisner algebra of K .

It is still an open question whether a Davis-Januszkiewicz space $DJ(K)$ is, up to homotopy, uniquely determined by the Stanley-Reisner algebra $R[K]$. Notbohm and Ray [NR] showed that rationally the Davis-Januszkiewicz space $DJ(K)$ is determined by its cohomology ring. However, there are few homotopy equivalent models of a Davis-Januszkiewicz space. The first topological model of $DJ(K)$ was given by Davis and Januszkiewicz (that's where the name comes from) as a Borel construction type model.

In our approach we use another model for a Davis-Januszkiewicz space given by Buchstaber and Panov [BP] through a simple colimit of nice building blocks.

We recall briefly Buchstaber-Panov model. Let R to be the ring of integers. Identify the classifying space of the circle S^1 with the infinite-dimensional projective space $\mathbb{C}P^\infty$, and thus the classifying space BT^n of the n -torus with the n -fold product of $\mathbb{C}P^\infty$.

For an arbitrary subset ω of the index set $[n]$, define the subproduct

$$BT^\omega := \{(x_1, \dots, x_n) \in BT^n : x_i = * \text{ if } i \notin \omega\}$$

so that the coordinates that are labeled by non-elements of σ are identified with the based point.

Definition 4.4. Let K be a simplicial complex on the index set $[n]$. The *Davis-Januszkiewicz space* of a simplicial complex K on $[n]$ is given by the colimit over the face category of K of BT^σ , that is,

$$DJ(K) := \bigcup_{\sigma \in K} BT^\sigma \subset BT^n.$$

Buchstaber and Panov proved that his model is homotopy equivalent to Davis-Januszkiewicz's original model.

4.3.2. The moment-angle complex. There is yet another topological object associated to a simplicial complex K and its face ring $R[K]$. We recall its construction which is due to Buchstaber and Panov.

Realise the torus T^n as a subspace of the unit polydisc

$$T^n \subset (D^2)^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| \leq 1, \forall i\}.$$

For an arbitrary subset $\sigma \subset [n]$, define

$$B_\sigma := \{(z_1, \dots, z_n) \in (D^2)^n : |z_i| = 1 \quad i \notin \sigma\}.$$

Obviously, B_σ is homeomorphic to $(D^2)^{|\sigma|} \times T^{n-|\sigma|}$.

Definition 4.5. Let K be a simplicial complex on the index set $[n]$. Define the *moment-angle complex* \mathcal{Z}_K as the colimit over the face category of K of B_σ , that is,

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^n.$$

The torus T^n acts on the polydisc $(D^2)^n$ coordinatewise, and each B_σ is invariant under that action. Therefore, the moment angle complex \mathcal{Z}_K is acted by the torus. We can look at the T^n -equivariant cohomology of \mathcal{Z}_K and show that is isomorphic to the Stanley-Reisner ring of K .

Proposition 4.6 (Buchstaber-Panov [BP]). $H_{T^n}^*(\mathcal{Z}_K) = \mathbb{Z}[K]$

The moment-angle complex has another nice description.

Proposition 4.7 (Buchstaber-Panov [BP]). *The moment-angle complex \mathcal{Z}_K is the homotopy fibre of the embedding*

$$i: DJ(K) \longrightarrow BT^n.$$

4.4. Arrangements. A *complex coordinate subspace* of \mathbb{C}^n is given by

$$L_\sigma = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_{i_1} = \dots = z_{i_k} = 0\}$$

where $\sigma = \{i_1, \dots, i_k\}$ is a subset of $[n] = \{1, \dots, n\}$. Let

$$\mathcal{CA} = \{L_{\sigma_1}, \dots, L_{\sigma_r}\}$$

be a *complex coordinate subspace arrangement* in \mathbb{C}^n , that is, a finite set of complex coordinate subspaces in \mathbb{C}^n .

For such an arrangement \mathcal{CA} , define its *support* $|\mathcal{CA}|$ as $|\mathcal{CA}| = \bigcup_{i=1}^r L_{\sigma_i} \subset \mathbb{C}^n$.

The main topological space we study, naturally associated to the complex coordinate subspace arrangement \mathcal{CA} , is its complement $U(\mathcal{CA})$ in \mathbb{C}^n , that is,

$$U(\mathcal{CA}) = \mathbb{C}^n \setminus |\mathcal{CA}|.$$

Let K be a simplicial complex on the vertex set $[n]$. Every simplicial complex K on the vertex set $[n]$ defines a complex arrangement of coordinate subspaces in \mathbb{C}^n via the correspondence

$$K \ni \sigma \mapsto \text{span}\{e_i : i \notin \sigma\}$$

where $\{e_i\}_{i=1}^n$ is the standard basis for \mathbb{C}^n . Equivalently, for each simplicial complex K on the set $[n]$, we associate the complex coordinate subspace arrangement

$$\mathcal{CA}(K) = \{L_\sigma \mid \sigma \notin K\}$$

and its complement

$$(3) \quad U(K) = \mathbb{C}^n \setminus \bigcup_{\sigma \notin K} L_\sigma.$$

If $L \subset K$ is a subcomplex, then $U(L) \subset U(K)$.

Proposition 4.8. *This assignment $K \mapsto U(K)$ defines a one-to-one order preserving correspondence*

$$\left\{ \begin{array}{c} \text{simplicial} \\ \text{complexes on } [n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{complements of} \\ \text{coordinate subspace} \\ \text{arrangements in } \mathbb{C}^n \end{array} \right\}.$$

5. THE HOMOTOPY TYPE OF THE COMPLEMENT OF AN ARRANGEMENT

We aimed at finding those coordinate arrangements whose complements are homotopy equivalent to a wedge of spheres. Our results on the homotopy type of the complement of a complex coordinate arrangement are obtained by studying the topological and combinatorial structures of $U(K)$ with the help of commutative and homological algebra, combinatorics and homotopy theory.

The following theorem gives as a connection between \mathcal{Z}_K and $U(K)$.

Theorem 5.1 (Buchstaber, Panov [BP]). *There is an equivariant deformation retraction*

$$U(K) \xrightarrow{\cong} \mathcal{Z}_K.$$

Further on Buchstaber and Panov identified as algebras the cohomology ring of the complement of the coordinate arrangement associated to a simplicial complex K , that is of \mathcal{Z}_K , with a Tor algebra on one hand, and on the other hand with the homology of the appropriate Koszul resolution.

Theorem 5.2 (Buchstaber, Panov [BP]). *The following isomorphism of graded algebras holds*

$$(4) \quad H^*(U(K); k) \cong \mathrm{Tor}_{k[v_1, \dots, v_n]}(k[K], k) \cong H[\Lambda[u_1, \dots, u_n] \otimes k[K], d].$$

From this cohomology calculation it is clear that finding $U(K)$ which are homotopy equivalent to a wedge of sphere is not a trivial task. We can see that the cohomology of $U(K)$ might not be torsion free, more precisely whenever K has torsion, the cohomology $U(K)$ has it as well. On the other hand, hyperplane arrangements have a torsion free cohomology and after suspending their complement they become homotopy equivalent to a wedge of spheres [S].

Our first step was find those simplicial complexes K for which there are no non-trivial cup products in cohomology ring (4).

5.1. Hints from algebra and combinatorics. The Stanley-Reisner ring of K , $k[K]$ is called *Golod* if all Massey products in $\mathrm{Tor}_{k[v_1, \dots, v_n]}(k[K], k)$, that is in (4), vanish. Having a Golod Stanley-Reisner ring gives us that $U(K)$ is a topological space with all Massey products trivial, including cup products as well. This is still not enough to say that the complement of the arrangement is a wedge of spheres, but so far we have not found algebraic obstructions to that. Although being Golod is an important property, not much has been known about it. Combinatorists have studied a shifted complex in a connection to Golod property.

Definition 5.3. A simplicial complex K is *shifted* if there is an ordering

$$\sigma \in K, v' < v \Rightarrow (\sigma - v) \cup v' \in K.$$

Proposition 5.4 (Gasharov, Peeva, Welker [GPW]). *If K is shifted, then its face ring $k[K]$ is Golod.*

In this case we obtain a much stronger result by determining the homotopy type of $U(K)$

Theorem 5.5. *Let K be a shifted complex. Then $U(K)$ is homotopy equivalent to wedge of spheres.*

The proof of the Theorem is carefully carried out in [GT2]. In this paper we outline the main methods used on several examples in the following section.

6. EXAMPLES

Previously, the only known examples of $U(K)$ having the homotopy type of a wedge of spheres occurred when K was a disjoint union of 2 or 3 vertices.

Example 6.1. Let K be a disjoint union of two points. The associated complex coordinate subspace arrangement in \mathbb{C}^2 is given by the equation

$$z_1 = z_2 = 0.$$

The complement $\mathbb{C}^2 \setminus \{z_1 = z_2 = 0\}$ is homotopy equivalent to $\mathbb{R}^4 \setminus \{pt\} \simeq S^3$.

The other way to calculate the homotopy type of the complement of this arrangement is to use homotopy theory. By Lemma 2.6, there is a fibration

$$\Omega\mathbb{C}P^\infty * \Omega\mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

where $*$ denotes the joint of two spaces. Therefore the complement of the arrangement is homotopy equivalent to $\Omega\mathbb{C}P^\infty * \Omega\mathbb{C}P^\infty \simeq \Sigma S^1 \wedge S^1 \simeq S^3$.

Example 6.2. Let K be a disjoint union of three points. Then the corresponding complex coordinate subspace arrangement in \mathbb{C}^3 is given by the coordinate subspaces

$$\{z_1 = z_2 = 0, \quad z_1 = z_3 = 0, \quad z_2 = z_3 = 0\}.$$

The complement of this arrangement

$$\begin{aligned} U(K) &\simeq \mathbb{C}^n \setminus \{z_1 = z_2 = 0, \quad z_1 = z_3 = 0, \quad z_2 = z_3 = 0\} \\ &\simeq S^5 \setminus \{z_1 = z_2 = 0, \quad z_1 = z_3 = 0, \quad z_2 = z_3 = 0\} \\ &\simeq \Sigma^3(S^2 \setminus \{S^1, S^1, S^1\}). \end{aligned}$$

As the three circles are not linked, after removing them from the sphere S^2 we have

$$\begin{aligned} U(K) &\simeq \Sigma^3(S^0 \vee S^0 \vee S^0 \vee S^1 \vee S^1) \\ &\simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4. \end{aligned}$$

6.1. Examples for shifted complexes. It is easy to see that any i^{th} -skeleton $\Delta^i(n)$ of the standard simplex $\Delta(n)$ on n vertices is shifted. Other examples are easy to construct; we give two to illustrate.

Example 6.3. Let K be the simplicial complex consisting of vertices $\{1, 2, 3, 4\}$ and edges $\{12, 13, 14, 23, 24\}$. Then K is shifted.

Example 6.4. Let K be the simplicial complex consisting of vertices $\{1, 2, 3, 4, 5\}$ and edges $\{12, 13, 14, 15, 23, 24, 25, 34, 35\}$. Then K is shifted. Note that $K' = K \cup \{123\}$ is shifted, but $K'' = K \cup \{124\}$ is not shifted.

Two additional definitions we need are the following. Let K be a simplicial complex. The *link* and the *star* of a simplex $\sigma \in K$ are the subcomplexes

$$\text{link}_K \sigma = \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\};$$

$$\text{star}_K \sigma = \{\tau \in K \mid \sigma \cup \tau \in K\}.$$

It is well known (and easy to prove) that if K is shifted then each of $\text{link}(1)$, $\text{star}(1)$, and $\text{rest}\{2, \dots, n\}$ is shifted, $\text{star}(1) = (1) * \text{link}(1)$, and there is a topological pushout

$$\begin{array}{ccc} \text{link}(1) & \longrightarrow & \text{rest}\{2, \dots, n\} \\ \downarrow & & \downarrow \\ \text{star}(1) & \longrightarrow & K. \end{array}$$

This results in a corresponding homotopy pushout of Davis-Januszkiewicz spaces

$$\begin{array}{ccc} DJ(\text{link}(1)) & \longrightarrow & DJ(\text{rest}\{2, \dots, n\}) \\ \downarrow & & \downarrow \\ DJ(\text{star}(1)) & \longrightarrow & DJ(K) \end{array}$$

where $DJ(\text{star}(1)) = BT \times DJ(\text{link}(1))$. Mapping the four corners into $\prod_{i=1}^n BT$ and taking homotopy fibres gives a cube as in Lemma 3.1, and in particular a homotopy pushout of fibres

$$(5) \quad \begin{array}{ccc} S^1 \times \mathcal{Z}_{\text{link}(1)} & \longrightarrow & S^1 \times \mathcal{Z}_{\text{rest}\{2, \dots, n\}} \\ \downarrow & & \downarrow \\ \mathcal{Z}_{\text{star}(1)} & \longrightarrow & \mathcal{Z}_K. \end{array}$$

We wish to show that each of $\mathcal{Z}_{\text{link}(1)}$, $\mathcal{Z}_{\text{rest}\{2, \dots, n\}}$, and $\mathcal{Z}_{\text{star}(1)}$ is homotopy equivalent to a wedge of spheres, and then identify the maps in the homotopy pushout in order to show that \mathcal{Z}_K is also homotopy equivalent to a wedge of spheres.

Example 6.5. Let K be the simplicial complex in Example 6.3. Observe that

- (1) $\text{star}(1)$ consists of vertices $\{1, 2, 3, 4\}$ and edges $\{12, 13, 14\}$;
- (2) $\text{rest}\{2, 3, 4\}$ consists of vertices $\{2, 3, 4\}$ and edges $\{23, 24\}$;
- (3) $\text{link}(2)$ consists of vertices $\{2, 3, 4\}$ and no higher dimensional simplices;
- (4) $\text{star}(2)$ coincides with $\text{rest}\{2, 3, 4\}$;
- (5) $\text{rest}\{3, 4\}$ consists of vertices $\{3, 4\}$ and no higher dimensional simplices;
- (6) $\text{link}(2)$ coincides with $\text{rest}\{3, 4\}$.

The homotopy pushout for \mathcal{Z}_K in (5) refines to a homotopy pushout

$$\begin{array}{ccc} \Omega BT_1 \times \mathcal{Z}_{\text{link}(1)} & \xrightarrow{1 \times \gamma} & \Omega BT_1 \times \mathcal{Z}_{\text{rest}\{2, 3, 4\}} \\ \downarrow \pi_2 & & \downarrow \\ \mathcal{Z}_{\text{link}(1)} & \longrightarrow & \mathcal{Z}_K. \end{array}$$

As $\text{link}(1)$ consists only of three vertices, by Example 6.2 shows that $\mathcal{Z}_{\text{link}(1)} \simeq 3S^3 \vee 2S^4$. Similarly, as $\text{rest}\{2, 3, 4\}$ consists only of two vertices Example 6.1 shows that $\mathcal{Z}_{\text{rest}\{2, 3, 4\}} \simeq S^3$. By keeping track of coordinates, γ sends two of the S^3 summands and both S^4 summands of $\mathcal{Z}_{\text{link}(1)}$ to the basepoint and it sends the remaining S^3 summand identically onto itself. Substituting, the preceding homotopy pushout becomes

$$\begin{array}{ccc} S^1 \times (S^3 \vee 2S^3 \vee 2S^4) & \xrightarrow{1 \times (1 \vee * \vee *)} & S^1 \times S^3 \\ \downarrow \pi_2 & & \downarrow \\ 3S^3 \vee 2S^4 & \longrightarrow & \mathcal{Z}_K. \end{array}$$

Lemma 3.4 then says that $\mathcal{Z}_K \simeq (S^1 * (2S^3 \vee 2S^4)) \vee S^3 \simeq S^3 \vee 2S^5 \vee 2S^6$.

Example 6.6. Let K be the shifted complex in Example 6.4. Observe that

- (1) $\text{star}(1)$ consists of vertices $\{1, 2, 3, 4, 5\}$ and edges $\{12, 13, 14, 15\}$;

(2) $\text{rest}\{2, 3, 4, 5\}$ is the simplicial complex discussed in Examples 6.3 and 6.5;

(3) $\text{link}(1)$ consists of vertices $\{2, 3, 4, 5\}$ and no higher dimensional simplices.

As in Example 6.5, there is a homotopy pushout

$$\begin{array}{ccc} \Omega BT_1 \times \mathcal{Z}_{\text{link}(1)} & \xrightarrow{1 \times \gamma} & \Omega BT_1 \times \mathcal{Z}_{\text{rest}\{2,3,4,5\}} \\ \downarrow \pi_2 & & \downarrow \\ \mathcal{Z}_{\text{link}(1)} & \longrightarrow & \mathcal{Z}_K. \end{array}$$

As $\text{link}(1)$ consists only of four vertices, $\mathcal{Z}_{\text{link}(1)} \simeq 6S^3 \vee 8S^4 \vee 3S^5$. By Example 6.5, $\mathcal{Z}_{\text{rest}\{2,3,4,5\}} \simeq S^3 \vee 2S^5 \vee 2S^6$. Keeping track of the coordinates, we can see that γ sends five of the S^3 summands and all the S^4 and S^5 summands to the basepoint, and it sends the remaining S^3 by the identity map onto the S^3 summand of $\mathcal{Z}_{\text{rest}\{2,3,4,5\}}$. Substituting, the preceding homotopy pushout becomes

$$\begin{array}{ccc} S^1 \times (S^3 \vee 5S^3 \vee 8S^4 \vee 3S^5) & \xrightarrow{1 \times (1 \vee * \vee * \vee *)} & S^1 \times (S^3 \vee 2S^5 \vee 2S^6) \\ \downarrow \pi_2 & & \downarrow \\ 6S^3 \vee 8S^4 \vee 3S^5 & \longrightarrow & \mathcal{Z}_K. \end{array}$$

Applying Lemma 3.5 then shows that

$$\mathcal{Z}_K \simeq (S^1 * (5S^3 \vee 8S^4 \vee 3S^5)) \vee (S^1 \times (2S^5 \vee 2S^6)) \vee S^3 \simeq S^3 \vee 7S^5 \vee 12S^6 \vee 5S^7.$$

6.2. Arrangements associated to $\Delta^i(n)$. Let us recall that any i^{th} -skeleton $\Delta^i(n)$ of the standard simplex $\Delta(n)$ on n vertices is shifted. For this family of simplicial complexes, $U(\Delta^i(n))$ is homotopy equivalent to a wedge of spheres, and moreover we can write down a nice formula for its homotopy type. The proof of this problem although using the same methods as for any shifted simplicial complex is less technical and therefore easier to follow.

We can formulate the problem in the following way: Determine the homotopy type of the complement of arbitrary codimension coordinate subspace arrangements.

The strategy for solving this problem is as follows:

- Step 1) determine the simplicial complex K which corresponds to a codimension- i coordinate subspace arrangement, $U(K)$;
- Step 2) associate to the determined simplicial complex K its Davis-Januszkiewicz space, that is, $DJ(K)$;
- Step 3) looking at the fibration

$$\mathcal{Z}_K \longrightarrow DJ(K) \longrightarrow BT^n,$$

describe the homotopy type of \mathcal{Z}_K .

Look at an $i+2$ -coordinate subspace in \mathbb{C}^n , that is,

$$L_\omega = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_{j_1} = \dots = z_{j_{i+2}} = 0\}, \omega = \{j_1, \dots, j_{i+2}\}$$

we can conclude that all simplexes of dimension less or equal to i belong to K , while all the simplexes of dimension greater than i do not belong to K . Therefore the associated simplicial complex is $K = \Delta^i(n)$. Hence,

$$\mathbb{C}^n \setminus \mathcal{CA}^{i+2} = U(\Delta^i(n)).$$

Further on, as the colimit model of the Davis-Januszkiewicz space for any K on $[n]$ is given by

$$DJ(K) := \bigcup_{\sigma \in K} BT^\sigma \subset BT^n,$$

we have that

$$DJ(\Delta^i(n)) = T_{n-1-i}^n = \{(z_1, \dots, z_n) : \text{at least } n-1-i \text{ coordinates are } *\} \subset (\mathbb{C}P^\infty)^n.$$

According to the proposed strategy, our further aim is to determine the homotopy fibre $U(K)$ of the fibration sequence

$$U(K)_k^n \longrightarrow T_k^n \longrightarrow (\mathbb{C}P^\infty)^n \text{ for } 1 \leq k \leq n-1.$$

This can be done in a greater generality. Let X_1, \dots, X_n be path-connected spaces. There is a filtration of $X_1 \times \dots \times X_n$ given by

$$T_n^n \longrightarrow T_{n-1}^n \longrightarrow \dots \longrightarrow T_0^n$$

where $T_k^n = \{(x_1, \dots, x_n) \in X_1 \times \dots \times X_n : \text{at least } k \text{ of } x_i \text{'s are } *\}$.

Theorem 6.7 (Porter; Grbić, Theriault [GT2]). *For $n \geq 1$, and k such that $1 \leq k \leq n-1$, the homotopy fibre F_k^n of the inclusion*

$$i: T_k^n \longrightarrow X_1 \times \dots \times X_n$$

decomposes as

$$F_k^n \simeq \bigvee_{j+n-k+1}^n \left(\bigvee_{1 \leq i_1 < \dots < i_j \leq n} \binom{j-1}{n-k} \Sigma^{n-k} \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_j} \right).$$

□

To return to our problem, take that $X_1 = \dots = X_n = \mathbb{C}P^\infty$. Then we have the inclusion

$$i: T_k^n(\mathbb{C}P^\infty) \longrightarrow (\mathbb{C}P^\infty)^n.$$

From Theorem 6.7 it follows that

$$(6) \quad \begin{aligned} F_k^n &\simeq \bigvee_{j=n-k+1}^n \binom{n}{j} \binom{j-1}{n-k} \Sigma^{n-k} \underbrace{\Omega \mathbb{C}P^\infty \wedge \dots \wedge \Omega \mathbb{C}P^\infty}_j \\ &\simeq \bigvee_{j=n-k+1}^n \binom{n}{j} \binom{j-1}{n-k} S^{n+j-k}. \end{aligned}$$

Note that this might be thought of as a representation of the symmetric group.

7. TOPOLOGICAL EXTENSIONS

At this point, we have shown that if a simplicial complex K is shifted, then its moment-angle complex $U(K)$ is homotopy equivalent to a wedge of spheres. Next we want to consider other non-shifted simplicial complexes K for which $U(K)$ is homotopy equivalent to a wedge of spheres, or for which $\Sigma U(K)$ is homotopy equivalent to a wedge of spheres. Note again that torsion can occur in the cohomology ring of $U(K)$ for certain simplicial complexes K , making it impossible for $U(K)$ to be even stably homotopy equivalent to a wedge of spheres.

We consider how three combinatorial operations - the disjoint union of simplicial complexes, gluing along a common face and the join of simplicial complexes - alter the homotopy type of the moment-angle complex. Recall that for given simplicial

complexes K_1 and K_2 on sets \mathcal{S}_1 and \mathcal{S}_2 respectively, the *join* $K_1 * K_2$ is the simplicial complex

$$K_1 * K_2 := \{\sigma \subset \mathcal{S}_1 \cup \mathcal{S}_2 \mid \sigma = \sigma_1 \cup \sigma_2, \sigma_1 \in K_1, \sigma_2 \in K_2\}$$

on the set $\mathcal{S}_1 \cup \mathcal{S}_2$.

Theorem 7.1 (Grbić, Theriault [GT2]). *Let K_1 and K_2 be simplicial complexes such that \mathcal{Z}_{K_1} and \mathcal{Z}_{K_2} are homotopy equivalent to wedges of spheres. Then the following hold:*

- (1) *if $K = K_1 \amalg K_2$ is the disjoint union of simplicial complexes, then \mathcal{Z}_K is homotopy equivalent to a wedge of spheres, more precisely,*

$$\mathcal{Z}_K \simeq \left(\prod_{i=1}^m S^1 * \prod_{j=1}^n S^1 \right) \vee \left(\mathcal{Z}_{K_1} \times \prod_{i=1}^n S^1 \right) \vee \left(\prod_{i=1}^m S^1 \times \mathcal{Z}_{K_2} \right);$$

- (2) *if $K = K_1 \cup_{\sigma} K_2$ is obtained by gluing along a common face, then \mathcal{Z}_K is homotopy equivalent to a wedge of spheres, that is,*

$$\mathcal{Z}_K \simeq (M * N) \vee (M \times \mathcal{Z}_{K_2}) \vee (\mathcal{Z}_{K_1} \times N).$$

where M and N are finite products of circles;

- (3) *if $K = K_1 * K_2$ is the join of simplicial complexes, then \mathcal{Z}_K is not homotopy equivalent to a wedge of spheres but $\Sigma \mathcal{Z}_K$ is.*

Proof. Here we recall just the proof of part 3. The Davis-Januszkiewicz space of the join $K = K_1 * K_2$ of two simplicial complexes K_1 and K_2 on the index sets $[m]$ and $[n]$ has the following form:

$$\begin{aligned} DJ(K) &= \bigcup_{\sigma \in K} BT^{\sigma} = \bigcup_{\sigma_1 \cup \sigma_2 \in K} BT^{\sigma_1} \times BT^{\sigma_2} \\ &= \left(\bigcup_{\sigma_1 \in K_1} BT^{\sigma_1} \right) \times \left(\bigcup_{\sigma_2 \in K_2} BT^{\sigma_2} \right) = DJ(K_1) \times DJ(K_2). \end{aligned}$$

Therefore the fibration

$$DJ(K_1 * K_2) \longrightarrow BT^{m+n}$$

associated to the join of K_1 and K_2 is the product fibration

$$DJ(K_1) \times DJ(K_2) \longrightarrow BT^m \times BT^n.$$

Hence $\mathcal{Z}_{K_1 * K_2} \simeq \mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2}$. This proves part (3). \square

Theorem 7.1 can be applied to two shifted complexes K_1 and K_2 . However, the simplicial complex K in parts (1) and (2) need not be shifted. So the Theorem substantially extends the family of simplicial complexes for which the moment-angle complex is homotopy equivalent to a wedge of spheres.

Theorem 7.1 can also be useful for calculating the homotopy types of moment-angle complexes \mathcal{Z}_K . To illustrate, we give an alternative calculation of that in Example 6.5.

Example 7.2. Let K be the shifted complex in Example 6.3. Then K is obtained by gluing two copies of the 1-skeleton of the standard simplex $\Delta(3)$ (on three vertices) along a common edge. Specifically, $K = K_1 \cup_{\sigma} K_2$ where K_1 consists of vertices $\{1, 2, 3\}$ and edges $\{12, 13, 23\}$, K_2 consists of vertices $\{2, 3, 4\}$ and edges $\{23, 24, 34\}$, and σ is the common edge $\{23\}$. Using the notation in the proof of Theorem 7.1 (a), the formula $\mathcal{Z}_K \simeq (M * N) \vee (M \times \mathcal{Z}_{K_2}) \vee (\mathcal{Z}_{K_1} \times N)$ corresponds to $\mathcal{Z}_K \simeq (S^1 * S^1) \vee (S^1 \times \mathcal{Z}_{K_2}) \vee (\mathcal{Z}_{K_1} \times S^1)$. Both K_1 and K_2 are copies of $\Delta^1(3)$ and therefore $\mathcal{Z}_{K_1} \simeq \mathcal{Z}_{K_2} \simeq S^5$. Hence $\mathcal{Z}_K \simeq S^3 \vee 2S^5 \vee 2S^6$.

Example 7.3. In [GT] the homotopy type of the moment angle complex of the simplicial complex consisting of n disjoint union is calculated. Consider K as the null-skeleton of $\Delta(n)$. Then by formula 6, we have that

$$\mathcal{Z}_K \simeq \bigvee_{k=2}^n (k-1) \binom{n}{k} S^{k+1}.$$

We can also consider K as a disjoint union of n one point sets. Now iteratively applying Theorem 7.1 part 1, we obtain the same result.

8. APPLICATIONS

8.1. In topology. In connection to Example 7.3 we can ask a purely homotopy theoretical question: What is the map that maps the moment angle complex \mathcal{Z}_K to the Davis-Januszkiewicz space $DJ(K)$ in the fibration sequence $\mathcal{Z}_K \rightarrow DJ(K) \rightarrow BT^n$. When K is a disjoint union of two vertices, then Lemma 2.7 asserts that the map $\mathcal{Z}_K \rightarrow DJ(K)$ is given by the Whitehead product. It is not difficult to see that if K is a disjoint union of n vertices, we obtain iterated Whitehead products by looking at the following diagram

$$\begin{array}{ccc} \bigvee_{k=2}^n (k-1) \binom{n}{k} S^{k+1} & \xrightarrow{[\iota_1, \dots, \iota_n]} & S^2 \vee \dots \vee S^2 \longrightarrow S^2 \times \dots \times S^2 \\ \parallel & & \downarrow \qquad \qquad \qquad \downarrow \\ \bigvee_{k=2}^n (k-1) \binom{n}{k} S^{k+1} & \longrightarrow & \mathbb{C}P^\infty \vee \dots \vee \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \times \dots \vee \mathbb{C}P^\infty \end{array}$$

where each S^2 includes in $\mathbb{C}P^\infty$ as the bottom cell. Theriault and the author have been working on the case when $K = \Delta^i(n)$ where the corresponding maps should be given by certain higher Whitehead products.

8.2. In algebra. Let A be a polynomial ring on n variables $k[x_1, \dots, x_n]$ over a field k and let $R = A/I$, where I is homogeneous ideal. In this section we shall be interested in the nature of $\text{Tor}_R(k, k)$; specifically, in identifying a class of rings R for which all Massey products in $\text{Tor}_A(R, k)$ vanish and how this impacts upon the Poincaré series of R . Recall that the Poincaré series of R is the formal power series

$$P(R) = \sum_{i=0}^{\infty} b_i t^i$$

where $b_i = \dim_k \text{Tor}_R^i(k, k)$ are the Betti numbers of R . It has been conjectured by Kaplansky and Serre that $P(R)$ always represents a rational function. The regular local rings were the first rings for which $P(R)$ was explicitly computed. In this case Serre [Se] showed that $P(R) = (1+t)^n$. Tate [T] showed that if R is a complete intersection, then there exist non-negative integers m, n such that

$$P(R) = \frac{(1+t)^n}{(1-t^2)^m}.$$

Golod [G] made a far reaching contribution to the problem by showing that if certain homology operations on the Koszul complex vanish, then there exist non-negative integers n, c_1, \dots, c_n such that

$$P(R) = \frac{(1+t)^n}{1 - \sum_{i=1}^n c_i t^{i+1}}.$$

In general not much is known about the rationality of $P(R)$; although there is an inequality due to Golod [G] showing that $P(R)$ is always bounded (coefficient-wise) by a rational function.

In the past, describing various properties of $\mathrm{Tor}_R(k, k)$ has been largely an algebraic problem. Further on, we translate the problem of rationality of the Poincaré series into topology by using recent results of toric topology. Then by using our results on the homotopy type of the complement of a coordinate subspace arrangement, we find a class of rings R for which $P(R)$ is a rational function determined by $P(\mathrm{Tor}_A(R, k))$.

In what follows R will be the Stanley-Reisner ring $k[K]$ of an arbitrary simplicial complex K on n vertices. Recall that the Stanley-Reisner ring $k[K]$ is *Golod* if all Massey products in $\mathrm{Tor}_{k[v_1, \dots, v_n]}(k[K], k)$ vanish. Buchstaber and Panov [BP] proved that

$$\mathrm{Tor}_{k[K]}^*(k, k) \cong H^*(\Omega DJ(K); k).$$

This isomorphism now lets us exploit the topological properties of the loop space $\Omega DJ(K)$ to obtain further information about $\mathrm{Tor}_R(k, k)$. Looking at the split fibration

$$\Omega \mathcal{Z}_K \longrightarrow \Omega DJ(K) \longrightarrow T^n$$

we have

$$\mathrm{Tor}_R^*(k, k) \cong H^*(\Omega DJ(K)) = H^*(T^n) \otimes H^*(\Omega \mathcal{Z}_K).$$

A calculation using the bar resolution shows that

$$P(H^*(\Omega \mathcal{Z}_K)) \leq P(T(\Sigma^{-1} H^*(\mathcal{Z}_K)))$$

where $\Sigma^{-1} H^*(\mathcal{Z}_K)$ is the desuspension of the module $H^*(\mathcal{Z}_K)$. Therefore

$$P(R) \leq (1+t)^n P(T(\Sigma^{-1} H^*(\mathcal{Z}_K))) = \frac{t(1+t)^n}{t - P(H^*(\mathcal{Z}_K))}.$$

Looking at the Eilenberg-More spectral sequence (the bar resolution) that computes the cohomology of the fibre in the path-loop fibration $\Omega \mathcal{Z}_K \longrightarrow * \longrightarrow \mathcal{Z}_K$, we conclude that the above equality is reached when the differentials are trivial. According to May, the differentials are determined by the Massey products and therefore they are trivial when all the Massey products in $H^*(\mathcal{Z}_K)$ vanish. As $H^*(\mathcal{Z}_K) \cong \mathrm{Tor}_{k[v_1, \dots, v_n]}(k[K], k)$ [BP], an equality for $P(R)$ is obtained when the Stanley-Reisner ring $k[K]$ is Golod. This proves the following theorem.

Theorem 8.1. *For a simplicial complex K ,*

$$(7) \quad P(k[K]) \leq \frac{t(1+t)^n}{t - P(H^*(\mathcal{Z}_K))}.$$

Equality is obtained when $k[K]$ is Golod.

We proceed by describing a new class of Golod rings using topological methods.

Theorem 8.2. *Let K be a simplicial complex such that \mathcal{Z}_K is homotopy equivalent to a wedge of spheres, then $k[K]$ is a Golod ring.*

Proof. Since \mathcal{Z}_K is homotopy equivalent to a wedge of spheres, in the cohomology of \mathcal{Z}_K all cup products and higher Massey products are trivial. On the other hand, recall that Buchstaber and Panov [BP] proved that

$$H^*(\mathcal{Z}_K) \cong \mathrm{Tor}_{k[v_1, \dots, v_n]}(k[K], k).$$

Therefore in $\text{Tor}_{k[v_1, \dots, v_n]}(k[K], k)$ all Massey products are trivial. Now by definition, the ring $k[K]$ is Golod. \square

We finish by proving that the Poincaré series of a ring belonging to the class defined in Theorem 8.2 represents a rational function.

Corollary 8.3. *If $K \in \mathcal{F}_0$, then the Poincaré series of the ring $k[K]$ has the following form*

$$P(k[K]) = \frac{t(1+t)^n}{t - P(H^*(Z_K))}.$$

Proof. As $k[K]$ is a Golod ring, in (7) equality holds. \square

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SCHOOL OF MATHEMATICAL, UNIVERSITY OF MANCHESTER, OXFORD ROAD, MANCHESTER M13 9PL, UNITED KINGDOM

E-mail address: jelena@maths.abdn.ac.uk