NEW RATIONALITY PRINCIPLES IN PURE INDUCTIVE LOGIC

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Engineering and Physical Sciences

2015

Elizabeth Howarth
School of Mathematics

Contents

Word count $< 50,000$

The University of Manchester

Elizabeth Howarth
Doctor of Philosophy
New Rationality Principles in Pure Inductive Logic
July 2, 2015

We propose and investigate several new principles of rational reasoning within the framework of Pure Inductive Logic, PIL, where probability functions defined on the sentences of a first-order language are used to model an agent's beliefs. The *Elephant Principle* is concerned with how learning, modelled by conditioning, may be uniquely 'remembered'. The *Perspective Principle* requires that, from a given prior, conditioning on statistically similar experiences should result in similar assignments, and is found to be a necessary condition for Reichenbach's Axiom to hold. The *Abductive Inference Principle* and some variations are proposed as possible formulations of a restriction of C.S. Peirce's notion of *hypothesis* in the context of PIL, though characterization results obtained for these principles suggest that they may be too strong. The *Finite Values Property* holds when a probability function takes only finitely many values when restricted to sentences containing only constant symbols from some fixed finite set. This is shown to entail a certain systematic method of assigning probabilities in terms of possible worlds, and it is considered in this light as a possible principle of inductive reasoning. Classification results are given, stating which members of certain established families of probability functions satisfy each of these new principles.

Additionally, we define the *theory* of a principle $\mathcal{P}$ of PIL to be the set of those sentences which are assigned probability 1 by every probability function which satisfies $\mathcal{P}$. We investigate the theory of the established principle of Spectrum Exchangeability by finding separately the theories of heterogeneous and homogeneous functions. The theory of Spectrum Exchangeability is found to be equal to the theory of finite structures. The theory of Johnson's Sufficientness Postulate is also found. Consequently, we find that Spectrum Exchangeability, Johnson's Sufficientness Postulate and the Finite Values Property are all inconsistent with the principle of Super-Regularity: that any consistent sentence should be assigned non-zero probability.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

Acknowledgements

Grateful thanks to my supervisor Jeff Paris for his continued interest, guidance and encouragement throughout this project. Thanks also to other staff and students from the School of Mathematics at Manchester University, particularly to Malte Kließ, Tahel Ronel, Alena Vencovská and George Wilmers for helpful comments, and to my family for ongoing support.

This research project was funded by a Doctoral Training Studentship from the Engineering and Physical Sciences Research Council.

All results from chapter 2 appeared originally in [25], though the authors retain copyright.

All results from chapter 4, together with Proposition 55 and Corollary 58, appeared originally in [26], and are reproduced with permission.

Chapter 1

Introduction

Induction, the study of using evidence or observations to reason about uncertain hypotheses or events, is an important area in philosophy with a long and controversial history, and is intimately related with the study of probability. Developing an approach used by Keynes [31], Carnap [2] describes *Inductive Logic* as the theory of *degree of confirmation* between evidence and hypothesis, expressed as propositions or as sentences of a formal language. Later [5], he describes it more broadly as "a theory of logical probability providing rules for inductive thinking". In [7] Carnap identifies *Pure Inductive Logic*, PIL, as a distinct approach within Inductive Logic, whose aim is to separate the logical or rational considerations of inductive reasoning from any consideration of the context or interpretation of a particular inductive argument.

More recently, Paris & Vencovská have developed Carnap's conception of PIL as a branch of mathematical logic, as presented by Paris in [44] and Paris & Vencovská in [49]. According to them, PIL is concerned primarily with the mathematical formalization of

> "*assigning logical, as opposed to statistical, probabilities by attempting to formulate the underlying notions, such as symmetry, irrelevance, relevance, on which they appear to depend*".

Within PIL, reasoning is modelled by the choice of a single probability function [1], from

[1] The Dutch Book argument, originally due to de Finetti [12] and Ramsey [53], and developed by Kemeny [29] and Lehman [38], provides a justification of this approach in terms of betting behaviour.

all possible probability functions defined on the sentences of a first-order predicate language, so that an agent's 'belief' that a given sentence is true is represented by the value in the interval $[0, 1]$ assigned to that sentence by its chosen function. Any observations or other knowledge which the agent considers in assigning such probabilities are modelled as sentences of the language over which the conditional probability may be found. A crucial feature of Pure, as opposed to Applied, Inductive Logic is that the agent is assumed to have no interpretation of the language; so that its choice of probability function must be based on logical considerations alone.

The approach of PIL has been to consider certain 'principles of rationality', and to investigate how the adoption of these principles, singly and in combination, affects the choice of probability functions available to the agent. These principles are usually expressed in terms of desirable behaviour of a probability function used to model rational belief, for example to prescribe when certain sentences should be assigned equal probability, or when the probability assigned to one should not exceed that assigned to another, based solely on the logical form of the sentences.

This study, conducted under the supervision of Professor J.B. Paris, presents the results of several distinct but related investigations into certain newly proposed principles of PIL. Each of these new principles is intended to express some facet of rational reasoning which has not prevously received attention in the literature on induction, and some of which are based on other considerations than the usual candidates of symmetry, relevance and irrelevance. The aim of these investigations has been to discover how the adoption of such principles, in combination with other established principles, affects the choice of probability functions available, and what the consequences of adopting certain principles may be beyond what is explicitly stated therein. Such results may elucidate the different aspects of rationality which these principles attempt to express, and inform any judgement on how well these principles may be said to represent them.

We also present the results of an enquiry regarding an established, though relatively young, principle of PIL: Spectrum Exchangeability. This enquiry builds on existing results of Fagin [11], Landes, Nix, Paris, Rad & Vencovská [33], [35], [40], [42], [45],

[49], to gain a deeper understanding of what is entailed by adopting Spectrum Exchangeability by the identification of its *theory*: the set of sentences which must be accepted with certainty by any agent who adopts the principle.

Firstly, though, we set out in detail the framework which will be used.

1.1 Context and notation

This section sets down for reference the terminology and notation which will be used throughout the thesis, together with definitions of some established principles of inductive reasoning and of certain families of probability functions which will be considered, and some key results in the field to which we will need to refer.

Throughout, $\mathbb{R}$ will denote the set of real numbers and $\mathbb{N}$ the set of natural numbers, with $\mathbb{N}^+$ denoting the positive natural numbers and $\mathbb{N}_n$ the set $\{1, 2, \ldots, n\}$. For $x \in \mathbb{R}$, $[x]$ will denote the integer part of x. The symbol S_n will denote the set of all permutations of $\{1, 2, \ldots, n\}$.

The framework for this thesis, based on that considered by Paris & Vencovská in [49], consists of a first order language L containing finitely many relation symbols $R_1, R_2, \ldots, R_q$ of arities $r_1, r_2, \ldots, r_q$ respectively, and variables x_i and constant symbols a_i for $i \in \mathbb{N}^+$, with no equality nor any function symbols. Where convenient, alternative symbols are used for constants and variables, such as $b_1, \ldots, b_m$ to represent a sequence of m unspecified constant symbols $a_{i_1}, \ldots, a_{i_m}$, assumed to be distinct unless stated otherwise.

Let SL denote the set of first order sentences of L and $QFSL$ denote those sentences of SL which are quantifier free. Similarly, let $(QF)FL$ denote the (quantifier free) formulae of L. Where a sentence is denoted $\theta(b_1, \ldots, b_n)$, this expresses that θ mentions only constants from among $b_1, \ldots, b_n$, but not necessarily all (or any) of these. For $n \in \mathbb{N}$, define $SL^{(n)}$ to be the set of those sentences of L which mention only constant symbols from among $a_1, \ldots, a_n$. A sentence θ belongs to $SL^{(0)}$ if it does not mention any constant symbols, and $\theta \in SL^{(n)} \implies \theta \in SL^{(m)}$ for all $m \geq n$.

Let $\mathcal{T}L$ denote the set of structures for L with universe $\{a_1, a_2, a_3, \ldots\}$, where the symbol a_i is interpreted as the individual a_i, so that the constants a_i name all individuals in the universe.

The central question of PIL is how a supposedly rational agent inhabiting a structure M in $\mathcal{T}L$, but having no prior knowledge concerning which such structure, nor any interpretation of the language, should assign probabilities $w(\theta)$ to the sentences $\theta \in SL$. Probability functions defined on SL are used to model the agent's belief [2], so in these terms the essential question is to what extent the requirement of rationality limits the agent's choice of probability function.

A function $w : SL \to [0,1]$ is a *probability function* on SL just if it satisfies that for all $\theta, \phi, \exists x\, \psi(x) \in SL$:

(P1) If $\models \theta$ then $w(\theta) = 1$,

(P2) If $\models \neg(\theta \wedge \phi)$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$,

(P3) $$w(\exists x\, \psi(x)) = \lim_{m\to\infty} w(\bigvee_{i=1}^{m} \psi(a_i)), \tag{1.1}$$

where $\models$ is the logical consequence relation for L. Since any convex sum of functions satisfying P1-P3 will also satisfy these properties, any convex sum of probability functions is also a probability function.

Conditioning is used to model the process of the agent's learning, or imagining that it has learnt, that some sentence is true in M. For a probability function w on SL and any fixed $\phi \in SL$, the *conditional probability*, $w(\,\cdot \mid \phi)$, is defined to be a function satisfying

$$w(\theta \mid \phi) \cdot w(\phi) = w(\theta \wedge \phi)$$

for $\theta \in SL$. Therefore we take, for example

$$w(\theta \mid \phi) = w(\theta' \mid \phi')$$

[2] See for example [13], [29] for a justification of this approach.

to be equivalent to

$$w(\theta \wedge \phi)\, w(\phi') = w(\theta' \wedge \phi')\, w(\phi) \tag{1.2}$$

which holds even in case either or both of $w(\phi), w(\phi')$ is zero. Where $w(\phi) > 0$,

$$w(\theta \mid \phi) = \frac{w(\theta \wedge \phi)}{w(\phi)},$$

and $w(\,\cdot \mid \phi)$ is a probability function. [3]

A *state description* for $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ is a quantifier free sentence $\Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ of the form

$$\bigwedge_{k=1}^{q} \quad \bigwedge_{\vec{b} \in \{a_{i_1}, \ldots, a_{i_m}\}^{r_k}} \pm R_k(b_1, b_2, \ldots, b_{r_k})$$

where r_k is the arity of relation symbol R_k and the $\vec{b} = \langle b_1, b_2, \ldots, b_{r_k} \rangle$ range over all possible tuples from $\{a_{i_1}, \ldots, a_{i_m}\}^{r_k}$. Here $+R_k(\vec{b})$ stands for $R_k(\vec{b})$ while $-R_k(\vec{b})$ stands for $\neg R_k(\vec{b})$. Therefore, a state description $\Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ says all that can be said in L about how the constants $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ relate to each other (though, where L contains polyadic relations, it says nothing about how they relate to other constants). A formula of the same form

$$\bigwedge_{k=1}^{q} \quad \bigwedge_{\vec{y} \in \{x_{i_1}, \ldots, x_{i_m}\}^{r_k}} \pm R_k(y_1, y_2, \ldots, y_{r_k})$$

for distinct variables $x_{i_1}, \ldots, x_{i_m}$ is known as a *state formula* for $x_{i_1}, \ldots, x_{i_m}$. State descriptions which are logically equivalent are identified, unless stated otherwise, and similarly for logically equivalent state formulae. By convention, state descriptions for zero constants and state formulae for zero variables are taken to be equivalent to some fixed tautology, denoted $\top$, mentioning no constants. Upper case Greek letters will be used throughout to denote state descriptions or state formulae.

Where L is unary (consists of purely unary relations), a state description for a single constant a_i is a sentence of the form

$$\pm R_1(a_i) \wedge \pm R_2(a_i) \wedge \ldots \wedge \pm R_q(a_i), \tag{1.3}$$

[3] This is shown in [49, Proposition 4.1].

known as an *atom*[4] (a state formula of the same form (1.3) for a single variable is also known as an atom). There are 2^q such atoms, denoted $\alpha_1, \alpha_2, \ldots, \alpha_{2^q}$ according to some arbitrary fixed enumeration. Therefore, a state description for constants $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ may be expressed in the form

$$\bigwedge_{r=1}^{m} \alpha_{h_r}(a_{i_r}), \tag{1.4}$$

where $h_r \in \{1, 2, \ldots, 2^q\}$ for $r = 1, 2, \ldots, m$, and a similar form may be used for state formulae of a unary language.

A *restriction* of a state description $\Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ to a subset $\{a_{i_{s_1}}, \ldots, a_{i_{s_u}}\}$ of $\{a_{i_1}, \ldots, a_{i_m}\}$, i.e. the conjunction of those conjuncts of Θ which refer only to constants from among $\{a_{i_{s_1}}, \ldots, a_{i_{s_u}}\}$, will be denoted $\Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m})[a_{i_{s_1}}, \ldots, a_{i_{s_u}}]$, or just $\Theta[a_{i_{s_1}}, \ldots, a_{i_{s_u}}]$ if the tuple $\langle a_{i_1}, a_{i_2}, \ldots, a_{i_m}\rangle$ is clear from the context.

For example, if L consists of just a single binary relation symbol R_1 and $\Theta(a_1, a_2, a_3)$ is the conjunction of

$$\begin{array}{ccc} R_1(a_1, a_1) & R_1(a_1, a_2) & R_1(a_1, a_3) \\ \neg R_1(a_2, a_1) & R_1(a_2, a_2) & \neg R_1(a_2, a_3) \\ \neg R_1(a_3, a_1) & \neg R_1(a_3, a_2) & R_1(a_3, a_3) \end{array}$$

then $\Theta(a_1, a_2, a_3)[a_1, a_3]$ is the conjunction of

$$\begin{array}{cc} R_1(a_1, a_1) & R_1(a_1, a_3) \\ \neg R_1(a_3, a_1) & R_1(a_3, a_3). \end{array}$$

A state description $\Theta^+(a_{i_1}, \ldots, a_{i_m}, a_{i_{m+1}})$ *extends* $\Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ if $\Theta^+ \models \Theta$, that is if the restriction of Θ^+ to $a_{i_1}, \ldots, a_{i_m}$ is logically equivalent to Θ,

$$\Theta^+[a_{i_1}, \ldots, a_{i_m}] \equiv \Theta(a_{i_1}, a_{i_2}, \ldots, a_{i_m}).$$

The same notation is used for restrictions and extensions of state formulae.

[4]Our 'atoms' correspond to what Carnap et al. called 'molecular Q-predicates'.

For a fixed language L, any $n \geq m \geq 0$ and any state description $\Theta(a_{i_1}, \ldots, a_{i_m})$ of L, the number of state descriptions $\Phi(a_{i_1}, \ldots, a_{i_n})$ of L extending Θ (up to logical equivalence) depends only on m and n (and L, which we leave implicit). This number will be denoted $SD(m, n)$, while the total number of state descriptions in L for n constants is denoted $SD(n)$ $(= SD(0, n))$.

By the Disjunctive Normal Form Theorem, every $\phi(a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \in QFSL$ is logically equivalent to a disjunction of distinct (so necessarily pairwise disjoint) state descriptions, from which it follows that the probability of ϕ is the sum of the probabilities of these state descriptions. Furthermore, by Gaifman's Theorem [15], a probability function is completely determined on the whole of SL, not just on $QFSL$, by its values on state descriptions.

Suppose that a state description $\Theta(b_1, b_2, \ldots, b_m)$ is such that for some b_i, b_j

$$\Theta \models R(b_{k_1}, \ldots, b_{k_u}, b_i, b_{k_{u+2}}, \ldots, b_{k_r}) \leftrightarrow R(b_{k_1}, \ldots, b_{k_u}, b_j, b_{k_{u+2}}, \ldots, b_{k_r})$$

for any r-ary relation symbol from L, any $1 \leq u \leq r$ and not necessarily distinct $b_{k_1}, \ldots, b_{k_u}, b_{k_{u+2}}, \ldots, b_{k_r}$ from $\{b_1, b_2, \ldots, b_m\}$.[5] Then b_i is said to be *indistinguishable* from b_j according to Θ. This may be expressed using an equivalence relation

$$b_i \sim_\Theta b_j$$

where the equivalence classes of $\sim_\Theta$ partition $b_1, \ldots, b_m$ so that those in the same class are all indistinguishable from each other, but distinguishable from any member of another class, according to Θ. The *multiset* of the sizes of these equivalence classes is called the *spectrum* of Θ, denoted $\mathcal{S}(\Theta)$. The size of this multiset will be called the *spectrum length* and denoted $|\mathcal{S}(\Theta)|$. The spectrum length of Θ is therefore the number of equivalence classes of $\sim_\Theta$.

The set of spectra for m distinct constants will be denoted *Spec*(m). The spectrum consisting of m ones (corresponding to a state description where each of m constants is distinguishable from every other) will be denoted $\mathbf{1}_m$. The symbol $\emptyset$ will be used to

[5] Equivalently $\Theta \wedge b_i = b_j$ would be consistent if equality were added to the language.

denote the spectrum of a state description for zero constants (i.e. a tautology, by the above convention).

Given spectra $\tilde{m}, \tilde{n}$ and a state description Θ with spectrum $\tilde{m}$ it can be shown, see [33], [40], [42], [49], that the number of state descriptions with spectrum $\tilde{n}$ extending Θ depends only on $\tilde{m}, \tilde{n}$ and not on the particular choice of Θ. We denote this number by $\mathcal{N}(\tilde{m}, \tilde{n})$. The total number of state descriptions in L with spectrum $\tilde{n}$ is denoted $\mathcal{N}(\emptyset, \tilde{n})$.

1.2 Principles of PIL

The aim of PIL is to investigate how different purported requirements of rationality restrict an agent's choice of probability function. The usual approach is to propose principles of rational reasoning which such a function should satisfy, and investigate their consequences. Several such principles may be found in the literature and we state those which will feature in this thesis.

The following principle was proposed by Johnson [28] and adopted by Carnap (from [2] onwards, under the name Axiom of Symmetry), based on the symmetry between the constant symbols of L.

Constant Exchangeability, Ex.

For w a probability function on SL, $\theta(a_1, \ldots, a_m) \in SL$ and any permutation σ of $\mathbb{N}^+$,

$$w(\theta(a_{\sigma(1)}, \ldots, a_{\sigma(m)})) = w(\theta(a_1, \ldots, a_m)).$$

The justification for this as a principle of rationality is that, in the absence of any interpretation, there is complete symmetry between the constant symbols, and hence between $\theta(a_1, \ldots, a_m)$ and $\theta(a_{\sigma(1)}, \ldots, a_{\sigma(m)})$, so that it would be irrational to assign different probabilities to these two sentences. All probability functions considered in this thesis are assumed to satisfy Ex, unless stated otherwise.

Where L is unary, it follows from Ex that the probability assigned to a state description of the form given in (1.4) depends only on its *signature*: $\langle m_1, \ldots, m_{2^q} \rangle$ where $m_j =$

$|\{r \mid h_r = j\}|$, regardless of which constants instantiate which atoms. Therefore, the alternative notation

$$\bigwedge_{j=1}^{2^q} \alpha_j^{m_j}$$

may be used for state descriptions of a purely unary language where convenient.

The symmetry between the relation symbols of L (without any interpretation of the language) likewise gives rise to the following principle.

Predicate Exchangeability, Px

If R_i and R_j are distinct relation symbols of L with the same arity, then for $\theta \in SL$,

$$w(\theta) = w(\theta')$$

where θ' is the result of simultaneously replacing R_i by R_j and R_j by R_i throughout θ. (Repeated applications ensure that equal probability must be given to any two sentences where one is obtained from the other by some permutation of relation symbols of the same arities.)

Where L is unary the following principle, proposed by Carnap in [4] [6], is based on the symmetry between the atoms: [7]

Atom Exchangeability, Ax

A probability function w on SL satisfies Atom Exchangeability if, for any permutation τ of $\{1, 2, \ldots, 2^q\}$,

$$w\left(\bigwedge_{r=1}^{m} \alpha_{h_r}(a_{i_r})\right) = w\left(\bigwedge_{r=1}^{m} \alpha_{\tau(h_r)}(a_{i_r})\right).$$

This principle, which implies Px, may be justified by the observation that, with no interpretation of the language or other background knowledge, there is complete symmetry between the atoms, and therefore any two enumerations should be treated

[6] Carnap refers to symmetry 'with respect to the Q-predicates'.

[7] It is essentially similar to the *attribute symmetry* of [8].

equally.

The following principle, proposed by Nix and Paris in [42] as an extension of Ax to polyadic languages [8], is based on the symmetry between state descriptions of the same spectrum:

Spectrum Exchangeability, Sx

A probability function w on SL satisfies Spectrum Exchangeability if, for any state descriptions $\Theta(b_1, b_2, \ldots, b_m), \Phi(b_1, b_2, \ldots, b_m)$ such that $\mathcal{S}(\Theta) = \mathcal{S}(\Phi)$

$$w(\Theta) = w(\Phi).$$

It may be justified on the grounds that, with no interpretation of the language, any differences between state descriptions beyond their spectra are irrelevant, so there is no reason to think any one state description more probable than any other of the same spectrum.

Where a probability function w is assumed to satisfy Sx, we will use the notation $w(\tilde{m})$ to stand for $w(\Theta)$ where Θ is any state description such that $\mathcal{S}(\Theta) = \tilde{m}$.

The following principles are based on the idea that if zero probability is identified with impossibility, then any sentence which is consistent, and therefore theoretically possible, should receive non-zero probability. Regularity applies this argument to quantifier-free sentences, Super-Regularity extends it to all sentences of L.

Regularity, Reg

A probability function w on SL satisfies Regularity if $w(\phi) > 0$ for all consistent $\phi \in QFSL$.

Carnap adopts the principle of Regularity [9] in early work such as [2] and [3], with the justification that it is intuitively compelling. Shimony [54] and Kemeny [29] later

[8] If L is purely unary then Sx is equivalent to Ax.

[9] In fact, Carnap assumes only that state descriptions should be assigned non-zero probability, though Regularity follows from this assumption by the Disjunctive Normal Form Theorem.

develop a justification in terms of betting behaviour, which Carnap adopts in [5], [7]. Hintikka [24] adopts the stronger principle of:

Super-Regularity, SReg

A probability function w on SL satisfies Super-Regularity if $w(\theta) > 0$ for all consistent $\theta \in SL$.

These two properties have the practical advantage of ensuring that conditional probabilities are always well-defined for sentences from $QFSL$ and SL respectively.

The following principle was adopted by Carnap in [6], [8] under the name 'Axiom of Convergence' [10]. It refers to probability functions on unary languages, and is based on the idea that the probability assigned to an event should converge, eventually, with its observed frequency.

Reichenbach's Axiom, RA

For w a Regular probability function on a unary language

$$\lim_{n\to\infty}\left(w\left(\alpha_j \mid \bigwedge_{i=1}^{n} \alpha_{h_i}\right) - \frac{u_j\left(\bigwedge_{i=1}^{n} \alpha_{h_i}\right)}{n}\right) = 0 \tag{1.5}$$

where $u_j\left(\bigwedge_{i=1}^{n} \alpha_{h_i}\right) = |\{i \mid h_i = j\}|$.

The following principle, which applies to purely unary languages, was proposed by Johnson in [28]. It is based on the idea of irrelevance and plays an important role in Carnap's programme, from [4] onwards.

Johnson's Sufficientness Postulate, JSP

A probability function w on a unary language satisfies Johnson's Sufficientness Postulate if

$$w\left(\alpha_j(a_{n+1}) \;\middle|\; \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) \tag{1.6}$$

[10] It was suggested to Carnap, under the name 'Reichenbach's Axiom', by Hilary Putnam see [8, p120].

depends only on n *and* $m_j = |\{i \mid h_i = j\}|$.

This expresses the idea that, in assigning a probability to a particular outcome of an event, only the number of known instances of this outcome and the number of known instances of this event are relevant; all else is irrelevant and should be disregarded.

The following property refers to the ability to extend a probability function to a larger language, containing the one on which it is initially defined. It was proposed (for unary languages) by Carnap [4] and endorsed by Kemeny [30].

Language Invariance, Li

A probability function w *on* SL *satisfies Language Invariance if there is a family of probability functions* $w^{\mathcal{L}}$*, one on each (finite) language* $\mathcal{L}$*, satisfying Px (and Ex) and such that* $w^L = w$ *and whenever* $\mathcal{L} \subseteq \mathcal{L}'$*,*

$$w^{\mathcal{L}} = w^{\mathcal{L}'} \restriction S\mathcal{L}$$

(i.e. $w^{\mathcal{L}'}$ *restricted to* $S\mathcal{L}$*).*

Where only unary languages are considered, we refer to Unary Language Invariance, ULi. We say that 'w satisfies Language Invariance with $\mathcal{P}$', where $\mathcal{P}$ is some property, if the members $w^{\mathcal{L}}$ of this family also all satisfy the property $\mathcal{P}$.

Since any sentence θ of SL, is also a member of $S\mathcal{L}$ for any $\mathcal{L} \supseteq L$, the justification for Language Invariance is that it forces the agent to assign the same probability to a given sentence θ, regardless of which language is under consideration. It therefore permits the agent to reason simultaneously about sentences of all languages consisting of finitely many predicates.

1.3 Particular probability functions

We will frequently make reference to certain established families of probability functions, whose definitions are given below.

The V_M functions

For $M \in \mathcal{T}L$, the function $V_M : SL \to [0,1]$ defined by

$$V_M(\theta) = \begin{cases} 1, & M \models \theta \\ 0, & \text{otherwise} \end{cases} \tag{1.7}$$

is a probability function, see [49, chapter 3]. These functions do not satisfy Ex in general.

The $w_{\vec{c}}$ functions

We introduce an important family of probability functions on unary languages. Let L be a unary language with q predicates. Let

$$\mathbb{D}_{2^q} = \{\langle x_1, x_2, \dots, x_{2^q}\rangle \mid x_1, x_2, \dots, x_{2^q} \geq 0 \text{ and } \sum_{j=1}^{2^q} x_j = 1\}$$

and for $\vec{c} \in \mathbb{D}_{2^q}$ define

$$w_{\vec{c}}\left(\bigwedge_{r=1}^{m} \alpha_{h_r}(a_{i_r})\right) = \prod_{r=1}^{m} c_{h_r} = \prod_{j=1}^{2^q} c_j^{m_j} \tag{1.8}$$

where, as above, $m_j = |\{r \mid h_r = j\}|$. Then $w_{\vec{c}}$ extends to a probability function on SL[11], and clearly $w_{\vec{c}}$ satisfies Ex.

It is an important property of these functions, as noted by Hill et al. in [22], that where $\theta, \phi \in QFSL$ have no constant symbols in common,

$$w_{\vec{c}}(\theta \wedge \phi) = w_{\vec{c}}(\theta) \cdot w_{\vec{c}}(\phi). \tag{1.9}$$

The $v_{\vec{c}}$ functions

For $\vec{c} = \langle c_1, c_2, \dots, c_{2^q}\rangle \in \mathbb{D}_{2^q}$, the function $v_{\vec{c}}$ on a unary language L with q predicates is defined to be

$$v_{\vec{c}} = |\mathsf{S}_{2^q}|^{-1} \sum_{\sigma \in \mathsf{S}_{2^q}} w_{\langle c_{\sigma(1)}, c_{\sigma(2)}, \dots, c_{\sigma(2^q)}\rangle}, \tag{1.10}$$

so that $v_{\vec{c}}$ is the uniform mixture of the $w_{\langle c_{\sigma(1)}, c_{\sigma(2)}, \dots, c_{\sigma(2^q)}\rangle}$ as σ ranges over the set S_{2^q} of permutations of $\mathbb{N}_{2^q}$. It follows from this definition[12] that the $v_{\vec{c}}$ functions satisfy Ax.

[11] See, for example, [49, chapter 8].
[12] See [49, chapter 14].

Carnap's Continuum, c^L_λ

Where L is unary, the function c^L_λ is defined [13] for $0 < \lambda \leq \infty$ by

$$c^L_\lambda\left(\alpha_j \;\middle|\; \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right) = \frac{m_j + \lambda 2^{-q}}{n+\lambda} \tag{1.11}$$

where $m_j = \{i \mid h_i = j\}|$ and we identify $(2^{-q} \cdot \infty)/\infty$ with 2^{-q}.

c^L_0 is defined by

$$c^L_0\left(\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)\right) = \begin{cases} 2^{-q} & \text{if } h_1 = h_2 = \ldots = h_m, \\ 0 & \text{otherwise.} \end{cases} \tag{1.12}$$

c^L_0 therefore 'assumes' that all individuals are indistinguishable, i.e. all satisfy the same atom.

The Nix-Paris Continuum, w^δ_L

Where L is unary, the function w^δ_L [14] is defined for $0 \leq \delta \leq 1$ by

$$w^\delta_L = 2^{-q} \sum_{j=1}^{2^q} w_{\vec{e_j}(\delta)} \tag{1.13}$$

where $\vec{e_j}(\delta) = \langle \gamma, \ldots, \gamma, \gamma+\delta, \gamma, \ldots, \gamma\rangle \in \mathbb{D}_{2^q}$, with $\gamma+\delta$ in the jth position and, necessarily, $\gamma = 2^{-q}(1-\delta)$.

From this definition it follows that

$$w^\delta_L\left(\bigwedge_{i=1}^{m} \alpha_{h_i}(a_i)\right) = 2^{-q} \sum_{j=1}^{2^q} \gamma^{m-m_j}(\gamma+\delta)^{m_j} \tag{1.14}$$

where $m_j = |\{i \mid h_i = j\}|$. [15]

In fact the c^L_λ and w^δ_L agree at their end points, precisely $c^L_0 = w^1_L$ and $c^L_\infty = w^0_L$, but nowhere else. Both satisfy Ex and Ax, though in general they have rather different

[13] It is straightforward to show that this determines the value of c^L_λ on every state description, hence on every quantifier free sentence, and therefore on all of SL by Gaifman's Theorem [15], see [49, chapter 16] for details.

[14] These functions originally appeared, with a characterization, in [41].

[15] See [49, chapter 19] for details.

properties. [16]

The following two families of probability functions, the $u^{\bar{p},L}$ and $v^{\bar{p},L}$ functions for polyadic languages, form the building blocks of all probability functions which satisfy Spectrum Exchangeability.

The $u^{\bar{p},L}$ functions

Let

$$\mathbb{B} = \{\langle p_0, p_1, p_2, \ldots \rangle \mid p_1 \geq p_2 \geq \ldots \geq 0, \sum_{i=0}^{\infty} p_i = 1\}.$$

For a given state description $\Phi(a_{i_1}, \ldots, a_{i_m})$ and a given vector $\vec{c} \in \mathbb{N}^m$, Φ is said to be *consistent with* $\vec{c}$ if for $1 \leq j, k \leq m$, $c_j = c_k \neq 0 \implies a_{i_j} \sim_\Phi a_{i_k}$. The set of all state descriptions for $\vec{a}$ which are consistent with $\vec{c}$ is denoted $\mathcal{C}(\vec{c}, \vec{a})$. For $\bar{p} \in \mathbb{B}$, the probability function $u^{\bar{p},L}$ is defined on state descriptions $\Phi(a_{i_1}, \ldots, a_{i_m})$ by

$$u^{\bar{p},L}(\Phi(a_{i_1}, \ldots, a_{i_m})) = \sum_{\substack{\vec{c} \in \mathbb{N}^m \\ \Phi \in \mathcal{C}(\vec{c}, \vec{a})}} |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{i=1}^{m} p_{c_i}, \tag{1.15}$$

and this definition extends to a probability function on SL. [17]

The $v^{\bar{p},L}$ functions

For $t \in \mathbb{N}^+$, let

$$\mathbb{B}_t = \{\bar{p} \in \mathbb{B} \mid p_0 = 0, p_t > 0 = p_{t+1}\}.$$

For $\bar{p} \in \mathbb{B}_t$, the function $v^{\bar{p},L}$ is defined on state descriptions $\Phi(a_{i_1}, \ldots, a_{i_m})$ in terms of vectors $\vec{c} \in (\mathbb{N}_t)^m$ and a function $\mathcal{G}(\vec{c}, \Phi)$. For a fixed $\vec{c}$, if Φ is not consistent with $\vec{c}$, i.e. if for some $1 \leq j, k \leq m$, $c_j = c_k > 0$ but $a_{i_j} \not\sim_\Phi a_{i_k}$, then $\mathcal{G}(\vec{c}, \Phi)$ is zero. Otherwise let $c_{g_1}, \ldots, c_{g_r}$ be the first instance of each distinct colour in $\vec{c}$ and let $\Phi' = \Phi[a_{i_{g_1}}, \ldots, a_{i_{g_r}}]$. Then $\mathcal{G}(\vec{c}, \Phi)$ takes the value

$$\frac{\mathcal{N}(\mathcal{S}(\Phi'), \mathbf{1}_t)}{\mathcal{N}(\emptyset, \mathbf{1}_t)},$$

[16] For a comparison see [47].
[17] See [34], [35] or [49, chapter 29] for the details.

i.e. the number of $\mathbf{1}_t$ extensions of Φ' as a proportion of the total number of $\mathbf{1}_t$ state descriptions for L. The value of $v^{\bar{p},L}(\Phi(a_{i_1},\dots,a_{i_m}))$ is defined in these terms to be

$$v^{\bar{p},L}(\Phi(a_{i_1},\dots,a_{i_m})) = \sum_{\vec{c}\in(\mathbb{N}_t)^m} \mathcal{G}(\vec{c},\Phi)\prod_{i=1}^{m} p_{c_i}, \tag{1.16}$$

and this definition extends to a probability function on SL.[18]

1.4 Key results

The following result states some well-known properties of probability functions which follow from the definition (1.1). A proof can be found, for example, in [49].

Proposition 1. *Let w be a probability function on SL. Then for $\theta, \phi \in SL$,*

1. $w(\neg\theta) = 1 - w(\theta)$.
2. $\models \neg\theta \implies w(\theta) = 0$.
3. $\theta \models \phi \implies w(\theta) \le w(\phi)$.
4. $\theta \equiv \phi \implies w(\theta) = w(\phi)$.
5. $w(\theta \vee \phi) = w(\theta) + w(\phi) - w(\theta \wedge \phi)$.

These properties will be used frequently, and usually without explicit mention.

The following well-known result appears, for example, in [23], [49].

Proposition 2. *Where L is unary and $\theta \in SL$, by the Prenex and Disjunctive Normal Form Theorems and after rearrangement using logical equivalences, θ is logically equivalent to some sentence θ' of the form*

$$\bigvee_{k=1}^{l}\left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_{k_j}} x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{n} \alpha_{f_{k_i}}(a_i)\right),$$

where each $\vec{\epsilon_k} \in \{0,1\}^n$, $\exists^1$ stands for $\exists$, $\exists^0$ stands for $\neg\exists$, and the disjuncts are disjoint and satisfiable.

[18] See [34], [35] or [49, chapter 29] for the details.

We reproduce the following well-known result:

Proposition 3. *If $\phi \in SL$ and w is a probability function on SL such that $w(\phi) = 1$, then*

$$w(\theta \wedge \phi) = w(\theta)$$

for all $\theta \in SL$.

Proof. Let ϕ and w be as described and let $\theta \in SL$. Since

$$\theta \wedge \neg\phi \models \neg\phi$$

we have

$$w(\theta \wedge \neg\phi) \leq w(\neg\phi) = 0$$

so $w(\theta \wedge \neg\phi) = 0$, and

$$w(\theta) = w(\theta \wedge \phi) + w(\theta \wedge \neg\phi) = w(\theta \wedge \phi).$$

□

The following representation theorem of Paris & Vencovská [49] is a corollory of Gaifman's Theorem [15]. Here, $\mathcal{B}$ is the σ-algebra generated by the algebra of subsets $[\theta] = \{M \in \mathcal{T}L \mid M \models \theta\}$ of $\mathcal{T}L$, for each $\theta \in QFSL$.

Theorem 4. *If w is a probability function on SL, then for some countably additive measure*[19] *μ on the algebra $\mathcal{B}$ of subsets of $\mathcal{T}L$,*

$$w = \int_{\mathcal{T}L} V_M \ \ d\mu(M).$$

The following Representation Theorem by de Finetti [14] is used extensively when working with unary languages (recall that we only consider probability functions which satisfy Ex).

[19] All measures are taken to be normalized and countably additive.

Theorem 5. *Let L be unary and let w be a probability function on SL satisfying Ex. Then there is a measure μ on the Borel subsets*[20] *of $\mathbb{D}_{2^q}$ such that*

$$w\left(\bigwedge_{r=1}^{m} \alpha_{h_r}(a_{i_r})\right) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\left(\bigwedge_{r=1}^{m} \alpha_{h_r}(a_{i_r})\right) d\mu(\vec{x}),$$
$$= \int_{\mathbb{D}_{2^q}} \prod_{j=1}^{2^q} x_j^{m_j}\, d\mu(\vec{x}) \qquad (1.17)$$

where $m_j = |\{r \mid h_r = j\}|$.

Conversely, given a measure μ on the Borel subsets of $\mathbb{D}_{2^q}$ the function w defined by (1.17) extends (uniquely) to a probability function on SL satisfying Ex.

This theorem gives rise to some new terminology. The measure μ is known as the *de Finetti prior* of the function w, and furthermore when w additionally satisfies Ax, μ is invariant under permutations of the 2^q co-ordinates[21], so that for τ a permutation of $\{1, \dots, 2^q\}$ and $A \subseteq \mathbb{D}_{2^q}$

$$\mu(A) = \mu(\{\tau(\vec{x}) \mid \vec{x} \in A\})$$

where $\tau(\vec{x}) = \langle x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(2^q)} \rangle$.

For $\epsilon > 0$ and $\vec{c} \in \mathbb{D}_{2^q}$, the *$\epsilon$-neighbourhood of $\vec{c}$* is denoted by

$$B_\epsilon(\vec{c}) = \{\vec{x} \in \mathbb{D}_{2^q} \mid |\vec{x} - \vec{c}| < \epsilon\}.$$

For a given μ, any $\vec{x} \in \mathbb{D}_{2^q}$ such that $\mu(B_\epsilon(\vec{x})) > 0$ for all $\epsilon > 0$ is known as a *support point* of μ or, sometimes more conveniently, of the corresponding (unique) probabiilty function w. The set of all such points is called the *support* of μ (or of w).

The following result of Hill & Paris, relating to conditioning with probability functions on unary languages, is proved in [20].

Lemma 6. *Let $\vec{b} = \langle b_1, b_2, \dots, b_{2^q} \rangle \in \mathbb{D}_{2^q}$ be a support point of μ and $k_1, k_2, \dots, k_{2^q} \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{\int_{\mathbb{D}_{2^q}} \prod_{j=1}^{2^q} x_j^{[nb_j]+k_j} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{j=1}^{2^q} x_j^{[nb_j]} d\mu(\vec{x})} = \prod_{j=1}^{2^q} b_j^{k_j}.$$

[20] That is, the closure under complement and countable unions of the open subsets of the relativized topology on $\mathbb{D}_{2^q} \subseteq \mathbb{R}^{2^q}$. This ensures that the functions $\vec{x} \mapsto w_{\vec{x}}(\theta)$ are integrable with respect to μ for $\theta \in SL$.

[21] See [49, chapter 14].

The following corollary of Lemma 6 is proved in [49] and states that, if any point $\vec{c} \in \mathbb{D}_{2^q}$ is a member of the support of a probability function, then the function can eventually approximate $w_{\vec{c}}$ arbitrarily closely, given appropriate conditioning information.

Corollary 7. *Let w be a probability function on SL with de Finetti prior μ and let $\vec{c}$ be a support point of μ. Then there exist state descriptions $\Theta_m(a_1, \ldots, a_{s_m})$ such that for any $r_1, \ldots, r_{2^q} \in \mathbb{N}$,*

$$\lim_{m \to \infty} w\left(\bigwedge_{i=1}^{2^q} \alpha_i^{r_i} \mid \Theta_m(a_1, \ldots, a_{s_m})\right) = \prod_{i=1}^{2^q} c_i^{r_i}.$$

The following characterization, by Paris & Vencovská, of functions satisfying Li with Sx is proved in [49], building on previous results obtained jointly with Landes [35].

Theorem 8. *Let w^L be a probability function on SL satisfying Li with Sx. Then there is a measure μ on $\mathbb{B}$ such that*

$$w^L = \int_{\mathbb{B}} u^{\bar{p},L} \, d\mu(\bar{p}). \tag{1.18}$$

Conversely given such a measure μ, w^L defined by (1.18) *is a probability function on SL satisfying Li with Sx.*

Chapter 2

Principles of Remembering and Forgetting

[1] There have been several principles proposed in unary Inductive Logic which are intended to capture some aspect of the idea that the probabilities one assigns should be informed in some particular way by one's experiences. Examples include Carnap's Principle of Instantial Relevance [7], Reichenbach's Axiom (1.5) and Paris & Vencovská's Unary Principle of Induction [49]. The first and third of these express the notion that the more times one has seen something in the past, the more likely one is to see it in the future, while the second asserts more strongly that the probability one assigns to an event should shadow its observed frequency (whether or not this converges to a single value). Whichever of these or other formulations [2] is preferred, it is widely accepted that it is rational to alter the probabilities one assigns in light of acquired knowledge or observations.

We propose a related principle of a different sort: that a probability function should, after conditioning on different past observations, result in different predictions for future observations. This ensures that all learning is 'remembered' by being uniquely incorporated into the resulting assignment, and for this reason [3] we call it the *Elephant Principle*. It could be argued that such perfect recall is ideally rational, based on the idea that information is valuable and should never be discarded; that you cannot do

[1] The results from this chapter appeared originally in [25].
[2] See [49, Chapter 21] for other related principles.
[3] It is said that 'an elephant never forgets'!

better by knowing less.[4,5]

On the other hand, it would seem unreasonable if two sequences of observations which are essentially very similar could result in wildly different assignments. It seems rational to keep our adjustments proportionate somehow, and to expect that assignments formed by conditioning on sufficiently similar sequences of observations should converge in the long run. We therefore propose a 'counterbalance' to the Elephant Principle, intended to express this notion, which we call the *Perspective Principle*. In fact, we claim that the latter may also be considered desirable in its own right without reference to the former, and by the above arguments, both could be considered as principles of rationality.

The two main results of this chapter, Theorems 12 and 14, are characterization results giving conditions under which these principles hold. Since the proofs of these results rely on de Finetti's Representation Theorem for probability functions on unary languages satisfying Ex, we assume throughout this chapter that L is unary. We then apply these results to two families of probability functions, namely Carnap's well-known Continuum of Inductive Methods and the more recent Nix-Paris Continuum.

2.1 The Elephant Principle

The motivation for the Elephant Principle, defined below, is the idea that the probabilities assigned by a rational agent to future events should reflect its observations of past events. If this notion is taken to its extreme, the resulting principle is that *any* difference in observations should result in *some* difference in assignments. We formalize this idea using conditional probabilities, as follows, identifying (real or imagined) 'observations' with state descriptions.

Suppose that an agent, which had initially adopted a probability function w, makes an

[4]Good [19] gives an interesting justification of this.

[5]It may appear that this conflicts with those principles of Inductive Logic which prescribe certain sorts of information 'irrelevant', for example JSP (1.6). That Carnap's Continuum satisfies JSP was shown originally by Johnson in [28]; that it will be subsequently shown to satisfy EP as well shows that in this case the two principles relate to differing forms of 'information'.

observation Γ about individuals $a_1, \dots, a_g$ and consequently conditions on this evidence to form $w(\cdot \mid \Gamma(a_1, \dots, a_g))$. Because Γ is a state description (and L is unary), the agent is now in no doubt about the properties of $a_1, \dots, a_g$, so we are really only concerned with how the agent's updated probability function $w(\cdot \mid \Gamma)$ assigns probabilities to state descriptions involving constants from $a_{g+1}, a_{g+2}, a_{g+3}, \dots$. For this reason we define, for a given state description $\Gamma(a_1, \dots, a_g)$ and a probability function w on SL such that $w(\Gamma) > 0$, a function $w_{*\Gamma}$ on the state descriptions[6] $\Theta(a_1, \dots, a_n)$ of L by

$$w_{*\Gamma}(\Theta(a_1, \dots, a_n)) = w(\Theta(a_{g+1}, \dots, a_{g+n}) \mid \Gamma(a_1, \dots, a_g)).$$

Because of our standing assumption that w satisfies Ex, $w_{*\Gamma}$ also satisfies Ex.

We define the Elephant Principle to formalize the idea that $w_{*\Gamma}$ should uniquely 're-member' the information $\Gamma(a_1, \dots, a_g)$.

The Elephant Principle, EP

For $\Gamma = \bigwedge_{i=1}^{2^q} \alpha_i^{g_i}$ and $\Gamma' = \bigwedge_{i=1}^{2^q} \alpha_i^{h_i}$ state descriptions of L, a probability function w on SL satisfies EP if

$$w_{*\Gamma} = w_{*\Gamma'} \iff g_i = h_i \quad \textit{for } i = 1, 2, \dots, 2^q.$$

Thus EP ensures that $w_{*\Gamma} = w_{*\Gamma'}$ just if Γ and Γ' have the same signature, so that any acquired information is uniquely reflected in the agent's assignments regarding possible future observations (up to the order of the instantiating constants, which is irrelevant by Ex).

We work towards a characterization result, Theorem 12, for those probability functions on SL which satisfy Ax + EP, via a sequence of lemmas. Firstly, we introduce some notation.

For $S \subset \mathbb{N}_{2^q}$ let

$$N_S = \{\vec{x} \in \mathbb{D}_{2^q} \mid x_i = 0 \iff i \in S\},$$

[6]In fact, see [43], $w_{*\Gamma}$ extends to a probability function on SL and continues to satisfy the identity $w_{*\Gamma}(\theta(a_1, \dots, a_n)) = w(\theta(a_{g+1}, \dots, a_{g+n}) \mid \Gamma(a_1, \dots, a_g))$ for any sentence θ of L. This is not needed in what follows, however.

and note that these N_S partition $\mathbb{D}_{2^q}$. We shall use S' to denote $\mathbb{N}_{2^q} - S$. For $\vec{x} \in \mathbb{D}_{2^q}$, let $S_{\vec{x}}$ denote the unique S such that $\vec{x} \in N_S$, so $S_{\vec{x}} = \{i \in \mathbb{N}_{2^q} \mid x_i = 0\}$.

Let w be a probability function on SL with de Finetti prior μ, so $w = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\, d\mu(\vec{x})$. If w does not satisfy EP, there must exist $\Gamma(a_1, \dots, a_g) = \bigwedge_{i=1}^{2^q} \alpha_i^{g_i}$ and $\Gamma'(a_1, \dots, a_h) = \bigwedge_{i=1}^{2^q} \alpha_i^{h_i}$, with different signatures $\langle g_1, \dots, g_{2^q} \rangle \neq \langle h_1, \dots, h_{2^q} \rangle$, such that $w(\Gamma),\ w(\Gamma') > 0$ (otherwise EP holds trivially by our convention (1.2)) and by de Finetti's Theorem 5,

$$\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+n_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i}\, d\mu(\vec{x})} = \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+n_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i}\, d\mu(\vec{x})} \tag{2.1}$$

for any $n_1, n_2, \dots, n_{2^q} \in \mathbb{N}$.

Let G, H be the sets of indices of atoms mentioned in $\Gamma,\ \Gamma'$ respectively, so $G = \{i \in \mathbb{N}_{2^q} \mid g_i > 0\}$ and $H = \{i \in \mathbb{N}_{2^q} \mid h_i > 0\}$ and let G', H' be the complement in $\mathbb{N}_{2^q}$ of G, H respectively, so $G' = \{i \in \mathbb{N}_{2^q} \mid g_i = 0\}$ etc..

Lemma 9. *If w fails EP with $\Gamma = \bigwedge_{i \in G} \alpha_i^{g_i}, \quad \Gamma' = \bigwedge_{i \in H} \alpha_i^{h_i}$, then*

$$\mu\left(\bigcup_{S \subseteq G' \cap H'} N_S\right) > 0.$$

Proof. Suppose, on the contrary, that w fails EP with $\Gamma,\ \Gamma'$ as described and $\mu(N_S) = 0$ for each $S \subseteq G' \cap H'$. Then, since $\prod_{i \in S} 0^{g_i+n_i} = 0$ whenever $S \cap G \neq \emptyset$ and $0^{g_i+n_i} = 1$ whenever $g_i = n_i = 0$,

$$\begin{aligned}
&\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+n_i}\, d\mu(\vec{x}) \\
&\qquad = \sum_{S \subset \mathbb{N}_{2^q}} \int_{N_S} \prod_{i \in S \cap G} 0^{g_i+n_i} \prod_{i \in S \cap G'} 0^{0+n_i} \prod_{i \in S' \cap G} x_i^{g_i+n_i} \prod_{i \in S' \cap G'} x_i^{0+n_i}\, d\mu(\vec{x}) \\
&\qquad = \sum_{S \cap G = \emptyset} \int_{N_S} \prod_{i \in S \cap G'} 0^{0+n_i} \prod_{i \in S'} x_i^{g_i+n_i}\, d\mu(\vec{x}) \\
&\qquad = \sum_{\substack{S \cap G = \emptyset \\ S \cap H \neq \emptyset}} \int_{N_S} \prod_{i \in S \cap G'} 0^{0+n_i} \prod_{i \in S'} x_i^{g_i+n_i}\, d\mu(\vec{x}),
\end{aligned}$$

by our assumption that $\mu(N_S) = 0$ for all $S \subseteq G' \cap H'$. By a similar argument we obtain

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+n_i}\, d\mu(\vec{x}) = \sum_{\substack{S \cap H = \emptyset \\ S \cap G \neq \emptyset}} \int_{N_S} \prod_{i \in S \cap H'} 0^{0+n_i} \prod_{i \in S'} x_i^{h_i+n_i}\, d\mu(\vec{x}),$$

and so from (2.1) that

$$w(\Gamma')\left(\sum_{\substack{S\cap G=\emptyset\\ S\cap H\neq\emptyset}}\int_{N_S}\prod_{i\in S\cap G'}0^{0+n_i}\prod_{i\in S'}x_i^{g_i+n_i}d\mu(\vec{x})\right)$$
$$= w(\Gamma)\left(\sum_{\substack{S\cap H=\emptyset\\ S\cap G\neq\emptyset}}\int_{N_S}\prod_{i\in S\cap H'}0^{0+n_i}\prod_{i\in S'}x_i^{h_i+n_i}d\mu(\vec{x})\right). \quad (2.2)$$

Furthermore, it must be the case that

$$\mu\left(\bigcup_{\substack{S\cap H=\emptyset\\ S\cap G\neq\emptyset}}N_S\right)>0,$$

since otherwise

$$\begin{aligned} w(\Gamma') &= \sum_{S\subset\mathbb{N}_{2^q}}\int_{N_S}\prod_{i\in S\cap H}0^{h_i}\prod_{i\in S'\cap H}x_i^{h_i}\,d\mu(\vec{x})\\ &= \sum_{S\cap H=\emptyset}\int_{N_S}\prod_{i\in S'\cap H}x_i^{h_i}\,d\mu(\vec{x})\\ &= \sum_{\substack{S\cap H=\emptyset\\ S\cap G\neq\emptyset}}\int_{N_S}\prod_{i\in S'\cap H}x_i^{h_i}d\mu(\vec{x}) \;=\; 0\end{aligned}$$

(again by the assumption that $\mu(N_S)=0$ for all $S\subseteq G'\cap H'$), contradicting $w(\Gamma')>0$.

Therefore, letting $n_i>0$ for all $i\in H\cap G'$ and $n_i=0$ for all remaining $i\in\mathbb{N}_{2^q}$ gives a value of 0 on the left of (2.2) with a positive value on the right, contradicting (2.1). The result follows. □

Let $\mathcal{M}\subseteq\mathbb{D}_{2^q}$ be the set of support points of μ.

Lemma 10. *If w fails EP with $\Gamma=\bigwedge_{i\in G}\alpha_i^{g_i}$, $\Gamma'=\bigwedge_{i\in H}\alpha_i^{h_i}$, then for any $\vec{d}\in\mathcal{M}$ such that $S_{\vec{d}}\subseteq G'\cap H'$, and any $\vec{c}\in\mathcal{M}$*

$$\prod_{i=1}^{2^q}c_i^{g_i}d_i^{h_i}=\prod_{i=1}^{2^q}c_i^{h_i}d_i^{g_i}. \quad (2.3)$$

Proof. Suppose $\vec{d} = \langle d_1, d_2, \ldots, d_{2^q}\rangle \in \mathcal{M}$ is such that $S_{\vec{d}} \subseteq G' \cap H'$ and $\vec{c} = \langle c_1, c_2, \ldots, c_{2^q}\rangle \in \mathcal{M}$. Let $n \in \mathbb{N}$ be large, then letting n_i in (2.1) take values $[n\, c_i]$, $[n\, d_i]$ in turn, and dividing the first equation obtained by the second obtained gives

$$\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+[n\,c_i]} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+[n\,d_i]} d\mu(\vec{x})} = \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+[n\,c_i]} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+[n\,d_i]} d\mu(\vec{x})}. \tag{2.4}$$

Since $\vec{d} \in \mathcal{M}$, $\mu(B_\epsilon(\vec{d})) > 0$ for any $\epsilon > 0$. Let $0 < \epsilon < min\{d_i \mid d_i > 0\}$ and let $\vec{x} \in B_\epsilon(\vec{d})$. Then $S_{\vec{x}} \subseteq S_{\vec{d}}$, for otherwise there must exist some i such that $x_i = 0 < d_i$, giving $|\vec{x} - \vec{d}| \geq \sqrt{d_i^2} > \epsilon$, a contradiction. Therefore since $\vec{d} \in \mathcal{M}$, $\mu\left(\bigcup_{S \subseteq S_{\vec{d}}} N_S\right) > 0$, so for $T = \bigcup_{S \subseteq S_{\vec{d}}} N_S$,

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+[n\,d_i]} d\mu(\vec{x}) \geq \int_T \prod_{i \in S_{\vec{d}}} x_i^{g_i+0} \prod_{i \notin S_{\vec{d}}} x_i^{g_i+[n\,d_i]}\, d\mu(\vec{x}) > 0$$

since $d_i = g_i = 0$ for all $i \in S_{\vec{d}}$. Likewise $\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+[n\,d_i]} d\mu(\vec{x}) > 0$, so (2.4) is well-defined.

Dividing both sides of (2.4) by

$$\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[n\,d_i]}\, d\mu(\vec{x}) \cdot \int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[n\,c_i]}\, d\mu(\vec{x}),$$

which (by the above argument with all $g_i = 0$) is similarly well-defined, and rearranging gives

$$\begin{aligned} &\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+[n\,c_i]} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[n\,c_i]} d\mu(\vec{x})} \cdot \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+[n\,d_i]} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[n\,d_i]} d\mu(\vec{x})} \\ &\qquad = \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+[n\,d_i]} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[n\,d_i]} d\mu(\vec{x})} \cdot \frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+[n\,c_i]} d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{[n\,c_i]} d\mu(\vec{x})}. \end{aligned}$$

By Lemma 6, since $\vec{c}, \vec{d} \in \mathcal{M}$, taking limits as $n \to \infty$ then gives

$$\prod_{i=1}^{2^q} c_i^{g_i} d_i^{h_i} = \prod_{i=1}^{2^q} c_i^{h_i} d_i^{g_i}.$$

$\square$

Furthermore, whenever $S_{\vec{c}} \subseteq G' \cap H'$, both sides of (2.3) are positive, and it is equivalent to

$$\prod_{i=1}^{2^q} c_i^{k_i} = \prod_{i=1}^{2^q} d_i^{k_i}, \tag{2.5}$$

where $k_i = g_i - h_i$. (If $S_{\vec{c}} \not\subseteq G' \cap H'$ then both sides of (2.3) are zero).

Lemma 11. *If w satisfies Ax and fails EP with $\Gamma = \bigwedge_{i \in G} \alpha_i^{g_i}$, $\Gamma' = \bigwedge_{i \in H} \alpha_i^{h_i}$, then for any $S \subset \mathbb{N}_{2^q}$ such that $|S| \leq |G' \cap H'|$, there is some constant X_S such that*

$$\mu(\{\vec{x} \in N_S \mid \prod_{i \notin S} x_i = X_S\}) = \mu(N_S). \tag{2.6}$$

Proof. Let w, S be as described and assume that $\mu(N_S) > 0$, since otherwise (2.6) holds trivially. By the remark following Theorem 5 since w satisfies Ax, μ is invariant under permutations of the 2^q co-ordinates, so that for $\tau \in \mathsf{S}_{2^q}$ and A a Borel subset of $\mathbb{D}_{2^q}$

$$\mu(A) = \mu(\{\tau(\vec{x}) \mid \vec{x} \in A\})$$

where $\tau(\vec{x}) = \langle x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(2^q)} \rangle$.

If $|S| = 2^q - 1$, then N_S is a singleton and the result follows. Otherwise, since $|S| \leq |G' \cap H'|$, there must exist $T \subseteq G' \cap H'$ with $|T| = |S|$ and $\mu(N_T) = \mu(N_S) > 0$. Let $\vec{d} \in \mathcal{M} \cap N_T$. Let $r,\ s \in T'$ with $r \neq s$ and let $\sigma \in \mathsf{S}_{2^q}$ be the permutation which exchanges r and s and leaves all other values unchanged. Then $\sigma(\vec{d}) = \langle d_{\sigma(1)}, \ldots, d_{\sigma(2^q)} \rangle$ is also in $\mathcal{M} \cap N_T$ by the symmetry of μ. Since w does not satisfy EP, by (2.5)

$$d_r^{k_r} d_s^{k_s} \prod_{i \neq r,s} d_i^{k_i} = d_s^{k_r} d_r^{k_s} \prod_{i \neq r,s} d_i^{k_i},$$

and therefore

$$\left(\frac{d_r}{d_s}\right)^{k_r} = \left(\frac{d_r}{d_s}\right)^{k_s}$$

giving either $d_r = d_s$ or $k_r = k_s$. For each pair of co-ordinates in T', the permutation exchanging these while leaving all others unchanged may be used similarly to show that, for all $r,\ s \in T'$, either $d_r = d_s$ (so $d_i = 0$ for $i \in T$ and $d_i = |T'|^{-1}$ for $i \in T'$ is the sole support point of μ in N_T) or $k_r = k_s$ and hence for all $\vec{c},\ \vec{d} \in \mathcal{M} \cap N_T$

$$\prod_{i \notin T} c_i = \prod_{i \notin T} d_i.$$

In either case, (2.6) holds for N_T. Let $\tau \in \mathsf{S}_{2^q}$ be such that $\tau(i) \in T \iff i \in S$. Then by Ax

$$\vec{x} \in \mathcal{M} \cap N_S \implies \tau(\vec{x}) \in \mathcal{M} \cap N_T \implies \prod_{i \notin T} x_{\tau(i)} = X_T = \prod_{i \notin S} x_i.$$

Therefore, since $\mu(\mathcal{M} \cap N_S) = \mu(N_S)$,

$$\mu(\{\vec{x} \in N_S \mid \prod_{i \notin S} x_i = X_T\}) = \mu(N_S).$$

□

We are now in a position to give the following characterization theorem.

Theorem 12. *Suppose that w is a probability function satisfying Ax with de Finetti prior μ, and let $z = min\{\,|S| \mid \mu(N_S) > 0\}$. Then w fails EP just if there is some $X \in \mathbb{R}$ such that*

$$\mu(\{\vec{x} \in N_S \mid \prod_{i \notin S} x_i = X\}) = \mu(N_S),$$

for every $S \subset \mathbb{N}_{2^q}$ such that $|S| = z$.

In other words, if z is the size of the smallest $S \subset \mathbb{N}_{2^q}$ such that $\mu(N_S) > 0$, then w fails EP just if for every $S \subset \mathbb{N}_{2^q}$ of size z, all the measure in N_S is concentrated on those $\vec{x}$ for which the product of the positive co-ordinates, $\prod_{i \notin S} x_i$, equals some fixed X. It seems doubtful whether there is any worthwhile intuitive interpretation of this result, its use is to provide a necessary and sufficient criterion to aid the classification of which probability functions do and do not satisfy EP with Ax.

Proof. Suppose w, μ and z are as described. Suppose firstly that there is some $X \in \mathbb{R}$ such that $\mu(\{\vec{x} \in N_S \mid \prod_{i=1}^{2^q} x_i = X\}) = \mu(N_S)$ for every $S \subset \mathbb{N}_{2^q}$ such that $|S| = z$. Let $T \subset \mathbb{N}_{2^q}$ with $|T| = z$, so that $\mu(N_T) > 0$ while $\mu(N_S) = 0$ whenever $|S| < |T|$, and for all $\vec{d} \in \mathcal{M} \cap N_T$, $\prod_{i \notin T} d_i = X$. Let $\Gamma = \bigwedge_{i \notin T} \alpha_i^g$, $\Gamma' = \bigwedge_{i \notin T} \alpha_i^h$ for some $g, h \in \mathbb{N}$ with $g, h > 0$, $g \neq h$, and let $n_1, \ldots, n_{2^q} \in \mathbb{N}$. Since for every $S \subset T$, $\mu(N_S) = 0$ and for every $S \nsubseteq T$, each $\vec{x} \in N_S$ has some zero co-ordinate $x_i = 0$ with $i \notin T$, so that

$\prod_{i \notin T} x_i^g = 0$,

$$\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i+n_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{g_i}\, d\mu(\vec{x})} = \frac{\int_{N_T} \prod_{i \in T} 0^{0+n_i} \prod_{i \notin T} x_i^{g+n_i}\, d\mu(\vec{x})}{\int_{N_T} \prod_{i \in T} 0^0 \prod_{i \notin T} x_i^{g}\, d\mu(\vec{x})}$$
$$= \frac{X^g \int_{N_T} \prod_{i=1}^{2^q} x_i^{n_i}\, d\mu(\vec{x})}{X^g \int_{N_T} 1\, d\mu(\vec{x})}$$
$$= \frac{1}{\mu(N_T)} \int_{N_T} \prod_{i=1}^{2^q} x_i^{n_i}\, d\mu(\vec{x}).$$

Substituting h for g shows that

$$\frac{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i+n_i}\, d\mu(\vec{x})}{\int_{\mathbb{D}_{2^q}} \prod_{i=1}^{2^q} x_i^{h_i}\, d\mu(\vec{x})}$$

takes the same value, so w fails EP.

For the converse result, suppose w fails EP with $\Gamma = \bigwedge_{i \in G} \alpha_i^{g_i}$, $\Gamma' = \bigwedge_{i \in H} \alpha_i^{h_i}$. By Ax (and the associated symmetry of μ), $\mu(N_S) > 0$ for every $S \subset \mathbb{N}_{2^q}$ of size z, and by Lemma 9, $z \leq |G' \cap H'|$ so there is some such $S \subseteq G' \cap H'$. Therefore, by Lemma 11, the result follows. □

We now apply this theorem to two established families of probability functions.

Corollary 13.

- *Members of Carnap's Continuum, c_λ^L, satisfy EP for $0 < \lambda < \infty$, and fail to satisfy EP at the endpoints $\lambda \in \{0, \infty\}$.*
- *Members of the Nix-Paris continuum, w_L^δ, fail to satisfy EP for $0 \leq \delta \leq 1$.*

Proof. It is shown in [49] that, for $0 < \lambda < \infty$, c_λ^L has a representation of the form

$$c_\lambda^L = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}\, d\mu(\vec{x}),$$

where

$$d\mu(\vec{x}) = \kappa \prod_{i=1}^{2^q} x_i^{\lambda 2^{-q}-1}\, d\rho(\vec{x}),$$

ρ is Lebesgue measure and κ is a normalizing constant. Therefore, every point in $\mathbb{D}_{2^q}$ is a support point of c_λ^L for $0 < \lambda < \infty$. For $\vec{x} = \langle 2^{-q}, 2^{-q}, \ldots, 2^{-q} \rangle$ and $\vec{y} \in N_\emptyset$ with $\vec{y} \neq \vec{x}$, both $\vec{x}, \vec{y} \in \mathcal{M} \cap N_\emptyset$ but their co-ordinate products are not equal since

$\prod_{i=1}^{2^q} z_i$ has a strict maximum at $\vec{x}$ for $\vec{z} \in \mathbb{D}_{2^q}$. Therefore these functions satisfy EP by Theorem 12.

That EP fails to hold for the w_L^δ for $0 \leq \delta \leq 1$ (which includes $c_0^L = w_L^1$ and $c_\infty^L = w_L^0$) follows from Theorem 12, using the observation that in each case by (1.13), the support points of the de Finetti prior are all permutations of each other:

$$\mathcal{M} = \{\vec{e_1}(\delta), \vec{e_2}(\delta), \dots, \vec{e_{2^q}}(\delta)\} = \{\sigma(\vec{e_1}(\delta)) \mid \sigma \in \mathsf{S}_{2^q}\},$$

so all have the same co-ordinate product. □

It is well known that c_∞^L fails to learn from experience, so its failure to satisfy EP is unsurprising. That c_0^L fails EP is rather for the opposite reason, it 'assumes' that all the individuals will be the same as the first one observed. In consequence the corresponding $w_{*\Gamma}$ 'keeps no record' of the numbers of each atom instantiated by individuals so far observed, it has no need to since all possible observations are already determined. The failure of EP for the w_L^δ is also unsurprising given that these probability functions possess the property of *Recovery* discussed by Paris and Waterhouse in [50], whereby new observations can effectively 'cancel out' previous observations.

2.2 The Perspective Principle

As remarked above, the Perspective Principle was originally conceived as a counterbalance to EP, to ensure that where different observations lead to different probability assignments these differences are somehow 'proportional'. However, we claim that it may considered as a principle of rational reasoning in its own right, defined as follows.

The Perspective Principle, PP

A Regular probability function w satisfies PP if, for any state descriptions $\Theta(a_1, \dots, a_n)$, $\Phi(a_1, \dots, a_n)$, $\Psi(a_1, \dots, a_r)$ and any $\epsilon > 0$, there is some m such that for all state descriptions $\Xi(a_{n+1}, \dots, a_k)$ with $k \geq n + m$,

$$|w(\Psi(a_{k+1}, \dots, a_{k+r}) \mid \Xi \wedge \Theta) \;-\; w(\Psi(a_{k+1}, \dots, a_{k+r}) \mid \Xi \wedge \Phi)| \;<\; \epsilon. \tag{2.7}$$

The assumption that w is Regular, i.e. $w(\theta) > 0$ for all consistent $\theta \in QFSL$, is necessary to ensure that the expression (2.7) is well-defined.

In essence then, the Perspective Principle says that no matter what observations Θ, Φ we start with, subsequently receiving a sufficiently long stream of common observations Ξ will almost eradicate the significance of this initial difference, at least as far as the probability assigned to a state description Ψ involving just unseen individuals is concerned. In fact the standing assumption of Ex ensures that the order of observations is irrelevant, the important feature is that where two sequences of observations eventually contain so many matched pairs of outcomes compared to unmatched ones, the resulting assignments should, according to PP, become arbitrarily similar.

The argument for the rationality of this principle is based on the idea that predictions about future events should be continuous functions of past observations; in that agents who initially adopt the same probability function on the basis of no information should continue to assign similar probabilities if their subsequent observations are sufficiently similar. Put another way, it would seem unduly risky (and hence arguably irrational) to adopt a probability function on the basis of no knowledge which could be critically dependent for all time on the particular properties of a relatively small number of previously observed individuals.

The main result of this section shows that Reichenbach's Axiom (1.5) is a sufficient condition for PP to hold.

Theorem 14. *If w is a probability function satisfying (Reg and) RA then w satisfies PP.*

Proof. Let w be as described. If RA holds then it holds uniformly[7], so that for any $\nu > 0$ there is some $t \in \mathbb{N}$ such that for any sequence of atoms α_{g_i} for $i = 1, \dots, m$ with $m \geq t$,

$$\left| w\left(\alpha_j \mid \bigwedge_{i=1}^{m} \alpha_{g_i}\right) - \frac{u_j\left(\bigwedge_{i=1}^{m} \alpha_{g_i}\right)}{m} \right| < \nu. \tag{2.8}$$

[7] This was stated as a corollory of de Finetti's Theorem by Gaifman in [16], for a proof see [48] or [49, Chapter 15].

Let $n \in \mathbb{N}^+$ and let $\Theta(a_1, \ldots, a_n)$, $\Phi(a_1, \ldots, a_n)$ be arbitrary fixed state descriptions for $a_1, \ldots, a_n$. Let $r \in \mathbb{N}^+$ and $\Psi(a_1, \ldots, a_r) = \bigwedge_{i=1}^r \alpha_{s_i}$ be an arbitrary fixed state description for $a_1, \ldots, a_r$. Let $m \in \mathbb{N}$ and let $k = n+m$. Then for any $\Xi(a_{n+1}, \ldots, a_k)$

$$\begin{aligned}
&|w(\Psi(a_{k+1}, \ldots, a_{k+r}) \mid \Xi \wedge \Theta) \; - \; w(\Psi(a_{k+1}, \ldots, a_{k+r}) \mid \Xi)| \\
&\quad = \left| w\left(\bigwedge_{i=1}^{r} \alpha_{s_i} \;\middle|\; \Xi \wedge \Theta \right) \; - \; w\left(\bigwedge_{i=1}^{r} \alpha_{s_i} \;\middle|\; \Xi \right) \right| \\
&\quad = \left| \prod_{b=1}^{r} w\left(\alpha_{s_b} \;\middle|\; \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta \right) \; - \; \prod_{b=1}^{r} w\left(\alpha_{s_b} \;\middle|\; \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \right) \right|. \qquad (2.9)
\end{aligned}$$

For any fixed $b \in \{1, \ldots, r\}$,

$$\begin{aligned}
&\left| w\left(\alpha_{s_b} \;\middle|\; \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta \right) \; - \; w\left(\alpha_{s_b} \;\middle|\; \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \right) \right| \\
&\quad \le \left| w\left(\alpha_{s_b} \;\middle|\; \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta \right) \; - \; \frac{u_{s_b}\left(\bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta \right)}{k+b-1} \right| \\
&\qquad + \left| w\left(\alpha_{s_b} \;\middle|\; \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \right) - \frac{u_{s_b}\left(\bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \right)}{m+b-1} \right| \\
&\qquad\quad + \left| \frac{u_{s_b}\left(\bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta \right)}{k+b-1} \; - \; \frac{u_{s_b}\left(\bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \right)}{m+b-1} \right|. \qquad (2.10)
\end{aligned}$$

By (2.8) and since

$$u_{s_b}\left(\bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta \right) = u_{s_b}\left(\bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \right) + u_{s_b}(\Theta)$$

where $u_{s_b}(\Theta) \le n$, (2.10) is smaller than any given $\delta > 0$, provided that m is taken sufficiently large.

Let $P_b = \min\{w\left(\alpha_{s_b} \mid \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi\right),\ w\left(\alpha_{s_b} \mid \bigwedge_{i=1}^{b-1} \alpha_{s_i} \wedge \Xi \wedge \Theta\right)\} \le 1$. Then if δ is an upper bound for (2.10), the value of (2.9) is less than

$$\prod_{b=1}^{r}(P_b + \delta) - \prod_{b=1}^{r} P_b \le \delta r + \delta^2 \binom{r}{2} + \cdots + \delta^r.$$

Given any $\epsilon > 0$, δ may be chosen such that the above is less than $\epsilon/2$. The same argument may also be used with Φ in place of Θ to finally obtain

$$
\begin{aligned}
&|w(\Psi(a_{k+1},\ldots,a_{k+r}) \mid \Xi(a_{n+1},\ldots,a_k) \wedge \Theta(a_1,\ldots,a_n)) \\
&\qquad\qquad - \quad w(\Psi(a_{k+1},\ldots,a_{k+r}) \mid \Xi(a_{n+1},\ldots,a_k) \wedge \Phi(a_1,\ldots,a_n))| \ < \ \epsilon
\end{aligned}
$$

for any $\Xi(a_{n+1},\ldots,a_k)$ where k is sufficiently large. Therefore, w satisfies PP. □

The converse of Theorem 14 fails, a counter-example is c^L_∞.

It follows from (1.5) and (1.11) that for $0<\lambda<\infty$ the c^L_λ satisfy Reg and RA, which leads to the result (with $\lambda=\infty$ a trivial case) that

Corollary 15. *For $0<\lambda\le\infty$, c^L_λ satisfies PP.*

(Since c^L_0 does not satisfy Reg, it cannot satisfy PP.)

The previous result shows that w^0_L $(=c^L_\infty)$ satisfies PP (w^1_L can't, since it doesn't satisfy Reg.) We now consider whether the w^δ_L functions satisfy PP for the intermediate values of δ, and find on the contrary that

Proposition 16. *For $0<\delta<1$, w^δ_L fails to satisfy PP.*

Proof. Let $0<\delta<1$. From the definitions given above in (1.13) and (1.8), it follows that

$$
w^\delta_L\left(\bigwedge_{i=1}^m \alpha_{h_i}\right) = 2^{-q}\sum_{j=1}^{2^q}\gamma^{m-m_j}(\gamma+\delta)^{m_j} \tag{2.11}
$$

where $m_j = |\{i \mid h_i = j\}|$. Let $n\in\mathbb{N}^+$ and let $\Theta(a_1,\ldots,a_n) = \bigwedge_{i=1}^{2^q}\alpha_i^{t_i}$, and $\Phi(a_1,\ldots,a_n) = \bigwedge_{i=1}^{2^q}\alpha_i^{p_i}$ be state descriptions for $a_1,\ldots,a_n$. For any $m\in\mathbb{N}$, choose $h\ge m$ such that $h=2^q g$ for some $g\in\mathbb{N}$. Let $\Xi(a_{n+1},\ldots,a_{n+h}) = \bigwedge_{i=1}^{2^q}\alpha_i^g$. Let $r\in\mathbb{N}^+$ and let $\Psi(a_{n+h+1},\ldots,a_{n+h+r}) = \bigwedge_{i=1}^{2^q}\alpha_i^{r_i}$ be a state description for $a_{n+h+1},\ldots,a_{n+h+r}$. Then by (2.11),

$$
\begin{aligned}
&|w^\delta_L(\Psi \mid \Xi\wedge\Theta) - w^\delta_L(\Psi \mid \Xi\wedge\Phi)| \\
&\quad = \left|\frac{\sum_{i=1}^{2^q}\gamma^{r+h+n-(r_i+g+t_i)}(\gamma+\delta)^{r_i+g+t_i}}{\sum_{i=1}^{2^q}\gamma^{h+n-(g+t_i)}(\gamma+\delta)^{g+t_i}} - \frac{\sum_{i=1}^{2^q}\gamma^{r+h+n-(r_i+g+p_i)}(\gamma+\delta)^{r_i+g+p_i}}{\sum_{i=1}^{2^q}\gamma^{h+n-(g+p_i)}(\gamma+\delta)^{g+p_i}}\right| \\
&\quad = \left|\frac{\sum_{i=1}^{2^q}\gamma^{r+n-(r_i+t_i)}(\gamma+\delta)^{r_i+t_i}}{\sum_{i=1}^{2^q}\gamma^{n-t_i}(\gamma+\delta)^{t_i}} - \frac{\sum_{i=1}^{2^q}\gamma^{r+n-(r_i+p_i)}(\gamma+\delta)^{r_i+p_i}}{\sum_{i=1}^{2^q}\gamma^{n-p_i}(\gamma+\delta)^{p_i}}\right| \\
&\quad = |w^\delta_L(\Psi\mid\Theta) - w^\delta_L(\Psi\mid\Phi)|.
\end{aligned}
$$

Since $\delta > 0$, Θ, Φ and Ψ may be chosen such that this last value is greater than 0. Therefore, the value of $|w_L^\delta(\Psi \mid \Xi \wedge \Theta) - w_L^\delta(\Psi \mid \Xi \wedge \Phi)|$ is fixed, positive and independent of the value of h, which may be arbitrarily large, and w_L^δ fails PP. $\square$

We have produced partial characterization results for EP and PP in unary languages, and used these to show that, for $0 < \lambda < \infty$ the members c_λ^L of Carnap's Continuum satisfy both principles, while all members w_L^δ of the Nix-Paris Continuum for $0 < \delta \leq 1$ (including $c_0^L = w_L^1$), satisfy neither, and $c_\infty^L = w_L^0$ satisfies PP without EP. Therefore, if these principles are considered desirable in probability functions used to model rational belief, these results support the choice of the non-extreme (i.e. $0 < \lambda < \infty$) c_λ^L over the possible alternatives w_L^δ for such a model.

We have focused on EP and PP entirely within the context of Unary Inductive Logic in order to make use of de Finetti's Theorem, however these principles would seem to be just as rational (or not) for polyadic languages. One possible direction for future work would be to investigate whether the generalization of de Finetti's Theorem to polyadic languages given by Paris & Vencovská in [49] [8] could be applied in characterizing EP and PP in non-unary languages. Furthermore, it still remains to examine natural generalizations of EP and PP to all sentences rather than just state descriptions, and to find a necessary condition for PP; a plausible candidate seems to be that the support $\mathcal{M}$ is connected.

The ideas which underlie EP and PP are rather different in nature from the symmetry, relevance, irrelevance and analogy considerations which form the basis of most current rational principles in Pure Inductive Logic. Whether they have the same force as these stock notions is open to debate.

[8] This extends work done by Krauss in [32].

Chapter 3

Principles of Abductive Inference

The following argument is a mode of inference which Peirce [51] calls *hypothesis* (an example of what he calls more generally *abductive reasoning* [52]; a process of forming explanations to account for observations). We quote from [51, p.140]:

> " [*A population*] M *has, for example, the numerous marks* P', P'', P''', *etc.*
> [*A sample*] S *has the proportion* r *of the marks* P', P'', P''', *etc. :*
> *Hence, probably and approximately,* S *has an* r*-likeness to* M "

where the value r may take any value in $[0, 1]$, and an r*-likeness* is defined, imprecisely, to be the 'degree of resemblance' between the sample S and population M.

Taking 'marks' to mean some observable properties and fixing $r = 1$, the argument may be interpreted as follows.

> All members of some population M have the properties P', P'', P''', etc.
> All members of some sample S have all of the properties P', P'', P''', etc. :
> Hence, probably and approximately, S has a 1-likeness to M.

Peirce is clear that this last statement should be interpreted to mean that every member of S is a member of M. Thus, given some observed similarities between a sample and a population, and no other information (concerning properties *not* listed among P', P'', P''', etc.) we hypothesize that the members of the sample S all or almost all belong to the population M.[1]

[1] Peirce [51] was of the opinion that this argument (where r may vary) is *not* an instance of analogy, though where $r = 1$ it seems that it could be considered as such.

A probabilistic interpretation of the underlying intuitive reasoning is that, given that certain individuals have been found to possess certain distinguishing features of some population, the explanatory hypothesis that these individuals are in fact members of this population seems more likely than if the observation had not been made. In our framework, a similar idea may be expressed as follows: that given some observed similarity ψ between individuals in a sample

$$\bigwedge_{i=1}^{m} \psi(b_i)$$

(and no observed differences), it seems more likely that these individuals either all satisfy, or all fail to satisfy, some further property ϕ (consistent with membership of some hypothetical population)

$$\bigwedge_{i=1}^{m} \phi(b_i) \vee \bigwedge_{i=1}^{m} \neg\phi(b_i)$$

than if nothing were known about $b_1, \ldots, b_m$.

In this chapter we propose and investigate different formulations of Peirce's argument, where r is fixed to be 1, and investigate the consequences of adopting these as principles of rationality in PIL. In the first section we propose the *Abductive Inference Principle*, AIP, and proceed to give a characterization of those probability functions satisfying AIP with Spectrum Exchangeability, Sx (equivalent to Ax for unary languages). We use this to classify which members of the c_λ^L, w_L^δ and $u^{\bar{p},L}$ families of functions satisfy AIP, and then consider how the additional requirement of Language Invariance affects our classification. The results suggest that, in this setting, AIP may be rather unreasonable after all. The subsequent section presents some results concerning variations on AIP which allow for background information, involving extra constant symbols, to be considered. These too suggest that the formulations considered are rather demanding.

3.1 The Abductive Inference Principle

In order to simplify the problem, we take the sample size m to be just 2, and suppose that the observation formula $\psi(x)$ and the hypothetical similarity $\phi(x)$ are both quantifier-free. These limitations suggest the following formulation of Peirce's argument, which we will call the:

Abductive Inference Principle, AIP

For any $\phi(x), \psi(x) \in QFFL$

$$w(\phi(a_1) \leftrightarrow \phi(a_2)) \ \leq \ w(\phi(a_1) \leftrightarrow \phi(a_2) \mid \psi(a_1) \wedge \psi(a_2)). \tag{3.1}$$

By the convention above (1.2), we take this to be equivalent to

$$w(\phi(a_1) \leftrightarrow \phi(a_2)) \cdot w(\psi(a_1) \wedge \psi(a_2)) \ \leq \ w((\phi(a_1) \leftrightarrow \phi(a_2)) \wedge \psi(a_1) \wedge \psi(a_2)). \tag{3.2}$$

We begin by noting some simple cases where (3.2) holds.[2]

Lemma 17. *For $\phi(x), \psi(x) \in QFFL$ and w a probability function on SL,* (3.2) *holds*

1. *whenever $w(\phi(a_1)) \in \{0,1\}$ or $w(\psi(a_1)) \in \{0,1\}$;*
2. *whenever $w(\psi(a_1) \to \phi(a_1)) = 1$ or $w(\psi(a_1) \to \neg\phi(a_1)) = 1$;*
3. *when $q = 1$;*
4. *when w is such that*[3] $w\left(\bigvee_{\mathcal{S}(\Theta)=\{2\}} \Theta(a_1, a_2)\right) = 1$.

Proof. 1. If $w(\phi(a_1)) \in \{0,1\}$ (and by Ex $w(\phi(a_2)) = w(\phi(a_1))$) then by Proposition 3, $w(\phi(a_1) \leftrightarrow \phi(a_2)) = 1$, and therefore both sides of (3.2) are equal to $w(\psi(a_1) \wedge \psi(a_2))$. Similarly, if $w(\psi(a_1)) = w(\psi(a_2)) = 1$ then both sides of (3.2) are equal to $w(\phi(a_1) \leftrightarrow \phi(a_2))$, while if $w(\psi(a_1)) = w(\psi(a_2)) = 0$, both sides of (3.2) are zero.

2. If $w(\psi(a_1) \to \phi(a_1)) = 1$ or $w(\psi(a_1) \to \neg\phi(a_1)) = 1$ then

$$w(\phi(a_1) \leftrightarrow \phi(a_2) \wedge \psi(a_1) \wedge \psi(a_2)) = w(\psi(a_1) \wedge \psi(a_2)),$$

so (3.2) holds since $w(\phi(a_1) \leftrightarrow \phi(a_2)) \leq 1$.

3. If $q = 1$ then the above cases include all possibilities.

[2] Note that we have dropped the assumption from the previous chapter that L is unary.
[3] Recall the notation relating to spectra from §1.1.

4. Let $\phi(x) \in QFFL$. Since any state description $\Theta(a_1, a_2)$ which logically implies $\phi(a_1) \wedge \neg\phi(a_2)$ must have spectrum $\{1,1\}$, if $w\left(\bigvee_{\mathcal{S}(\Theta)=\{2\}} \Theta(a_1,a_2)\right) = 1$, then $w(\phi(a_1) \wedge \neg\phi(a_2)) = 0$ and similarly $w(\neg\phi(a_1) \wedge \phi(a_2)) = 0$. Therefore, $w(\phi(a_1) \leftrightarrow \phi(a_2)) = 1$, so by Lemma 3 both sides of (3.2) are equal to $w(\psi(a_1) \wedge \psi(a_2))$.

□

It follows from the previous result that all probability functions satisfy AIP on languages where $q = 1$, so assume for the rest of this chapter that $q \geq 2$. Let

$$r = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_q} \tag{3.3}$$

(where r_i is the arity of R_i). Then, using the notation from §1.1,

$$SD(1) = 2^q, \qquad SD(2) = 2^r, \quad \text{and} \quad SD(1,2) = 2^{r-q}. \tag{3.4}$$

Note that where w satisfies Sx we have

$$\mathcal{N}(\{2\})w(\{2\}) + \mathcal{N}(\{1,1\})w(\{1,1\}) = 1,$$

by (3.4) then

$$2^q w(\{2\}) + (2^r - 2^q)w(\{1,1\}) = 1,$$

and so

$$w(\{2\}) = 2^{-q} - (2^{r-q} - 1)w(\{1,1\}). \tag{3.5}$$

For $\phi(z),\ \psi(z) \in QFFL$, let $|\phi|$ denote the number of state formulae for z (up to logical equivalence) which logically imply $\phi(z)$:

$$|\phi| = |\{\Theta(z) \mid \Theta(z) \models \phi(z)\}|, \tag{3.6}$$

and similarly for $|\psi|,\ |\phi \wedge \psi|$ etc..

Theorem 18. *Let $q \geq 2$. If w is a probability function on SL satisfying Sx, then w satisfies AIP just if*

$$w(\{1,1\}) \leq 2^{q-r}(2^q - 1)^{-2}.$$

Proof. Suppose $q \geq 2$ and let w be a probability function on SL satisfying Sx, let X denote $w(\{2\})$ and Y denote $w(\{1,1\})$. Let $\phi(z),\ \psi(z) \in QFFL$ and let $n = |\phi|,\ k = |\psi|$

and $m = |\phi \wedge \psi|$.

For each state description $\Theta(a_1)$ which logically implies $\psi(a_1)$, the proportion $k/SD(1)$ of its extensions of the form $\Theta^+(a_1, a_2)$ logically imply $\psi(a_1) \wedge \psi(a_2)$. Of these just 1 has spectrum $\{2\}$ while all others have spectrum $\{1, 1\}$, so we obtain using (3.4) that

$$\begin{aligned} w(\psi(a_1) \wedge \psi(a_2)) &= \sum_{\Theta \models \psi(a_1) \wedge \psi(a_2)} w(\Theta(a_1, a_2)) \\ &= kX + k\left(\frac{k}{2^q} SD(1,2) - 1\right) Y \\ &= kX + k\left(2^{r-2q}k - 1\right) Y. \end{aligned} \tag{3.7}$$

Similar arguments concerning those state descriptions which logically imply $\phi(a_1)$, $\neg\phi(a_1)$, and $\psi(a_1) \wedge \phi(a_1)$ yield

$$\begin{aligned} &w(\phi(a_1) \leftrightarrow \phi(a_2)) \\ &\quad = 2^q X + \left(n\left(\frac{n}{2^q} SD(1,2) - 1\right) + (2^q - n)\left(\frac{2^q - n}{2^q} SD(1,2) - 1\right)\right) Y \\ &\quad = 2^q X + \left(n\left(2^{r-2q}n - 1\right) + (2^q - n)\left(2^{r-2q}(2^q - n) - 1\right)\right) Y, \end{aligned}$$

and

$$\begin{aligned} &w((\phi(a_1) \leftrightarrow \phi(a_2)) \wedge \psi(a_1) \wedge \psi(a_2)) \\ &\quad = kX + \left(m\left(\frac{m}{2^q} SD(1,2) - 1\right) + (k - m)\left(\frac{k - m}{2^q} SD(1,2) - 1\right)\right) Y \\ &\quad = kX + \left(m\left(2^{r-2q}m - 1\right) + (k - m)\left(2^{r-2q}(k - m) - 1\right)\right) Y. \end{aligned}$$

Condition (3.2)

$$w(\phi(a_1) \leftrightarrow \phi(a_2)) \cdot w(\psi(a_1) \wedge \psi(a_2)) \;\leq\; w((\phi(a_1) \leftrightarrow \phi(a_2)) \wedge \psi(a_1) \wedge \psi(a_2))$$

(repeated for convenience) becomes

$$\begin{aligned} &(2^q X + (n(2^{r-2q}n - 1) + (2^q - n)(2^{r-2q}(2^q - n) - 1))Y) \cdot (kX + k(2^{r-2q}k - 1)Y) \\ &\qquad \leq kX + (m(2^{r-2q}m - 1) + (k - m)(2^{r-2q}(k - m) - 1))Y. \end{aligned} \tag{3.8}$$

Since w satisfies Sx, by (3.5) we can substitute

$$X = 2^{-q} - (2^{r-q} - 1)Y$$

for X in (3.8) to give

$$\begin{aligned}&\big(\, 2^q(2^{-q} - (2^{r-q}-1)Y) \;+\; (n(2^{r-2q}n-1) + (2^q-n)(2^{r-2q}(2^q-n)-1))Y \,\big)\\ &\qquad\qquad\qquad \cdot (k(2^{-q} \;-\; (2^{r-q}-1)Y) \;+\; k(2^{r-2q}k-1)Y)\\ &\qquad \le k(2^{-q} - (2^{r-q}-1)Y) \;+\; (m(2^{r-2q}m-1) + (k-m)(2^{r-2q}(k-m)-1))Y.\end{aligned}$$

Collecting terms then gives

$$\begin{aligned}&(1 \;+\; (n(2^{r-2q}n-1) + (2^q-n)(2^{r-2q}(2^q-n)-1) - 2^q(2^{r-q}-1))Y)\\ &\qquad\qquad\qquad \cdot (k2^{-q} \;+\; (k(2^{r-2q}k-1) - k(2^{r-q}-1))Y)\\ &\qquad \le k2^{-q} \;+\; (m(2^{r-2q}m-1) + (k-m)(2^{r-2q}(k-m)-1) - k(2^{r-q}-1))Y,\end{aligned}$$

equivalently

$$\begin{aligned}&(1 - 2^{r-2q+1}n(2^q-n)Y)\cdot(k2^{-q} - 2^{r-2q}k(2^q-k)Y)\\ &\qquad\qquad\qquad\qquad \le k2^{-q} + 2^{r-2q}(m^2 + (k-m)^2 - k2^q)Y,\end{aligned}$$

and multiplying out then gives

$$\begin{aligned}&k2^{-q} \;-\; 2^{r-2q}k(2^q - k + 2^{1-q}n(2^q-n))Y \;+\; 2^{2r-4q+1}kn(2^q-k)(2^q-n)Y^2\\ &\qquad\qquad\qquad\qquad \le k2^{-q} \;+\; 2^{r-2q}(m^2 + (k-m)^2 - k2^q)Y.\end{aligned}$$

If $Y = 0$ then this is clearly satisfied, otherwise we can eliminate the constant term $k2^{-q}$, then divide through by $2^{r-2q}Y$ and rearrange to obtain

$$\begin{aligned}Y &\le \frac{m^2 \;+\; (k-m)^2 \;-\; k2^q \;+\; k(2^q - k + 2n(2^q-n)2^{-q})}{2^{r-2q+1}kn(2^q-k)(2^q-n)}\\ &= \frac{2(m(m-k) \;+\; kn(2^q-n)2^{-q})}{2^{r-2q+1}kn(2^q-k)(2^q-n)}\\ &= \frac{2^{-q}kn(2^q-n) \;-\; m(k-m)}{2^{r-2q}kn(2^q-k)(2^q-n)}.\end{aligned} \tag{3.9}$$

If this bound on the value of $Y = w(\{1,1\})$ holds for all possible values of n, k and m, then w satisfies AIP and conversely. We proceed to identify the values of n, k and m which give the smallest bound, and to calculate the bound in this case.

Suppose that n and k are fixed. Then the bound (3.9) for Y takes its smallest value where $m(k-m)$ is as large as possible. Given that $0 \le m \le \min\{k, n\}$, this occurs where m is as close as possible to $k/2$: so where $k = 2s$ we need only consider $m = s$, while where $k = 2s + 1$ letting either $m = s$ or $m = s + 1$ gives the same least value for (3.9).

Now, suppose that n is fixed. We will treat $n = 2^{q-1}$ as a separate case, but otherwise note that we need only consider $1 \leq n \leq 2^{q-1} - 1$, since we obtain the same values for $n = 2^q - j$ as for $n = j$ (if we replace ϕ by its negation). Given the conclusion above, we compare the value of (3.9) where $k = 2s,\ m = s$ with its value where $k = 2s + 2$, $m = s + 1$ for some $1 \leq s \leq n - 1$, and find that

$$\frac{2^{-q}2sn(2^q - n) - s^2}{2^{r-2q}2sn(2^q - 2s)(2^q - n)} > \frac{2^{-q}(2s+2)n(2^q - n) - (s+1)^2}{2^{r-2q}(2s+2)n(2^q - 2s - 2)(2^q - n)}, \tag{3.10}$$

since this expression may be simplified and rearranged as follows:

$$\begin{aligned}
&(2^{-q}2sn(2^q - n) - s^2)(2s+2)(2^q - 2s - 2) \\
&\qquad\qquad > (2^{-q}(2s+2)n(2^q - n) - (s+1)^2)2s(2^q - 2s) \\[1em]
\iff\ & 2^{1-q}n(2^q - n)(2s+2)(2^q - 2s - 2) - s(2s+2)(2^q - 2s - 2) \\
&\qquad\qquad > 2^{1-q}(2s+2)n(2^q - n)(2^q - 2s) - 2(s+1)^2(2^q - 2s) \\[1em]
\iff\ & 2(s+1)^2(2^q - 2s) - 2s(s+1)(2^q - 2s) + 4s(s+1) \\
&\qquad\qquad > 2^{2-q}(2s+2)n(2^q - n)
\end{aligned}$$

$$\begin{aligned}
\iff\ & 2(2^q - 2s)(s+1) + 4s(s+1) \ > \ 2^{3-q}(s+1)n(2^q - n) \\
\iff\ & 2^{q+1} \ > \ 2^{3-q}n(2^q - n) \\
\iff\ & 2^{2q-2} \ > \ n(2^q - n).
\end{aligned}$$

This holds, since $1 \leq n \leq 2^{q-1} - 1$, so that

$$n(2^q - n) \leq (2^{q-1} - 1)(2^{q-1} + 1) = 2^{2q-2} - 1. \tag{3.11}$$

Therefore (3.10) holds, meaning that, of the values of the bound (3.9) considered so far, the smallest occurs when $k = 2n, m = n$.

Still supposing that $1 \le n \le 2^{q-1} - 1$ is fixed, a similar comparison of the value of (3.9) where $k = 2s - 1,\ m = s$ with its value where $k = 2s + 1,\ m = s$ for some $1 \le s \le n$ yields that

$$\frac{2^{-q}(2s-1)n(2^q-n) - s(s-1)}{2^{r-2q}(2s-1)n(2^q-2s+1)(2^q-n)} > \frac{2^{-q}(2s+1)n(2^q-n) - s(s+1)}{2^{r-2q}(2s+1)n(2^q-2s-1)(2^q-n)}, \qquad (3.12)$$

since this expression simplifies to

$$\frac{2^q s(2^q s - 1)}{4s^2 - 1} > n(2^q - n).$$

By (3.11), the right hand side is bounded above by $2^{2q-2} - 1$, while the left hand side is strictly larger than this, since

$$\begin{aligned} &\frac{2^q s(2^q s - 1)}{4s^2 - 1} > 2^{2q-2} - 1 \\ \iff &2^{2q}s^2 - 2^q s > 2^{2q}s^2 - 4s^2 - 2^{2q-2} + 1 \\ \iff &4s^2 - 2^q s + 2^{2q-2} - 1 > 0. \end{aligned}$$

This quadratic expression in s has least value $3 \cdot 2^{2q-4} - 1$ (at $s = 2^{q-3}$), which is indeed positive for $q \ge 2$. Therefore, of the values of the bound (3.9) so far considered, the smallest occurs either where $k = 2n,\ m = n$ or where $k = 2n + 1,\ m = n$.

Now, still supposing that $1 \le n \le 2^{q-1} - 1$ is fixed, we consider values of k between $2n$ and $2^q - 1$, while keeping $m = n$ (as close as possible to $k/2$). Let $2n \le s \le 2^q - 2$. Then we compare the bound (3.9) for $k = s$ with that for $k = s + 1$, where $m = n$ in each case, and find that

$$\frac{2^{-q}sn(2^q-n) - n(s-n)}{2^{r-2q}sn(2^q-s)(2^q-n)} > \frac{2^{-q}(s+1)n(2^q-n) - n(s+1-n)}{2^{r-2q}(s+1)n(2^q-s-1)(2^q-n)} \qquad (3.13)$$

by the following rearrangement:

$$\begin{aligned} &(2^{-q}s(2^q-n) - (s-n))(s+1)(2^q-s-1) > (2^{-q}(s+1)(2^q-n) - (s+1-n))s(2^q-s) \\ \iff &(n - 2^{-q}sn)(s+1)(2^q-s-1) > (n - 2^{-q}n(s+1))s(2^q-s) \\ \iff &(1 - 2^{-q}s)(s+1)(2^q-s-1) > (1 - 2^{-q}(s+1))s(2^q-s) \\ \iff &(s+1)(2^q - s - 1 - 2^{-q}s(2^q-s) + 2^{-q}s) > s(2^q-s) - 2^{-q}(s+1)s(2^q-s) \\ \iff &(s+1)(2^q - s - 1 + 2^{-q}s) > s(2^q-s) \\ \iff &2^q - 2s - 1 + 2^{-q}s(s+1) > 0 \\ \iff &2^q + 2^{-q}s(s+1) > 2s + 1. \end{aligned}$$

Now, substituting $2^q - j$ for s (where $2 \leq j \leq 2^q - 2n$) gives that the above holds just if

$$\begin{aligned}
&2^q + 2^{-q}(2^q - j)(2^q - (j-1)) > 2(2^q - j) + 1 \\
\iff\; &2^q + 2^q - j - (j-1) + 2^{-q}j(j-1) > 2^{q+1} - 2j + 1 \\
\iff\; &2^{-q}j(j-1) > 0.
\end{aligned}$$

This holds for all possible values of j, so (3.13) holds. Taken together with (3.10) and (3.12), this tells us that for a fixed value of n, the smallest value of the bound (3.9) is obtained where $k = 2^q - 1$ and $m = n$.

We now use these values for k and m to compare the bound (3.9) with that obtained where $n+1$ is substituted for n in (3.9), where $1 \leq n \leq 2^{q-1} - 2$, and find that

$$\frac{2^{-q}(2^q-1)n(2^q-n) - n(2^q-1-n)}{2^{r-2q}(2^q-1)n(2^q-n)} < \frac{2^{-q}(2^q-1)(n+1)(2^q-n-1) - (n+1)(2^q-1-n-1)}{2^{r-2q}(2^q-1)(n+1)(2^q-n-1)}$$

by the following rearrangement:

$$\begin{aligned}
&(2^{-q}(2^q-1)(2^q-n) - (2^q-1-n))(n+1)(2^q-n-1) \\
&\qquad < (2^{-q}(n+1)(2^q-1)(2^q-n-1) - (n+1)(2^q-n-2))(2^q-n) \\
\iff\; &2^{-q}(2^q-1)(2^q-n)(n+1)(2^q-n-1) - (2^q-1-n)(n+1)(2^q-n-1) \\
&\qquad < 2^{-q}(n+1)(2^q-1)(2^q-n-1)(2^q-n) - (n+1)(2^q-n-2)(2^q-n) \\
\iff\; &(n+1)(2^q-n-2)(2^q-n) < (2^q-1-n)(n+1)(2^q-n-1) \\
\iff\; &(2^q-n)^2 - 2(2^q-n) < (2^q-n)^2 - 2(2^q-n) + 1.
\end{aligned}$$

It follows from this and previous remarks that for any $1 \leq n \leq 2^{q-1} - 1$ and any possible values of k and m, the smallest bound (3.9) is obtained where $n = 1$, $k = 2^q - 1$ and $m = 1$. The bound in this case is

$$w(\{1,1\}) = Y \leq 2^{q-r}(2^q - 1)^{-2}. \tag{3.14}$$

In the case where $n = 2^{q-1}$, we can apply the same arguments as used above for even and odd values of k up to $k = 2n-1$ to show that of these, either $k = 2n-2$ or $k = 2n-1$ yields the smallest bound. It then remains to compare these, and it is straightforward to show that the latter case provides the smaller bound of $(2^{r-q}(2^q-1))^{-1}$, though this is no smaller than that found in (3.14).

It has been shown, then, that for a probability function w which satisfies Sx, the 'toughest test' required to determine whether w satisfies AIP is whether (3.2) holds where $|\phi| = 1$, $|\psi| = 2^q - 1$ and $|\phi \wedge \psi| = 1$. Equivalently, whether

$$w(\Theta(a_1) \leftrightarrow \Theta(a_1)) \leq w((\Theta(a_1) \leftrightarrow \Theta(a_1)) \mid \neg\Psi(a_1) \wedge \neg\Psi(a_2)),$$

for any two distinct state formulae $\Theta(z)$, $\Psi(z)$. This is true regardless of the size $q \geq 2$ of the language (though as q increases, the bound as a proportion of the greatest possible value of $w(\{1,1\})$ decreases). The value of (3.9) in this case is given in (3.14), and the result follows. $\square$

We now examine the c_λ^L, w_L^δ and $u^{\bar{p},L}$ families of probability functions introduced in §1.3, using the bound obtained in the previous theorem to classify which members of these families satisfy AIP. Note that if L is unary, then $r = 2q$ by (3.3), and so by the previous theorem w satisfies AIP just if

$$w(\{1,1\}) \leq 2^{-q}(2^q-1)^{-2} \tag{3.15}$$

(where $w(\{1,1\}) = w(\alpha_i(a_1) \wedge \alpha_j(a_2))$ for $i \neq j$).

Corollary 19. *c_λ^L satisfies AIP just if $0 \leq \lambda \leq (2^q - 3 + 2^{-q})^{-1}$.*

Proof. By the definition of c_0^L (1.12),

$$c_0^L\left(\bigvee_{\mathcal{S}(\Theta)=\{2\}} \Theta(a_1, a_2)\right) = 1$$

so by part 4 of Lemma 17, (3.2) holds for every $\phi(x), \psi(x) \in QFFL$, and so c_0^L satisfies AIP.

For $\lambda > 0$, by (1.11),

$$c_\lambda^L(\{1,1\}) = 2^{-q}\frac{\lambda 2^{-q}}{1+\lambda}$$

so by (3.15), c^L_λ satisfies AIP just if

$$\begin{aligned}\frac{\lambda 2^{-2q}}{1+\lambda} &\leq \frac{1}{2^q(2^q-1)^2}\\ \iff \lambda(2^{-q}(2^q-1)^2-1) &\leq 1\\ \iff \lambda &\leq \frac{1}{2^q-3+2^{-q}}.\end{aligned}$$

□

From this it is clear that for any *fixed* $\lambda > 0$, c^L_λ will fail to satisfy AIP where L is sufficiently large. Since, for fixed λ, the functions $c^{\mathcal{L}}_\lambda$ form a Language Invariant family as $\mathcal{L}$ varies [4], it follows that $c^{\mathcal{L}}_0$ is the only such family to satisfy AIP on every language simultaneously.

Corollary 20. *w^δ_L satisfies AIP just if*

$$\delta^2 \geq 1 - \frac{2^q}{(2^q-1)^2}.$$

Proof. For $0 \leq \delta \leq 1$, by (1.14),

$$w^\delta_L(\{1,1\}) = 2^{-q}(2\gamma(\gamma+\delta) + (2^q-2)\gamma^2) = 2^{1-q}\gamma\delta + \gamma^2$$

and substituting $\gamma = \frac{1-\delta}{2^q}$ gives

$$w^\delta_L(\{1,1\}) = \frac{1-\delta^2}{2^{2q}}.$$

By (3.15) then, w^δ_L satisfies AIP just if

$$\begin{aligned}\frac{1-\delta^2}{2^{2q}} &\leq \frac{1}{2^q(2^q-1)^2}\\ \iff 1 - \frac{2^q}{(2^q-1)^2} &\leq \delta^2.\end{aligned}$$

□

Here too it is clear that, for any *fixed* $\delta < 1$, w^δ_L fails to satisfy AIP where L is sufficiently large. Similarly to the case with the $c^{\mathcal{L}}_\lambda$, where δ is fixed the $w^\delta_{\mathcal{L}}$ functions form a Language Invariant family as $\mathcal{L}$ varies [5], and it follows that w^1 ($= c_0$) is the only such family to satisfy AIP on each language simultaneously.

The following corollary refers to the $u^{\bar{p},L}$ functions defined at (1.15) on general (polyadic) as well as unary languages.

[4] See [49, chapter 16].
[5] See [49, chapter 19].

Corollary 21. *For $\bar{p} = \langle p_0, p_1, p_2, \ldots \rangle \in \mathbb{B}$, $u^{\bar{p},L}$ satisfies AIP just if*

$$\sum_{i=1}^{\infty} p_i^2 \geq 1 - \frac{2^q}{(2^q-1)^2}.$$

Proof. By (1.15),

$$\begin{aligned} u^{\bar{p},L}(\{1,1\}) &= \frac{1}{SD(2)}(p_0 + p_1(1-p_1) + p_2(1-p_2) + \ldots) \\ &= 2^{-r}\left(1 - \sum_{i=1}^{\infty} p_i^2\right) \end{aligned} \tag{3.16}$$

since by (3.4), $SD(2) = 2^r$. By Theorem 18 then, $u^{\bar{p},L}$ satisfies AIP just if

$$\begin{aligned} 2^{-r}\left(1 - \sum_{i=1}^{\infty} p_i^2\right) &\leq \frac{2^q}{2^r(2^q-1)^2} \\ \Longleftrightarrow 1 - \sum_{i=1}^{\infty} p_i^2 &\leq \frac{2^q}{(2^q-1)^2}. \end{aligned}$$

Since r does not appear in this bound, it applies both to unary and to strictly polyadic languages. □

Again it is clear that, for any fixed $\bar{p} \neq \langle 0, 1, 0, 0, \ldots \rangle$, $u^{\bar{p},L}(\{1,1\}) > 0$, so $u^{\bar{p},L}$ fails to satisfy AIP on sufficiently large languages. [6] The $u^{\bar{p},\mathcal{L}}$ are another example of functions which form Language Invariant families, obtained by fixing $\bar{p} \in \mathbb{B}$ as $\mathcal{L}$ varies [7]. Using the characterization in Theorem 8 of probability functions which satisfy Li with Sx, we obtain the following:

Corollary 22. *A probability function w on SL satisfies Li with Sx and AIP (that is, for each language $\mathcal{L}$ containing L there is a probability function $w^{\mathcal{L}}$ on $S\mathcal{L}$, such that for $\theta \in SL$, $w^{\mathcal{L}}(\theta) = w(\theta)$, and $w^{\mathcal{L}}$ satisfies Sx and AIP) just if*

$$w = u^{\langle 0,1,0,\ldots \rangle, L}.$$

Proof. If $w = u^{\langle 0,1,0,\ldots\rangle,L}$ then for each $\mathcal{L}$, $w^{\mathcal{L}} = u^{\langle 0,1,0,\ldots\rangle,\mathcal{L}}$ satisfies AIP by Corollary 21, and so w satisfies Li with Sx and AIP by Theorem 8. For the converse result, suppose w satisfies Li with Sx, so for each $\mathcal{L} \supseteq L$ by Theorem 8,

$$w^{\mathcal{L}} = \int_{\mathbb{B}} u^{\bar{p},\mathcal{L}} \, d\mu(\bar{p})$$

[6] It can be seen from (1.15) that for unary L, $u^{\langle 0,1,0,\ldots\rangle,L} = w_L^1 = c_0^L$.

[7] See [49, chapter 29].

for some fixed measure μ on $\mathbb{B}$. Suppose $\mu(\{\langle 0,1,0,0,\ldots\rangle\}) < 1$. Then there is some $\delta > 0$ such that

$$\mu(\{\bar{p} \in \mathbb{B} \mid p_1 \leq 1-\delta\}) > 0$$

and

$$\inf_{\substack{\bar{p}\in\mathbb{B}\\ p_1\leq 1-\delta}} \{1 - \sum_{i=1}^{\infty} p_i^2\} = (1-(1-\delta)^2 - \delta^2) > 0.$$

Therefore, the product

$$\mu(\{\bar{p} \in \mathbb{B} \mid p_1 \leq 1-\delta\}) \cdot \inf_{\substack{\bar{p}\in\mathbb{B}\\ p_1\leq 1-\delta}} \{1 - \sum_{i=1}^{\infty} p_i^2\}$$

is a fixed, positive value determined by μ, so by (3.16) for all sufficiently large q we have

$$\begin{aligned} w^{\mathcal{L}}(\{1,1\}) &\geq \int_{\substack{\bar{p}\in\mathbb{B}\\ p_1\leq 1-\delta}} u^{\bar{p},\mathcal{L}}(\{1,1\})\, d\mu(\bar{p}) \\ &\geq \mu(\{\bar{p} \in \mathbb{B} \mid p_1 \leq 1-\delta\}) \cdot \inf_{\substack{\bar{p}\in\mathbb{B}\\ p_1\leq 1-\delta}} \{1 - \sum_{i=1}^{\infty} p_i^2\} \cdot 2^{-r} \\ &> \frac{2^q}{2^r(2^q-1)^2} \end{aligned}$$

so $w^{\mathcal{L}}$ does not satisfy AIP where $\mathcal{L}$ is sufficiently large. The result follows. □

So, the combined requirements of Li, Sx and AIP limit the choice of probability functions to just one candidate! Unfortunately, this unique candidate may be found rather objectionable on the grounds that

$$u^{\langle 0,1,0,\ldots\rangle,L}(\Theta(a_1,\ldots,a_n)) = 0$$

whenever $|\mathcal{S}(\Theta)| > 1$, that is, whenever Θ makes any distinction between any of the constants $a_1,\ldots,a_n$. We would conclude that, however appealing these properties may be individually, in combination they place too severe a restriction on the choice of function, and should be rejected.

Even without the requirement of Language Invariance, however, Theorem 18 shows that AIP is rather a strong requirement of functions satisfying Sx on all but the smallest languages. Since the only possible spectra of state descriptions for 2 constants are

$\{1,1\}$ and $\{2\}$, where AIP forces the value of $w(\{1,1\})$ to be small the value of $w(\{2\})$ must be correspondingly large.

This observation is illustrated for the $v_{\vec{c}}$ functions on unary languages, defined at (1.10), by the following:

Proposition 23. *Let $q \geq 2$ and $\vec{c} \in \mathbb{D}_{2^q}$. If the probability function $v_{\vec{c}}$ on the unary language with q predicates satisfies AIP then the largest co-ordinate a in $\vec{c}$ satisfies*

$$2a(1-a) \leq \frac{1}{2^q - 1}.$$

Proof. Let $q \geq 2$ and $\vec{c} \in \mathbb{D}_{2^q}$, and let $0 < a \leq 1$ be the largest value of any co-ordinate in $\vec{c}$. Suppose, without loss of generality since $v_{\sigma(\vec{c})} = v_{\vec{c}}$ for any $\sigma \in \mathsf{S}_{2^q}$, that $c_1 = a$. Then by (1.10)

$$\begin{aligned} v_{\vec{c}}(\{1,1\}) &= v_{\vec{c}}(\alpha_1(a_1) \wedge \alpha_2(a_2)) \\ &= \frac{1}{2^q(2^q-1)} \sum_{j=1}^{2^q} (c_j - c_j^2). \end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned} \sum_{j=1}^{2^q} (c_j - c_j^2) &= 1 - a^2 - \sum_{j=2}^{2^q} c_j^2 \\ &\geq 1 - a^2 - (1-a)^2 \\ &= 2a(1-a), \end{aligned}$$

so that by (3.17),

$$v_{\vec{c}}(\{1,1\}) \geq \frac{2a(1-a)}{2^q(2^q-1)} = v_{\vec{a}}(\{1,1\})$$

where $\vec{a} = \langle a, 1-a, 0, \ldots, 0 \rangle \in \mathbb{D}_{2^q}$.

Therefore if $v_{\vec{c}}$ satisfies AIP, by (3.15),

$$v_{\vec{c}}(\{1,1\}) \leq 2^{-q}(2^q-1)^{-2}$$

and so by the above remark

$$v_{\vec{a}}(\{1,1\}) = \frac{2a(1-a)}{2^q(2^q-1)} \leq 2^{-q}(2^q-1)^{-2},$$

giving

$$2a(1-a) \leq \frac{1}{2^q - 1}.$$

□

Since for $0 < a \leq 1$, $a(1-a)$ takes its largest value at $a = 1/2$ and its smallest values at the endpoints close to 0 or 1 (and since the co-ordinates of $\vec{c}$ must sum to 1 and a is assumed to be the largest), the significance of this result is that for $v_{\vec{c}}$ to satisfy AIP, its largest co-ordinate must be rather close to 1, while all others must be very small. This illustrates the remarks above regarding the strength of AIP in the presence of Sx, even where Li is not assumed. What may seem to be a reasonable requirement in (3.1) has been shown to be much more demanding, at least in the presence of Sx, than it may at first appear.

3.2 Generalizations of AIP

We now proceed to investigate how AIP might be modified to take account of background information, mentioning more than just 2 constants. We limit this investigation to unary languages for simplicity, so we will assume for the rest of this chapter that L is unary. However, we no longer need assume that w satisfies Sx, subsequent results apply to all functions (satisfying our standing assumption of Ex) on unary languages.

We begin with a lemma concerning the $w_{\vec{c}}$ functions (1.8), which will be needed subsequently.

Lemma 24. *For $\vec{c} \in \mathbb{D}_{2^q}$, $w_{\vec{c}}$ satisfies AIP just if $\vec{c}$ has at most 2 non-zero co-ordinates.*

Proof. From right to left, if $\vec{c}$ has a single non-zero co-ordinate then $w_{\vec{c}}$ satisfies AIP by part 4 of Lemma 17. If $\vec{c}$ has exactly 2 non-zero co-ordinates then for any $\phi(x),\ \psi(x) \in QFFL$ either $w_{\vec{c}}(\psi(a_1)) = w_{\vec{c}}(\psi(a_2)) \in \{0,1\}$ and (3.2) holds by part 1 of Lemma 17, or $w(\psi(a_i) \to \alpha_j(a_i)) = 1$ for some unique $1 \leq j \leq 2^q$, and (3.2) holds by part 2 of the same result. Therefore $w_{\vec{c}}$ satisfies AIP.

In the opposite direction, suppose $\vec{c} \in \mathbb{D}_{2^q}$ has at least 3 non-zero co-ordinates. Let those co-ordinates with positive values be partitioned into 3 parts D_1, D_2, D_3 with the sum of the co-ordinate values in each part $d_1, d_2, d_3 > 0$ respectively. Without loss of generality, suppose that $d_3 \geq d_1, d_2$.

Let

$$\phi(x) = \bigvee_{i \in D_1} \alpha_i(x), \qquad \psi(x) = \bigvee_{i \in D_1 \cup D_3} \alpha_i(x).$$

Then by (1.8)

$$\begin{aligned} w_{\vec{c}}(\phi(a_1) \leftrightarrow \phi(a_2)) &= 1 \; - \; w_{\vec{c}}((\phi(a_1) \wedge \neg\phi(a_2)) \vee (\neg\phi(a_1) \wedge \phi(a_2))) \\ &= 1 - 2d_1(1 - d_1), \end{aligned}$$

while

$$w_{\vec{c}}(\psi(a_1) \wedge \psi(a_2)) = (d_1 + d_3)^2,$$

and

$$\begin{aligned} & w_{\vec{c}}((\phi(a_1) \leftrightarrow \phi(a_2)) \wedge \psi(a_1) \wedge \psi(a_2)) \\ & \quad = w_{\vec{c}}(\phi(a_1) \wedge \psi(a_1) \wedge \phi(a_2) \wedge \psi(a_2)) \; + \; w_{\vec{c}}(\neg\phi(a_1) \wedge \psi(a_1) \wedge \neg\phi(a_2) \wedge \psi(a_2)) \\ & \quad = d_1^2 + d_3^2. \end{aligned}$$

If $w_{\vec{c}}$ satisfies AIP then

$$\begin{aligned} & w_{\vec{c}}(\phi(a_1) \leftrightarrow \phi(a_2)) \; \cdot \; w_{\vec{c}}(\psi(a_1) \wedge \psi(a_2)) \; \leq \; w_{\vec{c}}((\phi(a_1) \leftrightarrow \phi(a_2)) \wedge \psi(a_1) \wedge \psi(a_2)) \\ & \quad \iff (1 - 2d_1(1 - d_1))(d_1 + d_3)^2 \; \leq \; d_1^2 + d_3^2 \\ & \quad \iff (d_1 + d_3)^2 - d_1^2 - d_3^2 \; \leq \; 2d_1(1 - d_1)(d_1 + d_3)^2 \\ & \quad \iff d_3 \; \leq \; (1 - d_1)(d_1 + d_3)^2 \\ & \quad \iff 1 - d_1 - d_2 \; \leq \; (1 - d_1)(1 - d_2)^2 \; = \; 1 - 2d_2 + d_2^2 - d_1 + 2d_1d_2 - d_1d_2^2 \\ & \quad \iff d_2 \; \leq \; d_2(d_2 + 2d_1 - d_1d_2) \\ & \quad \iff 1 + d_1d_2 \; \leq \; d_2 + 2d_1, \end{aligned}$$

but this is impossible since $d_1 \leq d_3$ so the right hand side is at most 1 while the left hand side is greater than 1. The result follows. $\square$

We now investigate different formulations of abductive inference in the presence of some background information. Two approaches are considered, the first is to condition a probability funcion w on a single sentence θ throughout (3.1), giving the following formulation:

Generalized Abductive Inference Principle (a), GAIP(a)

For $\vec{a} = \langle a_1, \dots, a_n \rangle$ and any $\phi(x), \psi(x) \in QFFL$ and $\theta(\vec{a}) \in SL$,

$$w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \theta(\vec{a})) \;\leq\; w(\phi(a_1) \leftrightarrow \phi(a_2) \mid \psi(a_1) \wedge \psi(a_2) \wedge \theta(\vec{a})). \tag{3.18}$$

The following result will show that for all but a very restricted class of functions, there is some background information $\theta(a_1, \dots, a_n) \in QFSL$ and some $\phi(x), \psi(x) \in QFFL$ such that (3.18) fails to hold, that is, w conditioned on θ fails to satisfy AIP.

Proposition 25. *If w is a probability function satisying GAIP(a), then every support point*[8] *of w contains at most 2 non-zero co-ordinates.*

Proof. Suppose w is a probability function with a support point $\vec{c} \in \mathbb{D}_{2^q}$ which has at least 3 non-zero co-ordinates. By Lemma 24 (and Ex) there exist $\phi(x), \psi(x) \in QFFL$, with $w(\psi(a_{n+1}) \wedge \psi(a_{n+2})) > 0$, such that

$$w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2})) \;>\; w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \psi(a_{n+1}) \wedge \psi(a_{n+2})) > 0.$$

Let δ be equal to

$$w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2})) \;-\; w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \psi(a_{n+1}) \wedge \psi(a_{n+2}))$$

so $\delta > 0$, and let ϵ be small enough that

$$\begin{aligned} 0 < \frac{w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \wedge \psi(a_{n+1}) \wedge \psi(a_{n+2})) + \epsilon}{w_{\vec{c}}(\psi(a_{n+1}) \wedge \psi(a_{n+2})) - \epsilon} & \\ - \; w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \psi(a_{n+1}) \wedge \psi(a_{n+2})) \;&<\; \delta - \epsilon. \end{aligned} \tag{3.19}$$

By Corollary 7 there exist n and $\theta(a_1, \dots, a_n) \in QFSL$ such that $w(\theta) > 0$, and for any $\eta \in QFSL$

$$|w(\eta \mid \theta) - w_{\vec{c}}(\eta)| < \epsilon. \tag{3.20}$$

[8] The definition follows Theorem 5.

Therefore by (3.19), (3.20) and the definition of δ,

$$w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \psi(a_{n+1}) \wedge \psi(a_{n+2}) \wedge \theta(a_1, \ldots, a_n))$$

$$= \frac{w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \wedge \psi(a_{n+1}) \wedge \psi(a_{n+2}) \mid \theta(a_1, \ldots, a_n))}{w(\psi(a_{n+1}) \wedge \psi(a_{n+2}) \mid \theta(a_1, \ldots, a_n))}$$

$$< \frac{w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \wedge \psi(a_{n+1}) \wedge \psi(a_{n+2})) + \epsilon}{w_{\vec{c}}(\psi(a_{n+1}) \wedge \psi(a_{n+2})) - \epsilon}$$

$$< w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \psi(a_{n+1}) \wedge \psi(a_{n+2})) + \delta - \epsilon$$

$$= w_{\vec{c}}(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2})) - \epsilon$$

$$< w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \theta(a_1, \ldots, a_n)).$$

$\square$

So even those functions which do satisfy AIP ‘at the outset’, such as those satisfying Sx and (3.15), may fail to do so after conditioning on certain background information. Those functions which do satisfy GAIP(a) on languages where $q \geq 2$ may be found objectionable, by the previous result, on the grounds that they will assign zero probability to any state description containing 3 or more distinct atoms.

A different approach to including background information would be to allow it to be contained in the ‘observation’ formula ψ in (3.1), as in the following formulation: [9]

Generalized Abductive Inference Principle (b), GAIP(b)

For $\vec{a} = \langle a_1, \ldots, a_n \rangle$ and any $\phi(x),\ \psi(x, \vec{a}) \in QFFL$

$$w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2})) \ \leq\ w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2}) \mid \psi(a_{n+1}, \vec{a}) \wedge \psi(a_{n+2}, \vec{a})). \quad (3.21)$$

[9] A similar principle, where ϕ as well as ψ is permitted to contain information relating to some additional constants $\vec{a}$, is considered in [27], known there as the Equivalence Analogy Principle.

By our convention (1.2) we take this to be equivalent to

$$\begin{aligned} w(\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2})) \cdot w(\psi(a_{n+1}, \vec{a}) \wedge \psi(a_{n+2}, \vec{a})) \\ \leq w((\phi(a_{n+1}) \leftrightarrow \phi(a_{n+2})) \wedge \psi(a_{n+1}, \vec{a}) \wedge \psi(a_{n+2}, \vec{a})). \end{aligned} \quad (3.22)$$

The following classification result for GAIP(b) shows that this too is a strong condition.

Theorem 26. *A probability function w on a unary language L with support $\mathcal{M}$ satisfies GAIP(b) just if $\mathcal{M}$ takes one of the following forms:*

1. *$\mathcal{M} = \{\vec{c}\}$ is a singleton where $\vec{c}$ has exactly 2 non-zero co-ordinates,*
2. *$\mathcal{M}$ is some permutation of $\{\langle c, 1-c, 0, \ldots, 0\rangle, \langle 1-c, c, 0, \ldots, 0\rangle\} \subset \mathbb{D}_{2^q}$ for some unique $0 < c < 1$,*
3. *$\mathcal{M} \subseteq \{\sigma(\langle 1, 0, \ldots, 0\rangle) \mid \sigma \in \mathsf{S}_{2^q}\}$.*

Proof. Let w be a probability function on a unary language L, with de Finetti prior μ and support $\mathcal{M}$. For $1 \leq j \leq 2^q$, let

$$\gamma_j = min\{|d_j - 1/2| \mid \vec{d} \in \mathcal{M}\}$$

(which exists since $\mathcal{M}$ is closed), and let

$$\Gamma_j = \{\vec{d} \in \mathcal{M} \mid |d_j - 1/2| = \gamma_j\}$$

(which is non-empty and closed for each j).

Suppose there is some $1 \leq j \leq 2^q$ such that $\mu(\Gamma_j) < 1$, and let $\phi(x) = \alpha_j(x)$. Let $m \in \mathbb{N}^+$ be large and for some fixed $\vec{c} \in \Gamma_j$, let $k = \sum_{i=1}^{2^q}[mc_i]$. Let $\vec{a} = \langle a_1, a_2, \ldots, a_k\rangle$ and let $\psi(x, \vec{a}) = \alpha_1^{[mc_1]} \wedge \ldots \wedge \alpha_{2^q}^{[mc_{2^q}]}$.

For $x \in \mathbb{D}_{2^q} - \Gamma_j$, $|x_j - 1/2| > \gamma_j$, and so

$$x_j(1 - x_j) < c_j(1 - c_j),$$

giving

$$1 - 2x_j(1 - x_j) > 1 - 2c_j(1 - c_j),$$

and

$$\int_{\mathbb{D}_{2^q}-\Gamma_j} 1-2x_j(1-x_j)\ d\mu(\vec{x}) > (1-\mu(\Gamma_j))(1-2c_j(1-c_j)).$$

Therefore

$$\begin{aligned} w(\phi(a_{m+1},\vec{a}) \leftrightarrow \phi(a_{m+2},\vec{a})) &= \int_{\mathbb{D}_{2^q}} 1-2x_j(1-x_j)\ d\mu(\vec{x}) \\ &= \int_{\Gamma_j} 1-2x_j(1-x_j)\ d\mu(\vec{x}) + \int_{\mathbb{D}_{2^q}-\Gamma_j} 1-2x_j(1-x_j)\ d\mu(\vec{x}) \\ &= \mu(\Gamma_j)(1-2c_j(1-c_j)) + \int_{\mathbb{D}_{2^q}-\Gamma_j} 1-2x_j(1-x_j)\ d\mu(\vec{x}) \\ &> 1-2c_j(1-c_j). \end{aligned}$$

By Lemma 6, $w(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2}) \mid \psi(a_{m+1},\vec{a}) \wedge \psi(a_{m+2},\vec{a}))$ becomes arbitrarily close to

$$w_{\vec{c}}(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) = 1-2c_j(1-c_j)$$

where m is sufficiently large, and so w fails to satisfy GAIP(b).

Now suppose that w satisfies GAIP(b). By the above argument, there can be no $1 \leq j \leq 2^q$ such that $\mu(\Gamma_j) < 1$, meaning that $\mu(\Gamma_j) = 1$ for each $1 \leq j \leq 2^q$. If $\vec{e} \in \mathcal{M}$ is not a limit point of Γ_j then it is possible to find $\epsilon > 0$ such that $B_\epsilon(\vec{e}) \subseteq \mathcal{M} - \Gamma_j$, contradicting $\mu(\Gamma_j) = 1$. Therefore any $\vec{e} \in \mathcal{M}$ is a limit point of Γ_j, and so must be a member of Γ_j (since Γ_j is closed). It follows that $\Gamma_i = \Gamma_j = \mathcal{M}$ for any $1 \leq i, j \leq 2^q$.

Suppose that $|\mathcal{M}| = 1$, so that $w = w_{\vec{c}}$ for some unique $\vec{c} \in \mathbb{D}_{2^q}$. If $\vec{c}$ has at least 3 non-zero co-ordinates, we know from Lemma 24 that w fails AIP (and therefore GAIP(b) with $\psi(x,\vec{a}) = \psi(x)$), contradicting our assumption. Therefore $\vec{c}$ can have at most 2 non-zero co-ordinates, and $\mathcal{M}$ is either of form 1 or form 3.

Now suppose otherwise that $|\mathcal{M}| \geq 2$, and let $\vec{c},\ \vec{d} \in \mathcal{M}$ be distinct support points. Let $S = \{i_1, \ldots, i_g\} \subseteq \mathbb{N}_{2^q}$ be the set of co-ordinates on which $\vec{c},\ \vec{d}$ differ:

$$c_j \neq d_j \iff j \in S.$$

If $\vec{c} = \langle c_1, c_2, \ldots, c_{2^q}\rangle$ then

$$d_j = \begin{cases} 1-c_j & j \in S \\ c_j & j \notin S \end{cases}$$

since $\vec{c},\ \vec{d} \in \Gamma_j$ for each $1 \leq j \leq 2^q$. The co-ordinates sum to 1 in each case which gives

$$1 - c_{i_1} + 1 - c_{i_2} + \ldots + 1 - c_{i_g} = c_{i_1} + \ldots + c_{i_g} \tag{3.23}$$

and so

$$g = 2(c_{i_1} + \ldots + c_{i_g}) \leq 2.$$

Since it is impossible for $\vec{c}$, $\vec{d}$ to differ in just 1 co-ordinate, g must be equal to 2, which substitution in (3.23) yields

$$c_{i_2} = 1 - c_{i_1}.$$

Where $c_{i_1} \in \{0, 1\}$, we conclude that $\vec{c},\ \vec{d}$ are distinct permutations of $\langle 1, 0, \ldots, 0\rangle$, as is any $\vec{e} \in \mathcal{M}$, so that $\mathcal{M}$ is of form 3. Suppose otherwise, that c_{i_1} takes the value $0 < c < 1$. Then any $\vec{e} \in \mathcal{M}$ distinct from $\vec{c}$ is either equal to $\vec{d}$, or differs from $\vec{c}$ at exactly 2 co-ordinates $\{u_1, u_2\} \neq \{i_1, i_2\}$. But

$$c_{u_1} + c_{u_2} < 1,$$

meaning that

$$e_{u_1} + e_{u_2} = (1 - c_{u_1}) + (1 - c_{u_2}) > 1,$$

which is impossible by the definition of $\mathbb{D}_{2^q}$. Therefore $\mathcal{M} = \{\vec{c},\ \vec{d}\}$ and so is of form 2. The left-to-right direction of the result follows.

In the other direction, we assume that w is a probability function with de Finetti prior μ, whose support $\mathcal{M}$ takes one of the forms in the statement of the result. Suppose that $\mathcal{M}$ is of the third form, so that

$$\mathcal{M} \subseteq \{\sigma(\langle 1, 0, \ldots, 0\rangle) \mid \sigma \in \mathsf{S}_{2^q}\}$$

and so

$$w = \sum_{j=1}^{2^q} \eta_j\, w_{\vec{b_j}}$$

where $\vec{b_j} \in \mathbb{D}_{2^q}$ has value 1 at co-ordinate j and 0 at all other co-ordinates, each $\eta_j \geq 0$ and $\sum_j \eta_j = 1$.

Let $\phi(x) \in QFFL$ and let $1 \leq j \leq 2^q$. If $\alpha_j(x) \models \phi(x)$ then (for any m)

$$w_{\vec{b_j}}(\phi(a_{m+1}) \wedge \phi(a_{m+2})) = 1, \qquad w_{\vec{b_j}}(\neg\phi(a_{m+1}) \wedge \neg\phi(a_{m+2})) = 0,$$

while if $\alpha_j(x) \models \neg\phi(x)$ then

$$w_{\vec{b_j}}(\phi(a_{m+1}) \wedge \phi(a_{m+2})) = 0, \qquad w_{\vec{b_j}}(\neg\phi(a_{m+1}) \wedge \neg\phi(a_{m+2})) = 1.$$

In either case,

$$\begin{aligned} & w_{\vec{b_j}}(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) \\ & \quad = w_{\vec{b_j}}(\phi(a_{m+1}) \wedge \phi(a_{m+2})) \ + \ w_{\vec{b_j}}(\neg\phi(a_{m+1}) \wedge \neg\phi(a_{m+2})) \\ & \quad = 1, \end{aligned}$$

and so

$$\begin{aligned} w(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) &= \sum_{j=1}^{2^q} \eta_j \, w_{\vec{b_j}}(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) \\ &= \sum_{j=1}^{2^q} \eta_j = 1, \end{aligned}$$

and by Lemma 3, (3.22) holds with equality for any $\phi(x), \ \psi(x, \vec{a})$.

Now suppose that $\mathcal{M}$ is of form 1 or 2, so that

$$w = \lambda w_{\vec{c}} + (1 - \lambda) w_{\vec{d}}$$

for some $0 < \lambda \leq 1$, where for some $0 < c < 1$, $\vec{c} \in \mathbb{D}_{2^q}$ has values $c, \ 1 - c$ at co-ordinates $s_1, \ s_2$ respectively, and 0 at all other co-ordinates, while $\vec{d} \in \mathbb{D}_{2^q}$ has value $1 - c$ at co-ordinate s_1 and c at s_2 etc.. Suppose, without loss of generality, that $s_1 = 1, s_2 = 2$ (the same argument applies for any other pair of distinct co-ordinates).

Let $\phi(x) \in QFFL$ so that

$$\phi(x) \equiv \bigvee_{j \in P_\phi} \alpha_j(x)$$

for some $P_\phi \subseteq \mathbb{N}_{2^q}$. Let $P_{\neg\phi}$ denote $\mathbb{N}_{2^q} - P_\phi$. Then (for any m)

$$\begin{aligned}
&w_{\vec{c}}(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2}))\\
&\quad = w_{\vec{c}}(\phi(a_{m+1}) \wedge \phi(a_{m+2})) \ + \ w_{\vec{c}}(\neg\phi(a_{m+1}) \wedge \neg\phi(a_{m+2}))\\
&\quad = w_{\vec{c}}\left(\bigvee_{j\in P_\phi} \alpha_j(a_{m+1}) \wedge \bigvee_{j\in P_\phi} \alpha_j(a_{m+2})\right) \ + \ w_{\vec{c}}\left(\bigvee_{j\in P_{\neg\phi}} \alpha_j(a_{m+1}) \wedge \bigvee_{j\in P_{\neg\phi}} \alpha_j(a_{m+2})\right)\\
&\quad = \left(\sum_{j\in P_\phi\cap\{1,2\}} c_j\right)^2 \ + \ \left(\sum_{j\in P_{\neg\phi}\cap\{1,2\}} c_j\right)^2\\
&\quad = \begin{cases} 1 & \text{if } \{1,2\} \subseteq P_\phi \text{ or } \{1,2\} \subseteq P_{\neg\phi}\\ c^2 + (1-c)^2 & \text{otherwise.}\end{cases}
\end{aligned}$$

A similar argument shows that likewise

$$w_{\vec{d}}(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) = \begin{cases} 1 & \text{if } \{1,2\} \subseteq P_\phi \text{ or } \{1,2\} \subseteq P_{\neg\phi}\\ c^2 + (1-c)^2 & \text{otherwise.}\end{cases}$$

If $\{1,2\} \subseteq P_\phi$ or $\{1,2\} \subseteq P_{\neg\phi}$ then

$$w(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) = \lambda + (1-\lambda) = 1,$$

and by Lemma 3 (3.22) holds with equality for any $\psi(x, \vec{a})$, so suppose otherwise that

$$\{1,2\} \cap P_\phi = \{1\}, \quad \{1,2\} \cap P_{\neg\phi} = \{2\} \tag{3.24}$$

or vice-versa. Let $\psi(x, \vec{a}) \in QFFL$ where $\vec{a} = \langle a_1, a_2, \ldots, a_m\rangle$ for some $m \geq 1$ (the same argument may be used, and becomes simpler, for the case where $\psi(x)$ does not refer to any constant symbols), so that

$$\psi(x, \vec{a}) \equiv \bigvee_{\Theta\in T_\psi} \left(\Theta(\vec{a}) \wedge \bigvee_{j\in Q_\Theta} \alpha_j(x)\right)$$

for some set T_ψ of state descriptions for $\vec{a}$ and subsets Q_Θ of $\mathbb{N}_{2^q}$.

Any disjunct which logically implies $\alpha_j(x)$ or $\alpha_j(a_i)$ for some $1 \leq i \leq m$ and any $j \notin \{1,2\}$ will be assigned probability 0 by w, so for each $\Theta \in T_\psi$, let $Q'_\Theta = Q_\Theta \cap \{1,2\}$, and let

$$T'_\psi = \{\Theta \in T_\psi \mid \Theta \models \bigwedge_{i=1}^{m} \bigvee_{j\in\{1,2\}} \alpha_j(a_i),\ Q'_\Theta \neq \emptyset\}.$$

Then

$$\begin{aligned}
&w(\psi(a_{m+1},\vec{a}) \wedge \psi(a_{m+2},\vec{a})) \\
&\quad = w\left(\bigvee_{\Theta\in T'_\psi}\left(\Theta(\vec{a}) \wedge \bigvee_{j\in Q'_\Theta} \alpha_j(a_{m+1})\right) \wedge \bigvee_{\Theta\in T'_\psi}\left(\Theta(\vec{a}) \wedge \bigvee_{j\in Q'_\Theta} \alpha_j(a_{m+2})\right)\right) \\
&\quad = w\left(\bigvee_{\Theta\in T'_\psi}\left(\Theta(\vec{a}) \wedge \bigvee_{\langle j_1,j_2\rangle\in(Q'_\Theta)^2} \alpha_{j_1}(a_{m+1}) \wedge \alpha_{j_2}(a_{m+2})\right)\right). \qquad (3.25)
\end{aligned}$$

For each $\Theta \in T'_\psi$, Q'_Θ takes one of the forms: $\{1\}$, $\{2\}$, or $\{1,2\}$, so partition T'_ψ into

$$T_1 = \{\Theta \in T'_\psi \mid Q'_\Theta = \{1\}\},$$

$$T_2 = \{\Theta \in T'_\psi \mid Q'_\Theta = \{2\}\},$$

$$T_{1,2} = \{\Theta \in T'_\psi \mid Q'_\Theta = \{1,2\}\}.$$

By (3.25), then

$$\begin{aligned}
&w(\psi(a_{m+1},\vec{a}) \wedge \psi(a_{m+2},\vec{a})) \\
&\quad = \sum_{\Theta\in T_1} w(\Theta(\vec{a}) \wedge \alpha_1(a_{m+1}) \wedge \alpha_1(a_{m+2})) + \sum_{\Theta\in T_2} w(\Theta(\vec{a}) \wedge \alpha_2(a_{m+1}) \wedge \alpha_2(a_{m+2})) \\
&\qquad + \sum_{\Theta\in T_{1,2}} w(\Theta(\vec{a}) \wedge ((\alpha_1(a_{m+1}) \wedge \alpha_1(a_{m+2})) \vee (\alpha_2(a_{m+1}) \wedge \alpha_2(a_{m+2})))) \\
&\qquad\quad + \sum_{\Theta\in T_{1,2}} w(\Theta(\vec{a}) \wedge ((\alpha_1(a_{m+1}) \wedge \alpha_2(a_{m+2})) \vee (\alpha_2(a_{m+1}) \wedge \alpha_1(a_{m+2})))).
\end{aligned}$$

Let β denote the sum of the first three terms above, and let δ denote the fourth term. By our assumption (3.24), the first three terms make up

$$w(\psi(a_{m+1}) \wedge \psi(a_{m+2}) \wedge (\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2}))),$$

and

$$w(\phi(a_{m+1}) \leftrightarrow \phi(a_{m+2})) = c^2 + (1-c)^2$$

so that (3.22) holds just if

$$(c^2 + (1-c)^2)(\beta + \delta) \leq \beta,$$

equivalently

$$(c^2 + (1-c)^2)\delta \leq 2c(1-c)\beta. \qquad (3.26)$$

Now, by the definition of β and by (1.9)

$$\begin{aligned}\beta &\geq \sum_{\Theta\in T_{1,2}} w(\Theta(\vec{a}) \wedge ((\alpha_1(a_{m+1}) \wedge \alpha_1(a_{m+2})) \vee (\alpha_2(a_{m+1}) \wedge \alpha_2(a_{m+2}))))\\ &= \sum_{\Theta\in T_{1,2}} \lambda w_{\vec{c}}(\Theta(\vec{a}))(c^2+(1-c)^2) \ + \ (1-\lambda)w_{\vec{d}}(\Theta(\vec{a}))((1-c)^2+c^2)\\ &= (c^2+(1-c)^2)\sum_{\Theta\in T_{1,2}} \lambda w_{\vec{c}}(\Theta(\vec{a})) + (1-\lambda)w_{\vec{d}}(\Theta(\vec{a})) \qquad (3.27)\end{aligned}$$

while

$$\begin{aligned}\delta &= \sum_{\Theta\in T_{1,2}} w(\Theta(\vec{a}) \wedge ((\alpha_1(a_1) \wedge \alpha_2(a_2)) \vee (\alpha_2(a_1) \wedge \alpha_1(a_2))))\\ &= \sum_{\Theta\in T_{1,2}} \lambda w_{\vec{c}}(\Theta(\vec{a}))2c(1-c) \ + \ (1-\lambda)w_{\vec{d}}(\Theta(\vec{a}))2c(1-c)\\ &= 2c(1-c)\sum_{\Theta\in T_{1,2}} \lambda w_{\vec{c}}(\Theta(\vec{a})) + (1-\lambda)w_{\vec{d}}(\Theta(\vec{a})). \qquad (3.28)\end{aligned}$$

Let κ denote $\sum_{\Theta\in T_{1,2}} \lambda w_{\vec{c}}(\Theta(\vec{a})) + (1-\lambda)w_{\vec{d}}(\Theta(\vec{a}))$. Then by (3.27) and (3.28)

$$\begin{aligned}2c(1-c)\beta &\geq 2c(1-c)(c^2+(1-c)^2)\kappa\\ &= (c^2+(1-c)^2)\delta,\end{aligned}$$

and so (3.26) holds and therefore (3.22) holds. The result follows. □

None of the three possible forms necessary for w to satisfy GAIP(b) seems very satisfactory; functions of form 1 or 2 only assign non-zero probability to 2 out of a possible 2^q atoms, while functions of form 3 are subject to the same problems already discussed in regard to c_0^L.

To summarize, then, though we have not been able to provide a characterization theorem for AIP without additionally assuming Sx, and have only considered GAIP for unary languages, our limited findings suggest that the formulations (3.1) for AIP and (3.18) and (3.21) for GAIP are rather too demanding. Large classes of probability functions have been found *not* to satisfy these formulations of Peirce's 'abductive inference', while those which do may be found undesirable on other grounds. We conclude that these formulations are unsatisfactory as principles of PIL, in that they do not correspond to desirable properties of probability functions used to model rational belief.

Returning to the motivation discussed in the introduction to this chapter, the aim of AIP and GAIP was to formalize within PIL a probabilistic interpretation of Peirce's *hypothesis*: that observing certain similarities between individuals makes it seem more likely (or at least not less likely) that they are also similar in other (unobserved) respects. Expressed in everyday language, this seems to be in accord with our intuitions, though perhaps, in light of these results, it should be qualified to say 'in *some* other (unobserved) respects'.

Allowing the hypothetical similarity $\phi(x)$ (or its negation) to be any formula from $QFFL$ has the apparent advantage that it makes no reference to any *particular* underlying population, there may be many candidates (in our framework the exact number will depend on the size of the language). Peirce [52] held the view that "any hypothesis may be admissible in the absence of any special contrary reasons", though he was referring to real-world applications, where any 'contrary reasons' may depend on the specific context of the application, which information is unavailable to our agent.

Perhaps the requirement of AIP that (3.1) must hold for *any* quantifier-free observation ψ and hypothetical property ϕ may be asking too much. For example, the proof of Theorem 18 shows that, in the presence of Sx, the 'toughest test' set by AIP is the extreme case where the observation $\psi(x)$ consists of the smallest possible amount of information expressible in the language: that x does not satisfy 1 particular state formula, while those state formulae which imply respectively $\phi(x)$ and $\neg\phi(x)$ are in the extreme ratio $1 : 2^q - 1$. If such extreme cases were excluded, AIP would certainly be less demanding, although it seems unclear how this should be done in a way which is 'reasonable' rather than arbitrary.

One possible reformulation of AIP which avoids this problem would be to replace $\phi(a_1) \leftrightarrow \phi(a_2)$ in (3.1) by

$$\bigvee_{\Theta(x)} (\Theta(a_1) \wedge \Theta(a_2)), \tag{3.29}$$

where Θ runs over all state formulae for 1 variable. Note that $\phi(a_1) \leftrightarrow \phi(a_2)$ is a logical consequence of (3.29) for any $\phi(x) \in QFFL$. The resulting principle

$$w\left(\bigvee_{\Theta(x)} (\Theta(a_1) \wedge \Theta(a_2)) \;\middle|\; \psi(a_1) \wedge \psi(a_2)\right) \geq w\left(\bigvee_{\Theta(x)} (\Theta(a_1) \wedge \Theta(a_2))\right) \tag{3.30}$$

expresses the idea that observing some similarity between a_1 and a_2 does not diminish our belief that they will be found to satisfy the same 1-place state formula (atom where L is unary). This seems to say something essentially very similar to AIP, and yet it turns out that this condition is satisfied by *every* probability function satisfying Sx! This can be seen since by (3.5), (3.6), (3.7) and our convention (1.2), (3.30) is equivalent to

$$w\left(\bigvee_{\Theta(x) \models \psi(x)} (\Theta(a_1) \wedge \Theta(a_2))\right) \;\geq\; w\left(\bigvee_{\Theta(x)} (\Theta(a_1) \wedge \Theta(a_2))\right) \cdot w\left(\bigvee_{\substack{\Phi(a_1,a_2) \models \\ \psi(a_1) \wedge \psi(a_2)}} \Phi(a_1, a_2)\right)$$

$$\iff |\psi|\, w(\{2\}) \;\geq\; \mathcal{N}(\{2\}) \cdot w(\{2\}) \cdot \left(|\psi|\, w(\{2\}) + |\psi|\, (2^{r-2q}|\psi| - 1)\, w(\{1,1\})\right)$$

$$\iff 1 \;\geq\; 2^q \left(w(\{2\}) + (2^{r-2q}|\psi| - 1)\, w(\{1,1\})\right).$$

This holds with equality if $w(\{1,1\}) = 0$, and otherwise just if

$$2^{-q} \;\geq\; 2^{-q} - (2^{r-q} - 1)w(\{1,1\}) + (2^{r-2q}|\psi| - 1)w(\{1,1\})$$

$$\iff 2^{r-q} - 1 \;\geq\; 2^{r-2q}|\psi| - 1$$

$$\iff 2^q \;\geq\; |\psi|,$$

which always holds. In terms of the requirements placed on probability functions, then, this formulation could hardly be further removed from AIP, despite apparent similarity of 'meaning'.

A similar phenomenon has occurred with other attempts to formalize probabilistic analogical reasoning within PIL. The Strict Analogy Principle considered by Hill & Paris in [20] was found to be unsatisfiable by any probability function on a unary language where $q > 2$, while the Counterpart Principle considered by the same authors in [21] was found to be a consequence of Language Invariance. Similarly, each of the various formulations of reasoning by analogy considered in [27] is found to be either 'too strong', restricting the available choice of probability function to just c_0^L, or to be sufficiently 'weak' that it places no further restriction on the choice of function beyond those already imposed by the background assumptions.

Perhaps the main conclusion of this chapter, then, is that our intuitions regarding how to formalize abductive or analogical reasoning within PIL can be rather misleading. Bartha in [1] takes the view that probabilistic reasoning by analogy is too sensitive to context to be submitted to any general rule, and the results given in this chapter and mentioned above do not contradict this view.

Chapter 4

The theory of Spectrum Exchangeability

[1] One approach to the question of how far a principle, or combination of principles, of inductive reasoning may be said to be 'rational' is to examine the resulting *theory*; the set of sentences of L which must be assigned probability 1 by any probability function which satisfies the chosen principle(s). [2] If this set $Th(\mathcal{P})$ can be identified for a particular set of principles $\mathcal{P}$, this gives a kind of 'creed' according to $\mathcal{P}$, a set of sentences which must be accepted with certainty by any agent who adopts $\mathcal{P}$. By the definition of a probability function, $Th(\mathcal{P})$ must contain all tautologies, but where it additionally contains non-tautologous sentences, this surely says something interesting about $\mathcal{P}$, which may give a new perspective on the extent to which $\mathcal{P}$ is a 'good' choice of rational principles.

Several principles are known to have non-trivial theories, including Paris & Vencovská's Invariance Principle [49] and for unary languages Johnson's Sufficient Postulate, JSP (see Appendix A for an explicit description of $Th(JSP)$). The Finite Values Property, the subject of chapter 5, is shown there to be another example. It seems to be a common feature of such principles that they are rather powerful in limiting the agent's choice of probability function, which may suggest that their power comes at the price of the agent accepting a non-tautological creed.

[1]The results from this chapter appeared originally in [26].

[2]This definition is a natural extension of that used by Gaifman & Snir [17] in their discussion on classifying probability functions by their *theories*.

In this chapter we apply this technique to the principle of Spectrum Exchangeability, Sx. Sx first appeared in [42] and was conceived by Nix and Paris as an extension of Ax to polyadic languages. Since then, much work has been done by Paris & Vencovská et al.[3] to determine those probability functions which satisfy this principle, and to investigate its relationships to other principles of PIL. Compared with those principles considered in other chapters then, Sx is not so new, and rather a lot is already known about it, though the approach considered here of identifying its theory has not previously been taken.

It has been shown[4] that all probability functions which satisfy Sx are comprised of a mixture of two essential types: heterogeneous and homogeneous functions (definitions are given below). We will identify the theories of these two types, and show that the theory of heterogeneity is equal to the theory of finite structures for L, i.e. those sentences true in all finite structures for L, which in turn is equal to the theory of Sx. It follows as a corollary that the principle of Sx is incompatible with that of Super-Regularity (that all non-contradictory sentences should be assigned non-zero probability).

Recall from §1.1 that $\mathcal{T}L$ denotes the set of structures for L with universe $\{a_1, a_2, a_3, \ldots\}$, where the symbol a_i is interpreted as the individual a_i. Notice that by the Löwenheim-Skolem Theorems, if $\Gamma \subseteq SL$ and infinitely many constants are not mentioned by the sentences of Γ, then Γ has a model just if it has a model in $\mathcal{T}L$. With this limitation on Γ, $\mathcal{T}L$ is *complete*, in the sense that if a sentence ψ is true in all models of Γ in $\mathcal{T}L$ then it is true in all models of Γ, and hence formally provable from Γ (indeed from some finite subset of Γ). This property of $\mathcal{T}L$ will be needed in what follows.

Recall the notation and terminology from §1.1 concerning extensions and restrictions of state descriptions, and their spectra. From §1.2 we restate

[3] See [33], [35], [36], [37], [40], [46], or for an overview [49].
[4] See [33], [35], [40], [42], [49].

The Principle of Spectrum Exchangeability, Sx

A probability function w on SL satisfies Spectrum Exchangeability if, for any state descriptions $\Theta(b_1, b_2, \ldots, b_m), \Phi(b_1, b_2, \ldots, b_m)$ such that $\mathcal{S}(\Theta) = \mathcal{S}(\Phi)$

$$w(\Theta) = w(\Phi).$$

The forthcoming Theorem 27 states that any probability function on SL which satisfies Sx may be expressed as a convex sum of probability functions of two basic types: *heterogeneous* and *homogeneous* functions, defined as follows.

Homogeneity

A probability function w on SL is homogeneous (abbreviated hom) if it satisfies Sx and for each $t \in \mathbb{N}^+$

$$\lim_{n\to\infty} w\left(\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_n))|=t} \Phi(a_1, \ldots, a_n)\right) = 0. \tag{4.1}$$

The disjunction is taken over all state descriptions of L for constants $a_1, \ldots, a_n$ with spectrum length t.

Heterogeneity

For $t \in \mathbb{N}^+$, a probability function w on SL is t-heterogeneous (abbreviated t-het) if it satisfies Sx and

$$\lim_{n\to\infty} w\left(\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_n))|=t} \Phi(a_1, \ldots, a_n)\right) = 1. \tag{4.2}$$

Again, the disjunction is taken over all state descriptions of L for constants $a_1, \ldots, a_n$ with spectrum length t.

The following result, known as The Ladder Theorem, is due to Landes, Nix, Paris & Vencovská. [5]

[5] See [33], [35], [40], [42], [49].

Theorem 27. *Any probability function w satisfying Sx can be expressed in the form*

$$w = \eta_0 w^{[0]} + \sum_{t=1}^{\infty} \eta_t w^{[t]}$$

where each $\eta_i \geq 0$, $\sum_i \eta_i = 1$, $w^{[0]}$ is homogeneous and $w^{[t]}$ is t-heterogeneous for $t > 0$. Furthermore this representation is unique up to a free choice of the $w^{[i]}$ when $\eta_i = 0$.

We proceed to identify the theory of Sx, that is the set

$$Th(Sx) = \{\theta \in SL \mid w \text{ satisfies Sx} \implies w(\theta) = 1\},$$

by separate investigations into the theory of t-heterogeneity, $Th(t\text{-}het)$, and the theory of homogeneity, $Th(hom)$, defined analogously.

4.1 The theory of t-heterogeneity

We begin by stating some established results concerning t-heterogeneous probability functions, which will be needed below. The following theorem of Landes et al. [35] states that any convex mixture of the $v^{\bar{p},L}$ probability functions (1.16), as $\bar{p}$ ranges over $\mathbb{B}_t$, is t-heterogeneous, and conversely.

Theorem 28. *Let w be a t-heterogeneous probability function on SL. Then there is a measure μ on the Borel*[6] *subsets of $\mathbb{B}_t$ such that*

$$w = \int_{\mathbb{B}_t} v^{\bar{p},L} \, d\mu(\bar{p}).$$

Conversely, given such a measure μ, w defined as above is a t-heterogeneous probability function on SL.

The next result follows from the above theorem and the definition of the $v^{\bar{p},L}$ functions (1.16), using the fact that there will be some $\vec{c} \in (\mathbb{N}_t)^m$ consistent with $\Phi(b_1, \ldots, b_m)$ just if $|\mathcal{S}(\Phi)| \leq t$.

[6] That is, the closure under complement and countable unions of the open subsets of $\mathbb{B}_t$. This ensures that the functions $\bar{p} \mapsto v^{\bar{p},L}(\theta)$ are integrable with respect to μ for $\theta \in SL$.

Lemma 29. *If w is a t-heterogeneous probability function on SL then*

- $w(\Phi(b_1, \dots, b_m)) > 0$ *for any state description* $\Phi(b_1, \dots, b_m)$ *with spectrum length* $|\mathcal{S}(\Phi)| \leq t$,
- $w(\Psi(b_1, \dots, b_m)) = 0$ *for any state description* $\Psi(b_1, \dots, b_m)$ *with spectrum length* $|\mathcal{S}(\Psi)| > t$.

It follows immediately that heterogeneous functions do not satisfy Reg.

We now state a technical result proved by Paris & Rad in [45].

Lemma 30. *Let $t \in \mathbb{N}^+$, let g be the largest arity of any relation symbol in L, and let k be the largest of $t+1$ and g. Then for any $m \geq k$ and any state description $\Phi(a_1, \dots, a_m)$ with spectrum length $|\mathcal{S}(\Phi)| \geq k \geq t+1$, there exists some s with $k \leq s \leq k+g = max\{t+1+g,\ 2g\}$, and some distinct $1 \leq i_1, \dots, i_s \leq m$ such that*

$$|\mathcal{S}(\Phi[a_{i_1}, \dots, a_{i_s}])| = s.$$

The significance of this result is that if we let $s(t) = max\{t+1+g, 2g\}$, then any state description of L with spectrum length greater than t must have a restriction to $s(t)$ or fewer constants, with spectrum length greater than t. Therefore, the following (finite) sentence ζ_t may be used to express the idea that some state formula of spectrum length t is instantiated, and that any instantiated state formula for *any number of variables* must have spectrum length at most t.

Let ζ_t be the sentence

$$\bigvee_{\substack{\Theta(z_1,\dots,z_t) \\ \mathcal{S}(\Theta)=\mathbf{1}_t}} \left(\exists x_1, \dots, x_t \, \Theta(x_1, \dots, x_t) \wedge \forall y_1, \dots, y_{s(t)} \bigvee_{\substack{\sigma:\{y_1,\dots,y_{s(t)}\} \\ \to\{z_1,\dots,z_t\}}} \Theta_\sigma(y_1, \dots, y_{s(t)}) \right)$$

where the outermost disjunction is over all state formulae with spectrum $\mathbf{1}_t$, and for $\sigma : \{y_1, \dots, y_{s(t)}\} \to \{z_1, \dots, z_t\}$ with image $\{z_{i_1}, \dots, z_{i_m}\}$, $\Theta_\sigma(y_1, \dots, y_{s(t)})$ is the unique (up to logical equivalence) state formula $\Psi(y_1, \dots, y_{s(t)})$ such that

$$\Psi(\sigma(y_1), \dots, \sigma(y_{s(t)})) \equiv \Theta[z_{i_1}, \dots, z_{i_m}].$$

In more detail, ζ_t firstly says that there are some $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ satisfying some state formula Θ of spectrum length t, so the a_{i_j} are all distinguished from each other by Θ. Additionally, ζ_t then says that if we take any $b_1, b_2, \ldots, b_{s(t)}$ then (taken together) they all look like clones of certain of the $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$. As a consequence of the choice of $s(t)$, by Lemma 30, this forces that *any* number of $b_1, b_2, \ldots, b_m$ must look like clones of certain of the $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$, in other words the universe has just t distinguishable elements in it (for example these $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$) and all other elements are clones of these.

The following result shows that the sentence ζ_t belongs to $Th(t\text{-}het)$.

Lemma 31. *If w is a t-heterogeneous probability function, then*

$$w(\zeta_t) = 1.$$

Proof. Suppose w is a t-heterogeneous probability function on SL and let $\Phi(a_1, \ldots, a_n)$ be a state description of L with spectrum length t. Then, by restricting Φ to one representative, $a_{g_1}, \ldots, a_{g_t}$, of each equivalence class of $\sim_\Phi$, we obtain a state description $\Theta(a_{g_1}, \ldots, a_{g_t})$ with spectrum $\mathbf{1}_t$. Furthermore, since the members of each equivalence class of $\sim_\Phi$ are indistinguishable from each other according to Φ, there is a unique surjective map $\sigma : \{a_1, \ldots, a_n\} \to \{z_1, \ldots, z_t\}$ such that $\Phi(a_1, \ldots, a_n) \equiv \Theta_\sigma(a_1, \ldots, a_n)$. Therefore

$$\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_n))|=t} \Phi(a_1, \ldots, a_n) \models$$

$$\bigvee_{\mathcal{S}(\Theta(z_1,\ldots,z_t))=\mathbf{1}_t} \left(\exists x_1, \ldots, x_t\, \Theta(x_1 \ldots, x_t) \wedge \bigvee_{\substack{\sigma:\{a_1,\ldots,a_n\} \\ \to\{z_1,\ldots,z_t\}}} \Theta_\sigma(a_1, \ldots, a_n) \right).$$

In fact, this last point applies not just to $\langle a_1, \ldots, a_n \rangle$, but to any tuple of constants of any length taken from $\{a_1, \ldots, a_n\}$, regardless of ordering or repeats. So for any tuple $\langle a_{i_1}, \ldots, a_{i_{s(t)}} \rangle \in \{a_1, \ldots, a_n\}^{s(t)}$, there is always some mapping σ (not necessarily surjective) such that

$$\Phi[a_{i_1}, \ldots, a_{i_{s(t)}}] \equiv \Theta_\sigma(a_{i_1}, \ldots, a_{i_{s(t)}}),$$

and so

$$\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_n))|=t} \Phi(a_1,\ldots,a_n) \models$$

$$\bigvee_{\mathcal{S}(\Theta(z_1,\ldots,z_t))=\mathbf{1}_t} \left(\exists x_1,\ldots,x_t \, \Theta(x_1\ldots,x_t) \wedge \bigwedge_{i_1,\ldots,i_{s(t)}\leq n} \quad \bigvee_{\substack{\sigma:\{a_{i_1},\ldots,a_{i_{s(t)}}\} \\ \to\{z_1,\ldots,z_t\}}} \Theta_\sigma(a_{i_1},\ldots,a_{i_{s(t)}}) \right).$$

Therefore, by part 3 of Proposition 1,

$$w\left(\bigvee_{|\mathcal{S}(\Phi(a_1,\ldots,a_n))|=t} \Phi(a_1,\ldots,a_n) \right) \leq$$

$$w\left(\bigvee_{\mathcal{S}(\Theta(z_1,\ldots,z_t))=\mathbf{1}_t} (\exists x_1,\ldots,x_t \, \Theta(x_1\ldots,x_t) \ \wedge \bigwedge_{i_1,\ldots,i_{s(t)}\leq n} \ \bigvee_{\substack{\sigma:\{a_{i_1},\ldots,a_{i_{s(t)}}\} \\ \to\{z_1,\ldots,z_t\}}} \Theta_\sigma(a_{i_1},\ldots,a_{i_{s(t)}})) \right).$$

Taking the limit as $n\to\infty$ now gives

$$1 \leq w(\zeta_t)$$

by (4.2), since w is t-heterogeneous, and hence $w(\zeta_t)=1$ since $w(\zeta_t)\in[0,1]$. $\Box$

It follows immediately from this that any sentence which is a logical consequence of ζ_t is also a member of $Th(t\text{-}het)$:

$$\{\theta\in SL \mid \zeta_t\models\theta\}\subseteq Th(t\text{-}het).$$

We proceed to show that the converse also holds, via a sequence of lemmas, after introducing some notation.

Let T be the set of $\mathbf{1}_t$ state formulae of L, i.e. $T=\{\Theta(z_1\ldots,z_t)\mid\mathcal{S}(\Theta)=\mathbf{1}_t\}$, and define an equivalence relation $\approx$ on T by

$$\Theta\approx\Phi \iff \Theta(z_1\ldots,z_t)\equiv\Phi(z_{\tau(1)}\ldots,z_{\tau(t)})$$

for some permutation τ of $\mathbb{N}_t$. Let $T_1,\ldots,T_u$ denote the equivalence classes of $\approx$, and for $1\leq j\leq u$ let $\Theta_j\in T_j$ be some representative of its equivalence class.

Let

$$\eta_t^j = \exists x_1, \ldots, x_t \, \Theta_j(x_1, \ldots, x_t)$$

and

$$\xi_t^j = \forall y_1, \ldots, y_{s(t)} \bigvee_{\substack{\sigma:\{y_1,\ldots,y_{s(t)}\} \\ \to\{z_1,\ldots,z_t\}}} (\Theta_j)_\sigma(y_1, \ldots y_{s(t)})$$

and let

$$\zeta_t^j = \eta_t^j \wedge \xi_t^j.$$

Since

$$\exists x_1, \ldots, x_t \, \Theta(x_1, \ldots, x_t) \wedge \forall y_1, \ldots, y_{s(t)} \bigvee_{\substack{\sigma:\{y_1,\ldots,y_{s(t)}\} \\ \to\{z_1,\ldots,z_t\}}} \Theta_\sigma(y_1, \ldots, y_{s(t)})$$

is logically equivalent to

$$\exists x_1, \ldots, x_t \, \Phi(x_1, \ldots, x_t) \wedge \forall y_1, \ldots, y_{s(t)} \bigvee_{\substack{\sigma:\{y_1,\ldots,y_{s(t)}\} \\ \to\{z_1,\ldots,z_t\}}} \Phi_\sigma(y_1, \ldots, y_{s(t)})$$

whenever $\Theta,\ \Phi \in T$ and $\Theta \approx \Phi$, it follows that ζ_t is equivalent to the disjunction of the pairwise disjoint sentences ζ_t^j

$$\zeta_t \equiv \bigvee_{j=1}^{u} \zeta_t^j = \bigvee_{j=1}^{u} (\eta_t^j \wedge \xi_t^j). \tag{4.3}$$

Let $M^j \in \mathcal{T}L$ be a model of ζ_t^j. Then $M^j \models \exists x_1, \ldots, x_t \, \Theta_j(x_1, \ldots, x_t)$, so suppose that $M^j \models \Theta_j(a_{g_1}, \ldots, a_{g_t})$. Since

$$M^j \models \xi_t^j, \tag{4.4}$$

for any constant symbol a_i there exists a unique $\sigma(a_i) \in \{a_{g_1}, \ldots, a_{g_t}\}$ such that

$$M^j \models (\Theta_j)_\sigma(a_{g_1}, \ldots, a_{g_t}, a_i).$$

Note that $\sigma(a_k) = a_k$ for $a_k \in \{a_{g_1}, \ldots, a_{g_t}\}$, and so $\sigma^2 = \sigma$.

Furthermore, the $\sigma(a_i)$ and a_i are indistinguishable in M^j, in the sense that for any state formula $\Phi(x_1, \ldots, x_{v+1})$ and $a_{k_1}, \ldots, a_{k_v}$

$$M^j \models \Phi(a_i, a_{k_1}, \ldots, a_{k_v}) \iff M^j \models \Phi(\sigma(a_i), a_{k_1}, \ldots, a_{k_v}). \tag{4.5}$$

For if this were not the case, there must exist some constants $a_{k_1}, \ldots, a_{k_v}$ such that, for the unique state description $\Phi^+(a_i, \sigma(a_i), a_{k_i}, \ldots, a_{k_v})$ such that $M^j \models \Phi^+$,

$$a_i \not\sim_{\Phi^+} \sigma(a_i).$$

This would mean that, for the unique state description $\Psi(a_{g_1}, \ldots, a_{g_t}, a_i, a_{k_1}, \ldots, a_{k_v})$ such that $M^j \models \Psi$, $|\mathcal{S}(\Psi)| > t$, since $a_{g_1}, \ldots, a_{g_t}$ are all distinguishable in Ψ, and a_i is distinguishable from each of them. This contradicts (4.4) by the above discussion regarding the choice of $s(t)$.

This discussion now yields:

Lemma 32. *For any $\psi(a_1, \ldots, a_n) \in SL$*

$$M^j \models \psi(a_1, \ldots, a_n) \iff M^j \models \psi(\sigma(a_1), \ldots, \sigma(a_n)).$$

Proof. Straightforward by induction on the length of ψ using (4.5). □

We now define a new structure A^j for L with universe $|A^j| = \{a_{g_1}, \ldots, a_{g_t}\}$ by taking the interpretation of a_i in A^j to be $\sigma(a_i)$ and the interpretation of R_k in A^j to be the interpretation of R_k in M^j restricted to $|A^j|$. Essentially, A^j is M^j with all the indistinguishable elements of M^j amalgamated into a single element. From this follows

Lemma 33. *For any $\psi(a_1, \ldots, a_n) \in SL$*

$$M^j \models \psi(a_1, \ldots, a_n) \iff A^j \models \psi(a_1, \ldots, a_n).$$

Proof. Straightforward by induction on the length of ψ using Lemma 32 . □

Since $|A^j|$ is finite with every element named by a constant, when referring to the truth of sentences in A^j we can replace existential and universal quantifiers by finite disjunctions and conjunctions respectively. This observation gives us:

Lemma 34. *For any $n \in \mathbb{N}$ and any $\theta(a_1, \ldots, a_n) \in SL$, there exists a quantifier-free formula $\theta'(x_1, \ldots, x_t, a_1, \ldots, a_n)$ such that*

$$M^j \models \theta'(a_{g_1}/x_1, \ldots, a_{g_t}/x_t, a_1, \ldots, a_n) \iff M^j \models \theta(a_1, \ldots, a_n).$$

Proof. Let $\theta(a_1,\ldots,a_n) \in SL$ for some $n \in \mathbb{N}$ (possibly zero). Assume, without loss of generality, that θ is in Prenex Normal Form, so

$$\theta(a_1,\ldots,a_n) =$$
$$Q_p\, z_{p,1},\ldots,z_{p,n_p}\ \ldots, z_{p-1,n_{p-1}},\ldots, Q_1\, z_{1,1},\ldots,z_{1,n_1}\, \phi(z_{1,1},\ldots,z_{p,n_p},a_1,\ldots,a_n),$$

where each Q_i is either $\forall$ or $\exists$ and $\phi(\vec{z},a_1,\ldots,a_n) \in QFFL$.

Now let θ' be θ with each occurrence of

$$\forall z_{k,1},\ldots,z_{k,n_k} \qquad \text{replaced by} \qquad \bigwedge_{\langle z_{k,1},\ldots,z_{k,n_k}\rangle \in \{x_1,\ldots,x_t\}^{n_k}}$$

and each occurrence of

$$\exists z_{k,1},\ldots,z_{k,n_k} \qquad \text{replaced by} \qquad \bigvee_{\langle z_{k,1},\ldots,z_{k,n_k}\rangle \in \{x_1,\ldots,x_t\}^{n_k}}$$

for $k = 1,\ldots,p$, so that $\theta'(x_1,\ldots,x_t,a_1,\ldots,a_n)$ is a quantifier free formula mentioning constants from $a_1,\ldots,a_n$ and free variables $x_1,\ldots,x_t$ (only).

Then for M^j, A^j and σ as above,

$$\begin{aligned} M^j \models \theta'(a_{g_1}/x_1,\ldots,a_{g_t}/x_t,a_1,\ldots,a_n) &\iff A^j \models \theta'(a_{g_1}/x_1,\ldots,a_{g_t}/x_t,a_1,\ldots,a_n) \\ &\iff A^j \models \theta(a_1,\ldots,a_n) \\ &\iff M^j \models \theta(a_1,\ldots,a_n) \end{aligned}$$

where the first and third implications hold by Lemma 33 and the second holds since $|A^j| = \{a_{g_1},\ldots,a_{g_t}\}$. □

Lemma 35. *For $\theta(a_1,\ldots,a_n) \in SL$ and some fixed $\sigma : \{z_1,\ldots,z_t,a_1,\ldots,a_n\} \to \{z_1,\ldots,z_t\}$*

$$\zeta_t^j \wedge \exists\vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z},a_1,\ldots,a_n)) \models \theta(a_1,\ldots,a_n)$$

or

$$\zeta_t^j \wedge \exists\vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z},a_1,\ldots,a_n)) \models \neg\theta(a_1,\ldots,a_n).$$

Proof. The result is clear if the left hand side is inconsistent. Assume otherwise, so by the remark following (4.4), σ must be the identity on the z_i. Let $M^j \in \mathcal{T}L$ be a model of ζ_t^j such that

$$M^j \models \zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n)).$$

Then from Lemma 34, since the representatives $a_{g_1}, \ldots, a_{g_t}$ were arbitrary up to satisfying Θ_j in M^j, we have that for θ' as given there,

$$M^j \models \forall \vec{z}\,(\Theta_j(\vec{z}) \to (\theta(a_1, \ldots, a_n) \leftrightarrow \theta'(\vec{z}, a_1, \ldots, a_n)))$$

regardless of the particular map σ. By earlier remarks concerning the completeness of the structures in $\mathcal{T}L$ this gives

$$\zeta_t^j \models \forall \vec{z}\,(\Theta_j(\vec{z}) \to (\theta(a_1, \ldots, a_n) \leftrightarrow \theta'(\vec{z}, a_1, \ldots, a_n))). \tag{4.6}$$

Since θ' is quantifier free and

$$\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n)$$

decides all the $\pm R_k(u_1, \ldots, u_{r_k})$ for $u_1, \ldots, u_{r_k}$ from $a_1, \ldots, a_n, z_1, \ldots, z_t$, it also decides θ' so

$$\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n) \models \theta' \quad \text{or} \quad \Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n) \models \neg\theta'.$$

Hence by (4.6),

$$\zeta_t^j \wedge \Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n) \models \theta \quad \text{or} \quad \zeta_t^j \wedge \Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n) \models \neg\theta,$$

and since $\vec{z}$ does not appear in θ we obtain that

$$\zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n)) \models \theta(a_1, \ldots, a_n)$$
$$\text{or} \quad \zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n)) \models \neg\theta(a_1, \ldots, a_n).$$

□

We can now prove the converse to our earlier observation that for any t-heterogeneous probability function w, if $\zeta_t \models \theta(\vec{a})$ then $w(\theta(\vec{a})) = 1$.

Lemma 36. *If w is a t-heterogeneous probability function, $\theta(a_1, \dots, a_n) \in SL$ and $w(\theta(\vec{a})) = 1$ then $\zeta_t \models \theta(\vec{a})$.*

Proof. If $\zeta_t^j \not\models \theta(\vec{a})$ for some $1 \leq j \leq u$ then, since

$$\zeta_t^j \equiv \zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge \bigvee_\sigma (\Theta_j)_\sigma(\vec{z}, a_1, \dots, a_n)),$$

there must be some σ for which

$$\zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \dots, a_n)) \not\models \theta(\vec{a}).$$

Hence this left hand side must be consistent and by Lemma 35

$$\zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \dots, a_n)) \models \neg\theta(\vec{a}). \tag{4.7}$$

Any extension, $\Psi(a_1, \dots, a_r)$, of $(\Theta_j)_\sigma(a_1, \dots, a_n)$ which has spectrum length t and is consistent with ζ_t^j is inconsistent with each ζ_t^k for $k \neq j$. Therefore, if

$$w(\zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \dots, a_n))) = 0,$$

then $w(\Psi(a_1, \dots, a_r)) = 0$. But since Ψ is a state description with spectrum length t, this is false by Lemma 29. Therefore by (4.7),

$$w(\neg\theta(\vec{a})) \geq w(\zeta_t^j \wedge \exists \vec{z}\,(\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \dots, a_n))) > 0.$$

The result follows. □

Since ζ_t does not mention any constants we obtain from Lemmas 31 and 36 the following:

Corollary 37. *If w is a t-heterogeneous probability function, $\theta(\vec{a}) \in SL$ and $w(\theta(\vec{a})) = 1$ then $\zeta_t \models \forall \vec{x}\, \theta(\vec{x})$, and so $w(\forall \vec{x}\, \theta(\vec{x})) = 1$.*

Also from Lemmas 31 and 36 we now obtain:

Theorem 38.

$$Th(t\text{-}het) = \{\theta \in SL \mid \zeta_t \models \theta\}.$$

Let $Th(Fin)$ denote the theory of finite structures; the set of sentences of L true in every finite structure for L. We now show that this set is equal to the intersection over t of the theories of t-heterogeneity.

Theorem 39.

$$Th(Fin) = \bigcap_{t \in \mathbb{N}^+} Th(t\text{-}het).$$

Proof. Suppose that $\theta(a_1, \ldots, a_n) \in Th(Fin)$. Let $M^j \in \mathcal{T}L$ be an arbitrary model of ζ_t^j, and let A^j be defined in terms of M^j as above (p77). Then $|A^j|$ is finite, so $A^j \models \theta(a_1, \ldots, a_n)$. By Lemma 33 then $M^j \models \theta(a_1, \ldots, a_n)$, so since t, j and M^j are arbitrary, by earlier remarks concerning the completeness of $\mathcal{T}L$, $\zeta_t \models \theta(a_1, \ldots, a_n)$ for each $t \in \mathbb{N}^+$. Therefore by Theorem 38, $\theta(a_1, \ldots, a_n) \in \bigcap_{t \in \mathbb{N}^+} Th(t\text{-}het)$.

Conversely, suppose that $\theta(a_1, \ldots, a_n) \in Th(t\text{-}het)$ for each $t \in \mathbb{N}^+$. Let M be a finite structure for L, say M has exactly t distinguishable elements. Then M must be a model of ζ_t, so $M \models \theta(a_1, \ldots, a_n)$, giving $\theta(a_1, \ldots, a_n) \in Th(Fin)$. The result follows. $\square$

Notice that by Trakhtenbrot's Theorem [55], $Th(Fin)$ is complete Π_1^0, so cannot be recursively axiomatized.

We now move on to consider the other essential type of probability functions satisfying Sx, the homogeneous functions.

4.2 The theory of Homogeneity

It is apparent from the definition of homogeneity (4.1) that there can be no homogeneous function on a purely unary language, since the spectrum length of any state description of a unary language can never exceed 2^q. Therefore, assume for this section that the language L contains at least one non-unary relation symbol.

We begin by stating an established result concerning homogeneous probability functions, which will be needed subsequently. Let

$$\mathbb{B}_\infty = \{\langle p_0, p_1, p_2, \ldots \rangle \in \mathbb{B} \mid p_0 > 0 \text{ or } p_i > 0 \text{ for all } i > 0\}.$$

The following theorem of Landes et al. [35] states that any convex mixture of the $u^{\bar{p},L}$ probability functions (1.15), as $\bar{p}$ ranges over $\mathbb{B}_\infty$, is homogeneous, and conversely.

Theorem 40. *Let w be a homogeneous probability function on SL. Then there is a measure μ on the Borel subsets of $\mathbb{B}_\infty$ such that*

$$w = \int_{\mathbb{B}_\infty} u^{\bar{p},L}\, d\mu(\bar{p}).$$

Conversely given such a measure μ, w defined as above is a homogeneous probability function on SL.

We shall now show that $Th(hom)$ is actually a variation on a theory studied by Fagin [11], Gaifman [15] and Glebskii et al. [18]. Recall that r_e denotes the arity of relation R_e and let $\rho(z_1, z_2)$ be the formula

$$\bigwedge_{e=1}^{q} \bigwedge_{f=1}^{r_e} \forall x_1, \ldots, x_{f-1}, x_{f+1}, \ldots, x_{r_e}$$

$$(R_e(x_1, \ldots, x_{f-1}, z_1, x_{f+1}, \ldots, x_{r_e}) \leftrightarrow R_e(x_1, \ldots, x_{f-1}, z_2, x_{f+1}, \ldots, x_{r_e}))$$

which expresses that z_1 and z_2 are permanently indistinguishable from one another. For $\vec{S} = S_1, \ldots, S_h$ a partition of $\mathbb{N}_m$, let $\upsilon^{\vec{S}}(y_1, \ldots, y_m)$ be the formula

$$\bigwedge_{g=1}^{h} \bigwedge_{i,j \in S_g} \left(\rho(y_i, y_j) \wedge \bigwedge_{u \in \mathbb{N}_m - S_g} \neg \rho(y_i, y_u) \right), \tag{4.8}$$

and for $\Theta(x_1, \ldots, x_m)$ a state formula let

$$\Theta^{\vec{S}}(x_1, \ldots, x_m) = (\Theta(x_1, \ldots, x_m) \wedge \upsilon^{\vec{S}}(x_1, \ldots, x_m)). \tag{4.9}$$

The proof of the following result uses notation introduced with the definition of the $u^{\bar{p},L}$ functions (1.15).

Lemma 41. *Let w be a homogeneous probability function on language L. Then for a partition $\vec{S} = S_1, \ldots, S_h$ of $\mathbb{N}_m$, and $\Theta(x_1, \ldots, x_m)$, $\Psi(x_1, \ldots, x_m, x_{m+1})$ state formulae of L such that $\Psi \models \Theta$ and $\Psi^{\vec{S},\{m+1\}}(x_1, \ldots, x_{m+1})$ is consistent,*

$$w(\forall x_1, \ldots, x_m\, (\Theta^{\vec{S}}(x_1, \ldots, x_m) \to \exists x_{m+1}\, \Psi^{\vec{S},\{m+1\}}(x_1, \ldots, x_m, x_{m+1}))) = 1. \tag{4.10}$$

Proof. By Theorem 40 it is sufficient to show the result for $w = u^{\bar{p},L}$ where $\bar{p} \in \mathbb{B}_\infty$. Given $\bar{p} \in \mathbb{B}_\infty$, consider the following process for constructing a sequence of pairs, each consisting of a state description $\Phi_k(a_1, \ldots, a_k)$ and a sequence of 'colours' $\vec{c}_k \in \mathbb{N}^k$. At stage $k = 0$ choose $\vec{c}_0 = \emptyset$, the empty sequence, and $\Phi_0 = \top$, a tautology. At stage $k+1$ pick c_{k+1} from $\mathbb{N}$ with probability $p_{c_{k+1}}$, and then pick Φ_{k+1} from among those state descriptions consistent with $\vec{c}_{k+1}$ (i.e. those in $\mathcal{C}(\vec{c}_{k+1}, \langle a_1, \ldots, a_k, a_{k+1}\rangle)$) which extend Φ_k, according to the uniform distribution, i.e. with probability

$$|\mathcal{C}(\vec{c}_k, \langle a_1, \ldots, a_k\rangle)| \cdot |\mathcal{C}(\vec{c}_{k+1}, \langle a_1, \ldots, a_k, a_{k+1}\rangle)|^{-1}.$$

(Note that, where $c_{k+1} = 0$ or is previously unseen in $\vec{c}_k$ there is a free choice of all those extensions Φ_{k+1} of Φ_k consistent with $\vec{c}_k$, while if $c_{k+1} > 0$ has occurred previously in $\vec{c}_k$, so that $c_{k+1} = c_r$, say, then Φ_{k+1} must be the unique extension of Φ_k such that a_{k+1} is a clone of a_r, meaning that $a_{k+1} \sim_{\Phi_{k+1}} a_r$.)

It is straightforward to show (as for example in [49, chapter 30]) that the probability that this process results at stage n in a particular pair $\langle \vec{c}_n, \Phi_n\rangle$ is given by

$$|\mathcal{C}(\vec{c}_n, \vec{a})|^{-1} \prod_{i=1}^{n} p_{c_i}.$$

Therefore, the value of

$$u^{\bar{p},L}(\Phi_n(a_1, \ldots, a_n)) = \sum_{\substack{\vec{c} \in \mathbb{N}^n \\ \Phi_n \in \mathcal{C}(\vec{c}, \vec{a})}} |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{i=1}^{n} p_{c_i}$$

is the sum, over all $\vec{c} \in \mathbb{N}^n$ consistent with Φ_n, of the probability of obtaining the pair $\langle \vec{c}, \Phi_n\rangle$ by the process described.

Now suppose that $\vec{S}$ and Ψ are as in the statement of the lemma and that this process has produced the pair $\langle \vec{c}, \Theta(a_1, \ldots, a_m)\rangle$ with probability

$$|\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{i=1}^{m} p_{c_i}. \tag{4.11}$$

In the case where $\vec{c}$ 'matches' $\vec{S}$, in the sense that $c_i = c_j \neq 0$ just if i, j are in the same part of $\vec{S}$, then for any new (previously unseen in $\vec{c}$) colour c_{m+1}, or 0, there is a fixed probability $1/C$ of picking $\Psi(a_1, \ldots, a_m, a_{m+1})$ as the next state description, where $C \geq SD(1) > 1$ is the number of extensions allowed by the process, i.e.

those which are consistent with $\vec{c}$. Similarly there is this same probability at each further choice where c_{m+s} is new, or 0, that the chosen state description will imply $\Psi(a_1, \ldots, a_m, a_{m+s})$.

Since $\bar{p} \in \mathbb{B}_\infty$, either $p_0 > 0$ or there are infinitely many non-zero p_n, so such a sequence of choices will, with probability 1, eventually produce a witness to $\exists x_{m+1}\, \Psi(a_1, \ldots, a_m, x_{m+1})$ which is assigned a different colour from those occurring in $\vec{c}$. Furthermore, with probability 1, any two constants assigned different colours become distinguishable from each other eventually by this process, and no extension obtained by this process will ever witness $\neg \upsilon^{\vec{S}}(a_1, \ldots, a_m)$, since each is consistent with $\vec{c}$. Hence the probability (4.11) will all contribute to

$$u^{\bar{p},L}(\exists x_{m+1}\, (\Psi^{\vec{S},\{m+1\}}(a_1, \ldots, a_m, x_{m+1}))).$$

Otherwise, if for some i, j in different parts of $\vec{S}$ we have $c_i = c_j \neq 0$ then no extension obtained by this process from $\langle \vec{c}, \Theta(a_1, \ldots, a_m) \rangle$ can ever witness (4.8), while if for some i, j in the same S_g we have $c_i \neq c_j$, by this process a_i and a_j must eventually become distinguishable. In either case, the probability (4.11) will all contribute to

$$u^{\bar{p},L}(\neg \upsilon^{\vec{S}}(a_1, \ldots, a_m)).$$

Combining all of these probabilities now gives

$$u^{\bar{p},L}(\Theta^{\vec{S}}(a_1, \ldots, a_m) \to \exists x_{m+1}\, \Psi^{\vec{S},\{m+1\}}(a_1, \ldots, a_m, x_{m+1})) = 1 \tag{4.12}$$

and by Ex this also holds for any distinct $a_{i_1}, \ldots, a_{i_m}$ in place of $a_1, \ldots, a_m$. Where $b_1, \ldots, b_m \in \{a_1, a_2, \ldots\}^m$ are not all distinct, we apply the same reasoning to the restriction of $\Theta(b_1, \ldots, b_m)$ to its distinct arguments $\Theta[b_{j_1}, \ldots, b_{j_s}]$, and find that (4.12) holds also with $b_1, \ldots, b_m$ in place of $a_1, \ldots, a_m$. The result follows. $\square$

Let Δ be the set of all sentences of the form (4.10) with $\vec{S}$, Θ etc. as in Lemma 41. It follows from the previous result that if $\Delta \models \phi$ then $w(\phi) = 1$ for any homogeneous w. We proceed to prove the converse to this using the following result, the first part of which is shown by Fagin in [11], and the second part of which follows from a simple adaptation of his back-and-forth argument. Recall that $SL^{(n)}$ denotes the set of sentences of L which mention only constant symbols from among $a_1, \ldots, a_n$ (so that $SL^{(0)}$ is the set of sentences containing no constant symbols).

Lemma 42. *Δ is complete for $SL^{(0)}$, and if $\Phi(a_1, \ldots, a_n)$ is a state description and $\vec{S}$ is a partition of $\mathbb{N}_n$ then $\Delta \cup \{\Phi^{\vec{S}}(a_1, \ldots, a_n)\}$ is complete for $SL^{(n)}$.*

We now apply this to give the following:

Lemma 43. *If $\phi \in SL$ is such that $w(\phi) = 1$ for some homogeneous probability function w on SL then $\Delta \models \phi$.*

Proof. Suppose that w, a homogeneous probability function on SL, and $\phi \in SL$ are such that $w(\phi) = 1$. Let m be large enough that $\phi \in SL^{(m)}$ and let $L^{(m)}$ be L but with constant symbols a_i only for $i \leq m$. By Lemma 42, for any state description $\Theta(a_1, \ldots, a_m)$ and partition $\vec{S}$ of $\mathbb{N}_m$, $\Delta \cup \{\Theta^{\vec{S}}(a_1, \ldots, a_m)\}$ is complete for $L^{(m)}$ (i.e. decides any sentence of $SL^{(m)}$). Hence, if for any Θ, $\vec{S}$

$$\Delta,\ \Theta^{\vec{S}}(a_1, \ldots, a_m) \not\models \phi,$$

then it must be the case that $\upsilon^{\vec{S}}$ is consistent with Θ and

$$\Delta,\ \Theta^{\vec{S}}(a_1, \ldots, a_m) \models \neg\phi. \tag{4.13}$$

Let $\bar{p} \in \mathbb{B}_\infty$. By the account in the proof of Lemma 41, whenever $\vec{c} \in \mathbb{N}^m$ is such that $c_i = c_j$ just if $a_i,\ a_j$ are in the same part of $\vec{S}$, and $\Psi(a_1, \ldots, a_m, x_{m+1})$ is some extension of Θ, the probability $|\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{i=1}^m p_{c_i} > 0$ all contributes to

$$u^{\bar{p},L}(\exists x_{m+1} \Psi^{\vec{S},\{m+1\}}(a_1, \ldots, a_m, x_{m+1})),$$

and therefore to $u^{\bar{p},L}(\Theta^{\vec{S}}(a_1, \ldots, a_m))$. It follows that $u^{\bar{p},L}(\Theta^{\vec{S}}(a_1, \ldots, a_m)) > 0$ whenever $\upsilon^{\vec{S}}$ is consistent with Θ. Therefore, from (4.13), $u^{\bar{p},L}(\neg\phi) > 0$, so $u^{\bar{p},L}(\phi) < 1$. Hence by the first part of Theorem 40, $w(\phi) < 1$, a contradiction.

From this it follows that for every state description $\Theta(a_1, \ldots, a_m)$ and partition $\vec{S}$ of $\mathbb{N}_m$

$$\Delta,\ \Theta^{\vec{S}}(a_1, \ldots, a_m) \models \phi.$$

Hence

$$\Delta, \bigvee_{\Theta,\vec{S}} \Theta^{\vec{S}}(a_1,\ldots,a_m) \models \phi,$$

giving

$$\Delta \models \phi,$$

since $\bigvee_{\Theta,\vec{S}} \Theta^{\vec{S}}(a_1,\ldots,a_m)$ is a tautology. □

From Lemmas 41 and 43 we now obtain

Theorem 44.

$$Th(hom) = \{\theta \in SL \mid \Delta \models \theta\}.$$

We now combine results from this and the previous section to identify the theory of Sx.

4.3 The theory of Sx

By Theorem 27 it is clear that the theory of Sx must be equal to the intersection of $Th(hom)$ and $Th(t\text{-}het)$ for each $t \in \mathbb{N}^+$. By Theorem 39 this is equal to $Th(hom) \cap Th(Fin)$, and over the course of the next few lemmas we shall show that in fact this is equal to $Th(Fin)$.

Lemma 45. *When L is not purely unary,*

$$\lim_{t\to\infty} \frac{\mathcal{N}(\emptyset, \mathbf{1}_t)}{SD(t)} = 1.$$

Proof. Suppose that a state description $\Phi(a_1,\ldots,a_t)$ is chosen at random from among all state descriptions for t constants. Then for any distinct $1 \le i,j \le t$, the probability that $a_i \sim_\Phi a_j$ is at most 2^{1-2t} (with equality when L consists of a single binary relation). The number of ways of choosing a distinct pair i,j is $t(t-1)/2$, so that the proportion of all state descriptions for $a_1,\ldots,a_t$ where some pair of distinct constants is indistinguishable is bounded above by

$$\frac{t(t-1)}{2^{2t}}.$$

This value tends to zero as $t \to \infty$, and the result follows. □

The next two lemmas show that as $t \to \infty$, $v^{\bar{p},L}(\phi) \to 1$ for each $\bar{p} \in \mathbb{B}_t$ and each $\phi \in \Delta$, and so the probability assigned by these functions to the complete set of logical consequences of Δ also tends to 1. This fact will then be used to derive the main result of this chapter, Theorem 48.

Lemma 46. *For any $\delta > 0$ there exists $N \in \mathbb{N}$ such that for any $t \geq N$, any $\bar{p} \in \mathbb{B}_t$ and any $\theta \in SL$,*

$$|u^{\bar{p},L}(\theta) - v^{\bar{p},L}(\theta)| < \delta.$$

Proof. Let $\delta > 0$ be fixed and let $\theta \in SL$. By a result of Landes [33], for $t \in \mathbb{N}^+$ and any $\bar{p} \in \mathbb{B}_t$

$$u^{\bar{p},L}(\theta) = \frac{\mathcal{N}(\emptyset, \mathbf{1}_t)}{SD(t)} v^{\bar{p},L}(\theta) + \sum_G \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)} v^{G(\bar{p}),L} \tag{4.14}$$

where $G = \{E_1, \ldots, E_{|G|}\}$ runs over the set of partitions of $\mathbb{N}_t$ with $|G| < t$ and $G(\bar{p}) \in \mathbb{B}_{|G|}$ has co-ordinates $\sum_{s \in E_i} p_s$. Since

$$\mathcal{N}(\emptyset, \mathbf{1}_t) + \sum_G \mathcal{N}(\emptyset, 1_{|G|}) = SD(t),$$

by Lemma 45 for t sufficiently large

$$\frac{\mathcal{N}(\emptyset, \mathbf{1}_t)}{SD(t)} = 1 - \sum_G \frac{\mathcal{N}(\emptyset, 1_{|G|})}{SD(t)} > 1 - \delta,$$

and the result follows by (4.14). □

Lemma 47. *Let $\vec{S}$, Θ, Ψ etc. be as in Lemma 41 and let $\delta > 0$. Then for t sufficiently large and any $\bar{p} \in \mathbb{B}_t$,*

$$v^{\bar{p},L}(\forall x_1, \ldots, x_m \, (\Theta^{\vec{S}}(x_1, \ldots, x_m) \to \exists x_{m+1} \, \Psi^{\vec{S},\{m+1\}}(x_1, \ldots, x_m, x_{m+1}))) > 1 - \delta.$$

Proof. By Lemma 46 it is enough to prove the result for $u^{\bar{p},L}$ in place of $v^{\bar{p},L}$. The proof given here is a refinement of that used for Lemma 41, the main difference being that where we had probability 1 in that lemma we will now only have probability close to 1. We estimate just how close as we proceed.

Let t be fixed and large. We consider the process described in the proof of Lemma 41, used to construct a sequence of state descriptions $\Phi_k(a_1, \ldots, a_k)$ and sequences $\vec{c}_k$, this time from $(\mathbb{N}_t)^k$ since $\bar{p} \in \mathbb{B}_t$ is used. As before, the probability that this process results at stage r in a particular pair $\langle \vec{c}_r, \Phi_r \rangle$ is given by

$$|\mathcal{C}(\vec{c}_r, \vec{a})|^{-1} \prod_{i=1}^{r} p_{c_i}, \tag{4.15}$$

and

$$u^{\bar{p},L}(\Phi_r(a_1, \ldots, a_r)) = \sum_{\substack{\vec{c} \in (\mathbb{N}_t)^r \\ \Phi_r \in \mathcal{C}(\vec{c}, \vec{a})}} |\mathcal{C}(\vec{c}, \vec{a})|^{-1} \prod_{i=1}^{r} p_{c_i}$$

is the sum, over each $\vec{c} \in (\mathbb{N}_t)^r$ consistent with Φ_r, of the probability of obtaining the pair $\langle \vec{c}, \Phi_r \rangle$ by the process described.

Let r be sufficiently large that

$$\sum_{\substack{\vec{c} \in (\mathbb{N}_t)^r \\ \{c_1, \ldots, c_r\} \subset \mathbb{N}_t}} \prod_{i=1}^{r} p_{c_i} \;\leq\; \sum_{j=1}^{t} (1 - p_j)^r \;\leq\; \frac{\delta}{2} \tag{4.16}$$

(note that the subset relation here is strict).

Suppose that this process has produced the pair $\langle \vec{c}, \Phi_r \rangle$ where $\vec{c} = \langle c_1, \ldots, c_r \rangle \in (\mathbb{N}_t)^r$ is such that $\{c_1, \ldots, c_r\} = \mathbb{N}_t$, and $\Phi_r(a_1, \ldots, a_r) \in \mathcal{C}(\vec{c}, \vec{a})$. Notice that since all the available colours $1, 2, \ldots, t$ occur in $\vec{c}$, any continuation of this process can only produce clones of constants previously seen in Φ_r, so that Φ_{k+1} is uniquely determined by c_{k+1} and Φ_k for $k \geq r$. Therefore, there is some state formula $\Upsilon(z_1, \ldots, z_t)$ and some distinct $g_1, \ldots, g_t \leq r$ such that $c_{g_u} = u$ for $1 \leq u \leq t$ and $\Phi_r \models \Upsilon(a_{g_1}, \ldots, a_{g_t})$.

Let χ be the sentence

$$\exists z_1, \ldots, z_t \, \Upsilon(z_1, \ldots, z_t) \;\wedge\; \forall y_1, \ldots, y_{s(t)} \bigvee_{\substack{\sigma : \{y_1, \ldots, y_{s(t)}\} \\ \to \{z_1, \ldots, z_t\}}} \Upsilon_\sigma(y_1, \ldots, y_{s(t)})$$

where the notation is as defined above on p73. Then any structure in $\mathcal{T}L$ which models this process for these particular $\vec{c}$ and Φ_r is a model of χ. (The only difference between χ and the sentences ζ_t^j considered in §4.1 is that Υ has spectrum length *at most* t, not necessarily equal to t). It can be shown, as in Lemma 35 and its antecedents, that the sentence

$$\chi \wedge \Phi_r(a_1, \ldots, a_r)$$

is complete for $SL^{(r)}$. In the case that

$$\chi \wedge \Phi_r(a_1, \ldots, a_r) \models \phi(a_1, \ldots, a_r),$$

we say ϕ is *fixed* by Φ_r, and all of the probability (4.15) will contribute to $u^{\bar{p},L}(\phi)$.

We now consider cases. If $\Phi_r(a_1, \ldots, a_r) \not\models \Theta(a_1, \ldots, a_m)$ or $c_i = c_j$ for some $1 \leq i, j \leq m$ in different parts of $\vec{S}$, then $\neg\Theta^{\vec{S}}(a_1, \ldots, a_m)$ is fixed by Φ_r.

Otherwise, if $c_i \neq c_j$ for some $1 \leq i, j \leq m$ in the same S_g, then each time a new colour c_k was chosen for $m < k \leq r$, there was a probability at most C^{-1}, where $C = SD(2,3)/SD(1,2) > 1$ depends only on L, that the choice of $\Phi_k(a_1, \ldots, a_k)$ would not witness the failure of $\rho(a_i, a_j)$ (i.e. would not make a_i and a_j distinguishable). Hence the probability in this case that such Φ_r would *not* fix $\neg\Theta^{\vec{S}}(a_1, \ldots, a_m)$ is at most

$$\binom{m}{2}\left(\frac{1}{C}\right)^{t-m}.$$

Otherwise, if $\Phi_r \models \Theta(a_1, \ldots, a_m)$ and $c_i = c_j$ just if i and j are in the same part of $\vec{S}$ for $1 \leq i, j \leq m$, then with every choice of a new colour c_k for $m < k \leq r$, there was probability at least D^{-1}, where $D = SD(m, m+1) > 1$ depends only on m and L, that

$$\Phi_k(a_1, \ldots, a_k) \models \Psi(a_1, \ldots, a_m, a_k),$$

and a probability of at most E^{-1}, where $E = SD(1,2) > 1$, that a_k is indistinguishable from a_i for some $1 \leq i \leq m$. Therefore, the probability of Φ_r *not* fixing $\exists x_{m+1}\, \Psi^{\vec{S},\{m+1\}}(a_1, \ldots, a_m, x_{m+1})$ in this case is at most

$$\left(\frac{D-1}{D}\right)^{t-m} + m\left(\frac{1}{E}\right)^{t-m}.$$

These estimates have been obtained for the specific constants $a_1, \ldots, a_m$. However, as remarked above, according to Φ_r there are at most t distinguishable constants among $a_1, \ldots, a_r$, and for each of the t^m choices of these (including those with repeated parameters) the same estimates may be obtained, using similar arguments, as for $a_1, \ldots, a_m$. Altogether then, the probability that Φ_r does not fix

$$\forall x_1, \ldots, x_m\, (\Theta^{\vec{S}}(x_1, \ldots, x_m) \to \exists x_{m+1}\, \Psi^{\vec{S},\{m+1\}}(x_1, \ldots, x_m, x_{m+1}))$$

is at most

$$t^m\left(\binom{m}{2}\left(\frac{1}{C}\right)^{t-m}+\left(\frac{D-1}{D}\right)^{t-m}+m\left(\frac{1}{E}\right)^{t-m}\right). \tag{4.17}$$

Hence to within the $\delta/2$, from (4.16), this same upper bound holds for $u^{\bar{p},L}$, and the result follows since (4.17) tends to zero as $t \to \infty$. $\square$

We can now prove the main result of this chapter:

Theorem 48.

$$Th(Sx) = Th(Fin).$$

Proof. Since by Theorems 27 and 39

$$Th(Sx) = Th(hom) \cap \bigcap_{t \in \mathbb{N}^+} Th(t\text{-}het) = Th(hom) \cap Th(Fin),$$

it is enough to show that $Th(Fin) \subseteq Th(hom)$. So suppose $\eta(a_{i_1}, \ldots, a_{i_m}) \in Th(Fin)$. Then by Theorem 39, $w(\eta(a_{i_1}, \ldots, a_{i_m})) = 1$ for every t-heterogeneous w for each $t \in \mathbb{N}^+$. Therefore by Corollary 37, $w(\forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m)) = 1$ for each such w. Hence

$$\forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m) \in Th(Fin)$$

and

$$\eta(a_{j_1}, \ldots, a_{j_m}) \in Th(Fin) \tag{4.18}$$

for any $j_1, \ldots, j_m$, not necessarily distinct.

By Lemma 42, Δ is complete for sentences which do not contain constants, so either

$$\Delta \models \forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m) \quad \text{or} \quad \Delta \models \neg\forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m).$$

Suppose the latter holds. Then by the Compactness Theorem there is a finite subset $\{\phi_1, \ldots, \phi_r\}$ of Δ such that

$$\phi_1, \ldots, \phi_r \models \neg\forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m). \tag{4.19}$$

By Lemma 47 for large enough t and $\bar{p} \in \mathbb{B}_t$,

$$v^{\bar{p},L}(\phi_i) > 1 - (2r)^{-1}$$

for each $1 \leq i \leq r$, so

$$\begin{aligned} v^{\bar{p},L}\left(\bigwedge_{i=1}^{r} \phi_i\right) &= 1 - v^{\bar{p},L}\left(\bigvee_{i=1}^{r} \neg\phi_i\right) \\ &\geq 1 - \sum_{i=1}^{r} v^{\bar{p},L}(\neg\phi_i) \\ &> 1 - \sum_{i=1}^{r} (2r)^{-1} \\ &= 1/2. \end{aligned}$$

From (4.19) then

$$v^{\bar{p},L}(\neg\forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m)) > 1/2,$$

so

$$v^{\bar{p},L}(\neg\eta(a_{j_1}, \ldots, a_{j_m})) > 0$$

for some $a_{j_1}, \ldots, a_{j_m}$, which contradicts (4.18) and Theorem 39, since $v^{\bar{p},L}$ is t-heterogeneous.

Hence it must be that

$$\Delta \models \forall x_1, \ldots, x_m\, \eta(x_1, \ldots, x_m),$$

so

$$\Delta \models \eta(a_{i_1}, \ldots, a_{i_m})$$

and $\eta(a_{i_1}, \ldots, a_{i_m}) \in Th(hom)$ by Theorem 44, as required. □

In fact $Th(hom)$ is a strict superset of $Th(Fin)$ since, for example, the sentence $\forall x(\Theta^{\{1\}}(x) \to \exists y\, \Psi^{\{1\},\{2\}}(x, y))$, where Ψ is an extension of Θ with spectrum length 2, is in $Th(hom)$ but is assigned probability 0 by any 1-heterogeneous probability function.

By Theorem 48 then, $Th(Sx)$ contains more than just tautologies. For example where L contains a binary relation R_k (a similar example can be constructed for any polyadic relation), the conjunction ϕ of

$$\forall x_1 \neg R_k(x_1, x_1), \quad \forall x_1 \exists x_2 R_k(x_1, x_2),$$

$$\forall x_1, x_2, x_3((R_k(x_1, x_2) \wedge R_k(x_2, x_3)) \to R_k(x_1, x_3))$$

expresses that R_k is a strict partial ordering of the universe with no top element and therefore no finite model. Therefore $\neg\phi \in Th(Sx)$, and so $w(\phi) = 0$ for any w satisfying Sx. This gives the following:

Corollary 49. *For w a probability function on a not purely unary language L, if w satisfies Sx then w does not satisfy SReg.*

The identification of the theory of t-heterogeneity as being equal to the theory of Finite Structures seems intuitively reasonable, since it is clear from the definition of a t-heterogeneous function (4.2) that it 'assumes' that the universe is of finite size t. The identification of the theory of homogeneity as the set of logical consequences of Δ demonstrates that it is part of the creed of every homogeneous function that, concerning the properties of individuals expressible in L, 'anything that can occur, will occur'. Both results shed new light on how heterogeneous and homogeneous probability functions may be said to employ certain implicit assumptions in the ways that they assign probabilities.

The result that the intersection of these two theories, and therefore the theory of Sx, is equal to $Th(Fin)$ gives a new perspective on Sx as a principle of reasoning in PIL. It follows that, by adopting Sx, an agent effectively assigns zero belief to the notion that the universe is *demonstrably* infinite. However, this doesn't necessarily entail the view that the universe must be finite, since by (4.1), the homogeneous component of any Sx function permits belief in infinitely many distinguishable individuals.

As noted above, by Trakhtenbrot's Theorem [55], $Th(Fin)$ cannot be recursively axiomatized, so there is no decision procedure to determine whether a general given sentence is a member. However, an example of a non-tautological member has been given, showing that Sx is incompatible with SReg for not purely unary languages. It is curious that these two principles, each of which has a certain appeal as a requirement of rationality, cannot be jointly satisfied. An agent which adopts one must reject the other, which raises the question of how one should 'rationally' choose between them.

Chapter 5

The Finite Values Property

The work presented in this chapter extends observations made by Paris & Vencovská in [49]. They noted there that the set of sentences of a unary language L which mention only constants from among $a_1, \ldots, a_n$ is finite up to logical equivalence for each n, and therefore any probability function on a unary language can take only finitely many values when restricted to such a subset of SL. The same applies to any probability function on a polyadic language which may be expressed in terms of functions on a unary language, for example, by a result of the same authors [49], the t-heterogeneous probability functions introduced in the previous chapter.

The motivation to extend these results arose from the following observation regarding the consequences of Lemma 35 and other results in chapter 4 (the argument outlined here will be given in detail in the forthcoming proof of Proposition 55), which seems to give some insight into *why* this finiteness property is exhibited by the heterogeneous functions.

Recall that $SL^{(n)}$ denotes those sentences of L which contain only constant symbols from among $a_1, \ldots, a_n$. Lemma 35 states that there are certain consistent sentences $\psi^{t,j,\sigma}$ such that, for any sentence $\theta(a_1, \ldots, a_n) \in SL^{(n)}$, either

$$\psi^{t,j,\sigma}(a_1, \ldots, a_n) \models \theta(a_1, \ldots, a_n) \quad \text{or} \quad \psi^{t,j,\sigma}(a_1, \ldots, a_n) \models \neg\theta(a_1, \ldots, a_n). \tag{5.1}$$

(The exact definition of these sentences is not needed for the current discussion.)

Where $t \in \mathbb{N}^+$ and $n \in \mathbb{N}$ are fixed, the set of sentences $\psi^{t,j,\sigma}$ is finite and pairwise disjoint, with the property that for any t-heterogeneous probability function w,

$$w\left(\bigvee_{\langle j,\sigma\rangle} \psi^{t,j,\sigma}\right) = 1. \tag{5.2}$$

If we consider a particular sentence $\theta(a_1, \dots, a_n)$, then by (5.1), we can partition the $\psi^{t,j,\sigma}$ into those which logically imply θ and those which imply its negation, so that if

$$A_\theta = \{\psi^{t,j,\sigma} \mid \psi^{t,j,\sigma} \models \theta\},$$

then any $\psi^{t,j,\sigma}$ not in A_θ must belong to

$$A_{\neg\theta} = \{\psi^{t,j,\sigma} \mid \psi^{t,j,\sigma} \models \neg\theta\}.$$

Combined with (5.2), this leads to the observation that

$$w(\theta(a_1, \dots, a_n)) = \sum_{\psi^{t,j,\sigma} \in A_\theta} w(\psi^{t,j,\sigma}(a_1, \dots, a_n)),$$

that is, the probability assigned by w to θ is *determined by* the probabilities it assigns to the $\psi^{t,j,\sigma}$ and the partition $A_\theta,\ A_{\neg\theta}$.

Since $\theta(a_1, \dots, a_n)$ and w were arbitrary, this yields the conclusion that any t-heterogeneous probability function takes only finitely many values on sentences of $SL^{(n)}$, since there are only finitely many ways of partitioning the $\psi^{t,j,\sigma}$ into A_θ and $A_{\neg\theta}$. We define this property for $n \in \mathbb{N}$ as:

The Finite Values Property for $SL^{(n)}$, FVP$_n$.

A probability function w satisfies FVP_n just if the image of w restricted to $SL^{(n)}$ is finite:

$$|\{w(\theta(a_1, \dots, a_n)) \mid \theta \in SL^{(n)}\}| \leq K_n$$

for some $K_n \in \mathbb{N}^+$.[1,2]

For the t-heterogeneous functions, the above argument applies to any $n \in \mathbb{N}$, resulting in what we will call:

[1]Under the standing assumption of Ex, if a function w satisfies FVP$_n$ then the image of w restricted to the set of sentences containing any fixed $b_1, \dots, b_n$ from $a_1, a_2, \dots$, is also finite.

[2]It is not the case that FVP$_n$ implies FVP$_{n+1}$ in general, though we currently only have counter-examples for specific values of n. We hope that a general counter-example may be found in due course.

The Finite Values Property, FVP.

A probability function w satisfies FVP just if w satisfies FVP$_n$ for all $n \in \mathbb{N}$.[3]

This property may seem rather surprising, since although at each 'level' n, the number of constants in each sentence of $SL^{(n)}$ is bounded, no such restriction is placed on the length or complexity of these sentences. However, we will present several examples where it has been found to hold. We postpone a discussion of whether this property could be considered a principle of rationality until after the following result, which will inform the subject. The following characterization of FVP$_n$ shows that the role of the sentences $\psi^{t,j,\sigma}$ in the example above is essential, and that they have a counterpart wherever FVP$_n$ occurs.

Theorem 50. *A probability function w on SL satisfies FVP$_n$ just if there is a set of sentences*

$$B^{(n)} = \{\phi_1, \ldots, \phi_g\} \subset SL^{(n)}$$

such that

- $w(\phi_i \wedge \phi_j) = 0$ *for any* $1 \leq i < j \leq g$,
- $\sum_{i=1}^{g} w(\phi_i) = 1,$ *and*
- *for any* $\theta \in SL^{(n)}$ *there is a subset* $B_\theta^{(n)}$ *of* $B^{(n)}$ *such that*

$$w\left(\theta \leftrightarrow \bigvee_{\phi \in B_\theta^{(n)}} \phi\right) = 1.$$

Proof. From left to right, suppose that w satisfies FVP$_n$. Let $\psi \in SL^{(n)}$ and let $B' = \{\psi, \neg\psi\}$. If, for any $\phi \in B'$ there exists $\theta \in SL^{(n)}$ such that

$$0 \ < \ w(\phi \wedge \theta), \ w(\phi \wedge \neg\theta) \ < \ w(\phi),$$

then replace ϕ in B' by $\phi \wedge \theta$ and $\phi \wedge \neg\theta$, and repeat this step until no such θ remains. Note that at each stage of this process

$$w\left(\bigvee_{\phi \in B'} \phi\right) = 1$$

[3] The bound K_n may vary with n; the case where a fixed bound holds for all n is considered below as the Strong Finite Values Property (5.19).

and for any distinct $\phi, \eta \in B'$, $w(\phi \wedge \eta) = 0$.

Note also that since w satisfies FVP$_n$, the values taken by w on $SL^{(n)}$ may be listed in order $0 = v_1 < v_2 < \ldots < v_K = 1$. Each time a sentence with probability v_i is removed from B' it is replaced by two sentences of strictly smaller positive probability v_s, v_t with $1 < s, t < i$, so that this process must be finite.

Once this process is completed remove any (at most 1) $\phi \in B'$ such that $w(\phi) = 0$, to obtain a finite set of disjoint sentences

$$B^{(n)} = \{\phi_1, \phi_2, \ldots, \phi_g\} \subset SL^{(n)}$$

such that for any $\theta \in SL^{(n)}$ and any $\phi_j \in B^{(n)}$,

$$w(\theta \wedge \phi_j) \in \{0, w(\phi_j)\}.$$

For $\theta \in SL^{(n)}$ let $B_\theta^{(n)} = \{\phi_j \in B^{(n)} \mid w(\theta \wedge \phi_j) = w(\phi_j)\}$. Then

$$w(\theta) = w\left(\theta \wedge \bigvee_{\phi \in B^{(n)}} \phi\right) = \sum_{\phi \in B^{(n)}} w(\theta \wedge \phi) = \sum_{\phi \in B_\theta^{(n)}} w(\phi)$$

since $w(\theta \wedge \phi) = 0$ for $\phi \in B^{(n)} - B_\theta^{(n)}$.

Furthermore, since

$$w(\phi_j) = w(\phi_j \wedge \theta) + w(\phi_j \wedge \neg\theta)$$

we have

$$B_{\neg\theta}^{(n)} = B^{(n)} - B_\theta^{(n)},$$

so that

$$w(\neg\theta) = \sum_{\phi \in B^{(n)} - B_\theta^{(n)}} w(\phi).$$

Therefore

$$\begin{aligned} &w\left(\neg\theta \vee \bigvee_{\phi \in B_\theta^{(n)}} \phi\right) \\ &\quad = w\left(\bigvee_{\phi \in B^{(n)} - B_\theta^{(n)}} \phi\right) + w\left(\bigvee_{\phi \in B_\theta^{(n)}} \phi\right) - w\left(\bigvee_{\phi \in (B^{(n)} - B_\theta^{(n)}) \cap B_\theta^{(n)}} \phi\right) \\ &\quad = 1, \end{aligned}$$

and similarly

$$w\left(\left(\neg\bigvee_{\phi\in B_\theta^{(n)}}\phi\right)\vee\theta\right)$$
$$=w\left(\bigvee_{\phi\in B^{(n)}-B_\theta^{(n)}}\phi\right)+w\left(\bigvee_{\phi\in B_\theta^{(n)}}\phi\right)-w\left(\bigvee_{\phi\in (B^{(n)}-B_\theta^{(n)})\cap B_\theta^{(n)}}\phi\right)$$
$$=1.$$

The result follows.

In the other direction, it is clear that if $B^{(n)}=\{\phi_1,\ldots,\phi_g\}\subset SL^{(n)}$ is as described in the statement of the result, then for any $\theta\in SL^{(n)}$

$$w(\theta)=\sum_{\phi\in B_\theta^{(n)}}w(\phi),$$

and since the number of possible subsets $B_\theta^{(n)}$ of $B^{(n)}$ is finite, then so is the image of $w\restriction SL^{(n)}$. $\square$

We will call such a set $B^{(n)}\subset SL^{(n)}$ with the properties given in Theorem 50 a set of *n-ions* for w. Note that from the above result, where ϕ is an n-ion for w and $\theta\in SL^{(n)}$, $B_{\theta\wedge\phi}^{(n)}$ is either equal to $\{\phi\}$ or to $\emptyset$, so that

$$w(\theta\wedge\phi)\in\{0,w(\phi)\}. \tag{5.3}$$

Theorem 50 shows that if a function w satisfies FVP$_n$, its n-ions correspond to various ‘possible worlds’, in each of which w is able to ‘decide’ every $\theta\in SL^{(n)}$, so that the probability it assigns to any such θ is the sum of the probabilities assigned to those worlds where θ is decided positively. This demonstrates that there is an underlying simplicity to those functions which satisfy FVP, beyond the superficial simplicity evident in its definition, in that it entails a rather ‘neat’, and arguably natural, way of assigning probabilities.

An agent which employs a probability function satisfying FVP has a fixed, finite set of possible worlds with which to compare any $\theta\in SL^{(0)}$, and decides on doing so which of either θ or its negation must hold. Each of these 'worlds' can then be split into various further possible worlds according to the number of constants under consideration in a given sentence $\phi \in SL^{(n)}$, to obtain similar findings regarding where ϕ does and doesn't hold.

It seems unclear whether FVP, considered as a principle of rationality, can have the same status as other principles of PIL since it prescribes not how a function should behave but, in a sense, how it *works*. One might object that it is not really a principle to be adopted, but just a convenient technical feature of certain functions. However, the same could perhaps be said of the principle of Language Invariance, which nonetheless holds an established place in PIL.

Simplicity, as a feature of probability functions used to model rational belief, was endorsed by Kemeny in [30], and considered by Paris & Vencovská in [47], but seems otherwise to have received little attention in Inductive Logic. Kemeny is not explicit about what constitutes simplicity, and the notion discussed by Paris & Vencovská is rather different from that considered here in relation to FVP. With these and likely other different ideas of simplicity available it would be reckless to claim without qualification that simplicity is always a desirable feature of probability functions, in fact in §5.3 we reach the opposite conclusion in the case of the Strong Finite Values Property. However, the particular simplicity entailed by FVP and interpreted above in terms of systematic reasoning about 'possible worlds', seems to be an appealing and arguably a rational feature, which we proceed to investigate.

We begin by proving the results mentioned above, relating to FVP for probability functions on unary languages, and others expressible in terms of these. We subsequently obtain classification results regarding FVP for heterogeneous and homogeneous probability functions, and then for general functions satisfying Sx. From these we find that, like Sx, FVP is inconsistent with the principle of SReg. Finally, we consider the Strong Finite Values Property, where the same bound on the size of the image of $w \restriction SL^{(n)}$ holds for all n.

5.1 FVP for unary languages

The following unsurprising lemma will prove useful in what follows.

Lemma 51. *If w is a finite convex sum of probability functions on SL which all satisfy FVP_n (FVP), then w satisfies FVP_n (FVP).*

Proof. Suppose

$$w = \sum_{i=1}^{k} \lambda_i \, w_i$$

for some $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$, and some probability functions $w_1, \ldots, w_k$.

Suppose in the first case that each w_i satisfies FVP$_n$, and let K_n be an upper bound for the image of $w_i \restriction SL^{(n)}$ for each $i = 1, \ldots, k$. Then

$$|\{w(\theta) \mid \theta \in SL^{(n)}\}| = |\{\lambda_1 w_1(\theta) + \ldots + \lambda_k w_k(\theta) \mid \theta \in SL^{(n)}\}| \leq (K_n)^k$$

and so w satisfies FVP$_n$.

Suppose, in the second case, that each w_i satisfies FVP. Then each satisfies FVP$_n$ for each $n \in \mathbb{N}^+$ so by the above argument w satisfies FVP$_n$ for each $n \in \mathbb{N}^+$, and therefore w satisfies FVP. □

We now give a proof of a result mentioned by Paris & Vencovská in [49] (and above): that $SL^{(n)}$ is finite (up to logical equivalence) for each n. It follows that all probability functions on unary languages, or expressible in terms of such, must satisfy FVP.

Lemma 52. *If L is unary and $n \in \mathbb{N}$ then $SL^{(n)}$ is finite up to logical equivalence.*

Proof. Suppose L is a unary language and $\theta(a_1, \ldots, a_n) \in SL^{(n)}$. Since L is unary, by Proposition 2, θ is logically equivalent to some sentence θ' of the form

$$\bigvee_{k=1}^{l} \left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_{k_j}} x \, \alpha_j(x) \wedge \bigwedge_{i=1}^{n} \alpha_{f_{k_i}}(a_i) \right),$$

where each $\vec{\epsilon_k} \in \{0,1\}^n$, $\exists^1$ stands for $\exists$, $\exists^0$ stands for $\neg\exists$, and the disjuncts are disjoint and satisfiable.

Let D be the set of possible disjuncts (up to logical equivalence) $\bigwedge_{j=1}^{2^q} \exists^{\epsilon_j} x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{n} \alpha_{f_i}(a_i)$. Since there are 2 choices for ϵ_j for each $j = 1, \ldots, 2^q$, and 2^q choices for α_{f_i} for each $i = 1, \ldots, n$, this gives

$$|D| = 2^{2^q + qn}.$$

Since θ is logically equivalent to the disjunction of some subset of D, the size of $SL^{(n)}$ up to logical equivalence is bounded by the number $2^{|D|}$ of distinct subsets of D. Therefore the number of logical equivalence classes of $SL^{(n)}$ is finite. □

Since n may take any value from $\mathbb{N}$, by replacing $a_1, \ldots, a_n$ with any distinct n-tuple of constant symbols $b_1, \ldots, b_n$, Lemma 52 tells us that for any finite tuple $\langle b_1, \ldots, b_n \rangle$ of constants, despite the fact that the length and complexity of sentences mentioning these constants is unlimited, there are in fact only finitely many propositions which may be expressed about these constants in a unary language L. From this follows:

Corollary 53. *If L is unary and w is a probability function on SL then w satisfies FVP.*

Furthermore, these results are applicable in certain cases to probability functions on polyadic languages. As shown by Nix & Paris in [42], some such functions have a representation in terms of, also called *a reduction to*, a probability function on a unary language. Specifically, a probability function w on a polyadic language L has a reduction to a probability function on a unary language L_0 when there are probability functions $v_1, \ldots, v_k$ on SL_0, some $\vec{\rho}_1, \ldots, \vec{\rho}_k \in (FL_0)^q$ and $\lambda_1, \ldots, \lambda_k \geq 0$ with $\sum_i \lambda_i = 1$ such that for any $\psi(R_1, \ldots, R_q, \vec{a}) \in SL$,

$$w(\psi(R_1, \ldots, R_q, \vec{a})) = \sum_{i=1}^{k} \lambda_i\, v_i^{\vec{\rho}_i}(\psi(R_1, \ldots, R_q, \vec{a})), \tag{5.4}$$

where $v_i^{\vec{\rho}_i}$ is a probability function on SL defined by

$$v_i^{\vec{\rho}_i}(\psi(R_1, \ldots, R_q, \vec{a})) = v_i(\psi'(\rho_{i,1}, \ldots, \rho_{i,q}, \vec{a})) \tag{5.5}$$

and $\psi'(\rho_{i,1}, \ldots, \rho_{i,q}, \vec{a}) \in SL_0$ is formed by replacing each occurrence of $R_s(t_1, \ldots, t_{r_s})$ in ψ by $\rho_{i,s}(t_1, \ldots, t_{r_s})$ for $s = 1, \ldots, q$.

Corollary 54. *If w has a reduction to a probability function on a unary language then w satisfies FVP.*

Proof. Suppose w is a probability function on SL with a representation as in (5.4), and let $n \in \mathbb{N}$ and $\theta(a_1, \ldots, a_n) \in SL^{(n)}$. For each $i = 1, \ldots, k$, by Corollary 53, v_i satisfies FVP$_n$ since each is a probability function on a unary language, so that by (5.5), $v_i^{\vec{\rho}_i}$ must also satisfy FVP$_n$. Therefore, by Lemma 51, w satisfies FVP$_n$ since it is a finite convex sum of probability functions which do. This holds for each $n \in \mathbb{N}^+$ and the result follows. □

A general classification of those probability functions on polyadic languages which have a reduction to a unary language has not yet been found, though it seems an interesting question to consider: under what circumstances may polyadic relations between individuals be 'translated' into combinations of unary properties of the individuals?

5.2 FVP with Sx

We now proceed to classify those probability functions (on polyadic languages) satisfying Sx which additionally satisfy FVP, beginning with the result described in the introduction to this chapter.

Proposition 55. *For each $t \in \mathbb{N}^+$, every t-heterogeneous probability function w satisfies FVP.*[4]

Proof. Let $t \in \mathbb{N}^+$ and $n \in \mathbb{N}$ be fixed, and let F be the (finite) set of maps from $\{z_1, \ldots, z_t, a_1, \ldots, a_n\}$ to $\{z_1, \ldots, z_t\}$, such that z_i maps to itself for $i = 1, \ldots, t$. Let u and, for $1 \le j \le u$, the sentences ζ_t and ζ_t^j and the state formulae $\Theta_j(z_1, \ldots, z_t)$ be as defined in §4.1, and similarly for $(\Theta_j)_\sigma(\vec{z}, a_1, \ldots, a_n)$ where $\sigma \in F$. Then by (4.3),

$$\zeta_t \equiv \bigvee_{j=1}^{u} \zeta_t^j,$$

while for $1 \le j < k \le u$, by their definition,

$$\zeta_t^j \models \neg \zeta_t^k.$$

[4] This result is given in [49] as a consequence of the heterogeneous functions' having a reduction to probability functions on a unary language, also shown there. The proof given here originally appeared in [26], and is somewhat more explanatory, with the advantage that it yields a set of n-ions.

Therefore by Lemma 31, for any t-heterogeneous probability function w,

$$w(\zeta_t) = \sum_{j=1}^{u} w(\zeta_t^j) = 1. \tag{5.6}$$

For $\sigma \in F$ let

$$\psi^{t,j,\sigma}(a_1, \dots, a_n) = \zeta_t^j \wedge \exists \vec{z}\, (\Theta_j(\vec{z}) \wedge (\Theta_j)_\sigma(\vec{z}, a_1, \dots, a_n)).$$

Then for any j,

$$\zeta_t^j \equiv \bigvee_{\sigma \in F} \psi^{t,j,\sigma}(a_1, \dots, a_n),$$

while for any fixed j and distinct $\sigma, \tau \in F$,

$$\psi^{t,j,\sigma}(a_1, \dots, a_n) \models \neg \psi^{t,j,\tau}(a_1, \dots, a_n), \tag{5.7}$$

so that

$$w(\zeta_t^j) = \sum_{\sigma \in F} w(\psi^{t,j,\sigma}(a_1, \dots, a_n)). \tag{5.8}$$

From Lemma 35, for any $\theta(a_1, \dots, a_n) \in SL^{(n)}$ and any fixed $\sigma \in F$, either

$$\psi^{t,j,\sigma}(a_1, \dots, a_n) \models \theta(a_1, \dots, a_n) \qquad \text{or} \qquad \psi^{t,j,\sigma}(a_1, \dots, a_n) \models \neg\theta(a_1, \dots, a_n). \tag{5.9}$$

Therefore, the (finite) set of pairs $\langle j, \sigma \rangle$ may be partitioned into two as follows: let A_θ be the set of pairs $\langle j, \sigma \rangle$ such that

$$\psi^{t,j,\sigma}(a_1, \dots, a_n) \models \theta(a_1, \dots, a_n).$$

Then by (5.9), any pair $\langle j, \sigma \rangle$ not in A_θ must be in $A_{\neg\theta}$.

Since by (5.6) and (5.8),

$$\sum_{\langle j,\sigma \rangle \in A_\theta \cup A_{\neg\theta}} w(\psi^{t,j,\sigma}(a_1, \dots, a_n)) = 1, \tag{5.10}$$

it follows that

$$w(\theta(a_1, \dots, a_n)) = \sum_{\langle j,\sigma \rangle \in A_\theta} w(\psi^{t,j,\sigma}(a_1, \dots, a_n)), \tag{5.11}$$

that is, $w(\theta(a_1, \dots, a_n))$ is determined by the probabilities assigned to the sentences $\psi^{t,j,\sigma}(a_1, \dots, a_n)$ and by the partition $A_\theta, A_{\neg\theta}$.

Therefore, there can only be finitely many values of $w(\theta(a_1 \dots, a_n))$ as θ ranges over $SL^{(n)}$, since there are only finitely many possible pairs $\langle j, \sigma\rangle$ and hence only finitely many possible ways to form the partition $A_\theta, A_{\neg\theta}$. Therefore, w satisfies FVP$_n$, and since n was arbitrary, the result follows. □

Furthermore, we obtain a set of n-ions for any t-heterogeneous probability function. Using the notation from the previous proof, by (5.7), (5.10) and (5.9) we have

Corollary 56. *For $t \in \mathbb{N}^+$ and $n \in \mathbb{N}$, the set of sentences*

$$\{\psi^{t,j,\sigma}(a_1, \dots, a_n) \mid j = 1, \dots, u,\ \sigma \in F\}$$

forms a set of n-ions for any t-heterogeneous probability function.

In order to reach a similar result concerning FVP for homogeneous functions, we recall the following notation from §4.2. As before, where $m \in \mathbb{N}$ and $\vec{S} = S_1, \dots, S_h$ is a partition of $\mathbb{N}_m$, let Δ be the set of sentences of the form

$$\forall x_1, \dots, x_m\, (\Phi^{\vec{S}}(x_1, \dots, x_m) \to \exists x_{m+1}\, \Psi^{\vec{S},\{m+1\}}(x_1, \dots, x_m, x_{m+1})),$$

where Φ is a state formula of L and Ψ is an extension of Φ to one extra variable, such that $\Psi^{\vec{S},\{m+1\}}(x_1, \dots, x_m, x_{m+1})$, as defined at (4.9), is consistent.

It is shown in [26], using a proof very similar to that of Proposition 55, that any homogeneous probability function w satisfies FVP by Lemmas 41 and 42. However, we will instead derive it as a corollory of the following, apparently more general, result.

Proposition 57. *If w is a probability function on SL, then $w(\phi) = 1$ for each $\phi \in \Delta$ just if w satisfies Reg + FVP, with n-ions* [5]

$$B^{(n)} = \{\Phi^{\vec{S}}(a_1, \dots, a_n) \mid \Phi \text{ a state description, } \vec{S} \text{ a partition of } \mathbb{N}_n\}$$

(with logically equivalent members identified) for each $n \in \mathbb{N}$.

[5] If the form of the n-ions were not specified, the converse direction would not hold. For example, the probability function $w = \frac{1}{2}w_{\langle\frac{1}{2},\frac{1}{2}\rangle} + \frac{1}{2}w_{\langle 1,0\rangle}$ on a unary language with 1 predicate satisfies Reg (by [49, Corollary 10.3]) and FVP (by Corollory 53), but $w(\forall x\,(\alpha_1(x) \to \exists y\, \alpha_1(x) \wedge \alpha_2(y))) = \frac{1}{2}$.

Proof. From left to right, suppose that w is a probability function on SL such that $w(\phi) = 1$ for each $\phi \in \Delta$. If w does not satisfy Reg then there is some state description $\Psi(a_1, \dots, a_{m+1})$ such that

$$0 = w(\Psi(a_1, \dots, a_{m+1})) < w(\Psi[a_1, \dots, a_m]). \tag{5.12}$$

Let $\Phi(\vec{a})$ denote $\Psi[a_1, \dots, a_m]$ (recall that where $m = 0$, we take $\Phi \equiv \top$).

Since

$$w(\Phi^{\vec{S}}(\vec{a}) \to \exists x_{m+1}\, \Psi^{\vec{S},\{m+1\}}(\vec{a}, x_{m+1})) = 1$$

for every partition $\vec{S}$ of $\mathbb{N}_m$, and since $\Psi^{\vec{S},\{m+1\}}(\vec{a}, a_i)$ is inconsistent for $i \in \mathbb{N}_m$, we have

$$w(\neg\Phi^{\vec{S}}(\vec{a})) + \lim_{n\to\infty} w\left(\bigvee_{i=m+1}^{n} \Psi^{\vec{S},\{m+1\}}(\vec{a}, a_i)\right) = 1.$$

It can't be the case that $w(\neg\Phi^{\vec{S}}(\vec{a})) = 1$ for every $\vec{S}$, since $w(\Phi(\vec{a})) > 0$, so there must be some partition $\vec{T}$ such that

$$\lim_{n\to\infty} w\left(\bigvee_{i=m+1}^{n} \Psi^{\vec{T},\{m+1\}}(\vec{a}, a_i)\right) > 0,$$

and so

$$w\left(\Psi^{\vec{T},\{m+1\}}(\vec{a}, a_{m+1})\right) > 0,$$

by Ex. Since

$$\Psi^{\vec{T},\{m+1\}}(\vec{a}, a_{m+1}) \models \Psi(\vec{a}, a_{m+1}),$$

this contradicts (5.12), so that w must satisfy Reg.

Suppose $n \in \mathbb{N}$ and let $B^{(n)}$ be as described in the statement of the result (where $B^{(0)}$ is taken to be $\{\top\}$ for some fixed tautology $\top \in SL^{(0)}$). Then for any $\Phi^{\vec{S}}, \Theta^{\vec{T}} \in B^{(n)}$ such that $\langle \Phi, \vec{S} \rangle \neq \langle \Theta, \vec{T} \rangle$, we have

$$\Phi^{\vec{S}}(a_1, \dots, a_n) \models \neg\Theta^{\vec{T}}(a_1, \dots, a_n),$$

and

$$\sum_{\Phi^{\vec{S}} \in B^{(n)}} w(\Phi^{\vec{S}}(a_1, \dots, a_n)) = 1.$$

By Lemma 42, for each $\Phi^{\vec{S}}(a_1,\dots,a_n) \in B^{(n)}$ and any $\theta \in SL^{(n)}$,

$$\Delta,\ \Phi^{\vec{S}}(a_1,\dots,a_n) \models \theta \iff \Delta,\ \Phi^{\vec{S}}(a_1,\dots,a_n) \not\models \neg\theta. \tag{5.13}$$

Let P_θ be the set $\{\Phi^{\vec{S}}(a_1,\dots,a_n) \in B^{(n)} \mid \Delta, \Phi \models \theta\}$. Then

$$\Delta, \bigvee_{\Phi^{\vec{S}} \in P_\theta} \Phi^{\vec{S}}(a_1,\dots,a_n) \models \theta,$$

(where $\bigvee_{\Phi^{\vec{S}} \in \emptyset} \Phi^{\vec{S}}(a_1,\dots,a_n)$ is taken to be a contradiction) and by the Compactness Theorem there is some finite $\Delta' \subset \Delta$ such that

$$\Delta', \bigvee_{\Phi^{\vec{S}} \in P_\theta} \Phi^{\vec{S}}(a_1,\dots,a_n) \models \theta.$$

Therefore

$$\begin{aligned}
\sum_{\Phi^{\vec{S}} \in P_\theta} w(\Phi^{\vec{S}}(a_1,\dots,a_n)) &= w(\bigvee_{\Phi^{\vec{S}} \in P_\theta} \Phi^{\vec{S}}(a_1,\dots,a_n)) \\
&= w(\bigwedge_{\phi \in \Delta'} \phi \wedge \bigvee_{\Phi^{\vec{S}} \in P_\theta} \Phi^{\vec{S}}(a_1,\dots,a_n)) \\
&\leq w(\theta).
\end{aligned} \tag{5.14}$$

Furthermore, by (5.13), $P_{\neg\theta} = B^{(n)} - P_\theta$, and again by the Compactness Theorem there is some finite $\Delta'' \subset \Delta$ such that

$$\Delta'', \bigvee_{\Phi^{\vec{S}} \in P_{\neg\theta}} \Phi^{\vec{S}}(a_1,\dots,a_n) \models \neg\theta,$$

so that

$$\begin{aligned}
\sum_{\Phi^{\vec{S}} \in P_{\neg\theta}} w(\Phi^{\vec{S}}(a_1,\dots,a_n)) &= w(\bigvee_{\Phi^{\vec{S}} \in P_{\neg\theta}} \Phi^{\vec{S}}(a_1,\dots,a_n)) \\
&= w(\bigwedge_{\phi \in \Delta''} \phi \wedge \bigvee_{\Phi^{\vec{S}} \in P_{\neg\theta}} \Phi^{\vec{S}}(a_1,\dots,a_n)) \\
&\leq w(\neg\theta).
\end{aligned} \tag{5.15}$$

Since

$$\sum_{\Phi^{\vec{S}} \in P_\theta} w(\Phi^{\vec{S}}(a_1,\dots,a_n)) + \sum_{\Phi^{\vec{S}} \in P_{\neg\theta}} w(\Phi^{\vec{S}}(a_1,\dots,a_n)) = 1 = w(\theta) + w(\neg\theta),$$

each of (5.14) and (5.15) must hold with equality. This gives

$$w\left(\left(\bigvee_{\Phi^{\vec{S}}\in P_\theta}\Phi^{\vec{S}}(a_1,\dots,a_n)\right)\to\theta\right)=1=w\left(\theta\to\left(\bigvee_{\Phi^{\vec{S}}\in P_\theta}\Phi^{\vec{S}}(a_1,\dots,a_n)\right)\right),$$

and so by Theorem 50, w satisfies FVP_n with n-ions $B^{(n)}$. This holds for all $n\in\mathbb{N}$.

In the other direction, suppose that w is a probability function on SL which satisfies Reg and FVP with n-ions $B^{(n)}$ as defined in the statement of the result for each $n\in\mathbb{N}$. Suppose $n\in\mathbb{N}$ and let $\Psi(x_1,\dots,x_n,x_{n+1})$ be a state formula and $\vec{S}$ a partition of $\mathbb{N}_n$ such that $\Psi^{\vec{S},\{n+1\}}(x_1,\dots,x_{n+1})$ is consistent. Let $\Phi(x_1,\dots,x_n)\equiv\Psi[x_1,\dots,x_n]$.

For any state formula $\Theta(x_1,\dots,x_m)$ and any partition $\vec{T}$ of $\mathbb{N}_n$ such that $\langle\Theta,\vec{T}\rangle\neq\langle\Phi,\vec{S}\rangle$,

$$\Theta^{\vec{T}}(a_1,\dots,a_n)\models\neg\Phi^{\vec{S}}(a_1,\dots,a_n)$$

so that

$$\begin{aligned}w(\Theta^{\vec{T}}(a_1,\dots,a_n)\wedge(\Phi^{\vec{S}}(a_1,\dots,a_n)\to\exists x_{n+1}\,\Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,x_{n+1})))&\\ =w(\Theta^{\vec{T}}(a_1,\dots,a_n)).&\quad(5.16)\end{aligned}$$

Since $\Phi^{\vec{S}}(a_1,\dots,a_n)$ is an n-ion for w, by (5.3),

$$\begin{aligned}w(\Phi^{\vec{S}}(a_1,\dots,a_n)\wedge(\Phi^{\vec{S}}(a_{i_1},\dots,a_{i_n})\to\exists x_{n+1}\,\Psi^{\vec{S},\{n+1\}}(a_{i_1},\dots,a_{i_n},x_{n+1})))&\\ \in\{0,w(\Phi^{\vec{S}}(a_1,\dots,a_n))\}.&\quad(5.17)\end{aligned}$$

Since

$$\Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,a_{n+1})\models\exists x_{n+1}\,\Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,x_{n+1})$$

we have

$$\begin{aligned}&w(\Phi^{\vec{S}}(a_1,\dots,a_n)\wedge(\Phi^{\vec{S}}(a_1,\dots,a_n)\to\exists x_{n+1}\,\Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,x_{n+1})))\\ &\quad\geq\ w(\Phi^{\vec{S}}(a_1,\dots,a_n)\wedge\Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,a_{n+1}))\\ &\quad>\ 0\end{aligned}$$

by Reg since $\Phi^{\vec{S}}(a_1,\dots,a_n)$ is consistent. Therefore, by (5.17)

$$w(\Phi^{\vec{S}}(a_1,\dots,a_n) \wedge (\Phi^{\vec{S}}(a_{i_1},\dots,a_{i_n}) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(a_{i_1},\dots,a_{i_n},x_{n+1}))) \\ = w(\Phi^{\vec{S}}(a_1,\dots,a_n))$$

and so by (5.16)

$$\begin{aligned} & w(\Phi^{\vec{S}}(a_1,\dots,a_n) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,x_{n+1})) \\ &= w(\Phi^{\vec{S}}(a_1,\dots,a_n) \wedge (\Phi^{\vec{S}}(a_1,\dots,a_n) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,x_{n+1}))) \\ &\quad + \sum_{\langle\Theta,\vec{T}\rangle\neq\langle\Phi,\vec{S}\rangle} w(\Theta^{\vec{T}}(a_1,\dots,a_n) \wedge (\Phi^{\vec{S}}(a_1,\dots,a_n) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(a_1,\dots,a_n,x_{n+1}))) \\ &= w(\Phi^{\vec{S}}(a_1,\dots,a_n)) + \sum_{\langle\Theta,\vec{T}\rangle\neq\langle\Phi,\vec{S}\rangle} w(\Theta^{\vec{T}}(a_1,\dots,a_n)) \\ &= 1. \end{aligned}$$

It follows by Ex that

$$w(\Phi^{\vec{S}}(b_1,\dots,b_n) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(b_1,\dots,b_n,x_{n+1})) = 1$$

for any distinct $b_1,\dots,b_n \subset \{a_1,a_2,\dots\}$.

In the case where $\{b_1,\dots,b_n\} \subset \{a_1,a_2,\dots\}$ contains exactly $k<n$ distinct constants, we can apply the above argument to a suitable restriction of Φ and $\vec{S}$. Specifically, restrict $\Phi(b_1,\dots,b_n)$ to its k distinct arguments $b_{j_1},\dots,b_{j_k}$, and substitute $\langle a_1,\dots,a_k\rangle$ for $\langle b_{j_1},\dots,b_{j_k}\rangle$, giving $\Phi[a_1/b_{j_1},\dots,a_k/b_{j_k}]$. Similarly, restrict $\vec{S}$ to give a partition of $\{j_1,\dots,j_k\}$, then use the corresponding partition of $\mathbb{N}_k$. In this case too, we obtain

$$w(\Phi^{\vec{S}}(b_1,\dots,b_n) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(b_1,\dots,b_n,x_{n+1})) = 1.$$

Therefore,

$$\begin{aligned} & w(\forall x_1,\dots,x_n\, (\Phi^{\vec{S}}(x_1,\dots,x_n) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(x_1,\dots,x_n,x_{n+1})) \\ &\quad = \lim_{r\to\infty} w\left(\bigwedge_{i_1,\dots,i_n\le r} (\Phi^{\vec{S}}(a_{i_1},\dots,a_{i_n}) \to \exists x_{n+1}\, \Psi^{\vec{S},\{n+1\}}(a_{i_1},\dots,a_{i_n},x_{n+1}))\right) \\ &\quad = \lim_{r\to\infty} 1 = 1. \end{aligned}$$

Since $n\in\mathbb{N}$, Φ and $\vec{S}$ were arbitrary, the result follows. □

The following result appears in [26] and follows easily from Proposition 57 and Lemma 41.

Corollary 58. *If w is a homogeneous probability function on some not purely unary language L, then w satisfies (Reg and)*[6] *FVP, with n-ions*

$$B^{(n)} = \{\Phi^{\vec{S}}(a_1, \ldots, a_n) \mid \Phi \text{ a state description, } \vec{S} \text{ a partition of } \mathbb{N}_n\}$$

(where logically equivalent members are identified) for each $n \in \mathbb{N}$.

However, although it is the case that all heterogeneous and homogeneous probability functions satisfy FVP, and by Theorem 27 these are the 'building blocks' of all Sx functions, not every Sx function satisfies FVP. The following classification result for Sx with FVP uses the fact that, by Theorem 27, any probability function w which satisfies Sx has a representation of the form

$$w = \eta_0 w^{[0]} + \sum_{t=1}^{\infty} \eta_t w^{[t]} \tag{5.18}$$

where $w^{[0]}$ is some homogeneous probability function, each $w^{[t]}$ is a t-heterogeneous probability function, and $\eta_0 + \sum_{t=1}^{\infty} \eta_t = 1$. Furthermore, this representation is unique up to a free choice of the $w^{[i]}$ when $\eta_i = 0$. We will say that 'the ladder representation of w is finite' (respectively 'infinite') to mean that the set

$$\{t \in \mathbb{N}^+ \mid \eta_t > 0\}$$

is finite (respectively infinite).

Theorem 59. *A probability function satisfying Sx satisfies FVP just if its ladder representation is finite.*

Proof. Suppose w is a probability function satisfying Sx, so by Theorem 27 it has a ladder representation as in (5.18), which is unique up to a free choice of the $w^{[i]}$ when $\eta_i = 0$ for each $i \in \mathbb{N}$. If its ladder representation is finite, then by Lemma 51, Proposition 55 and Corollary 58, w satisfies FVP.

[6] Their Regularity is already established by Landes, see [33] for example.

Otherwise, if its ladder representation is infinite, let $W = \{t \in \mathbb{N}^+ \mid \eta_t > 0\}$. By Lemma 31, for each $t \in W$ there is a sentence $\zeta_t \in SL^{(0)}$ such that $w^{[t]}(\zeta_t) = 1$, and for any $s \neq t$, $w^{[s]}(\zeta_t) = 0$ since $\zeta_s \models \neg\zeta_t$. Furthermore, where $\Theta(x_1, \ldots, x_t)$ has spectrum $\mathbf{1}_t$ and $\Phi(x_1, \ldots, x_{t+1})$ is an extension of Θ with spectrum $\mathbf{1}_{t+1}$ and $\vec{S}$ is the partition of $\mathbb{N}_t$ into t parts with one member each,

$$\forall x_1, \ldots, x_t \, (\Theta^{\vec{S}}(x_1, \ldots, x_t) \to \exists x_{t+1} \Phi^{\vec{S}, \{t+1\}}(x_1, \ldots, x_{t+1})) \models \neg\zeta_t,$$

so that by Lemma 41, $w^{[0]}(\zeta_t) = 0$ for each $t \in \mathbb{N}^+$.

Therefore, for each $t \in W$, $w(\zeta_t) = \eta_t$, so

$$\{\eta_t \mid t \in W\} = \{w(\zeta_t) \mid t \in W\} \subset \{w(\theta) \mid \theta \in SL^{(0)}\}.$$

Since W is infinite and $\sum_{t \in W} \eta_t \leq 1$, the set $\{\eta_t \mid t \in W\}$ must be infinite. Therefore w does not satisfy FVP_0, and so fails to satisfy FVP. □

The following result shows that for strictly polyadic languages, the simplicity of FVP comes (like the principles Sx and JSP) at the expense of SReg. That is, some consistent sentences must be assigned zero probability by any probability function satisfying FVP (in fact by any satisfying FVP_n for some n).

Corollary 60. *If L is strictly polyadic and w is a probability function on SL satisfying FVP_n for some $n \in \mathbb{N}$, then w does not satisfy SReg.*[7]

Proof. Suppose L and w are as described. Then by Proposition 50, there is some set of n-ions for w

$$B^{(n)} = \{\phi_1, \ldots, \phi_g\} \subset SL^{(n)}$$

such that $w(\phi_i \wedge \phi_j) = 0$ for $1 \leq i < j \leq g$,

$$\sum_{i=1}^{g} w(\phi_i) = 1,$$

and for every $\theta \in SL^{(n)}$ there is some $B_\theta^{(n)} \subseteq B^{(n)}$ such that

$$w\left(\theta \leftrightarrow \bigvee_{\phi_i \in B_\theta^{(n)}} \phi_i\right) = 1.$$

[7]This result would fail to hold with Reg in place of SReg, since any homogenous function satisfies both Reg and FVP. The converse does not hold, since by Theorem 59 and Corollory 49, some probability functions satisfy neither SReg nor FVP.

Suppose that

$$\models \bigvee_{i=1}^{g} \phi_i$$

and for each $\theta \in SL^{(n)}$

$$\models \theta \leftrightarrow \bigvee_{\phi_i \in B_\theta^{(n)}} \phi_i.$$

Then by Theorem 50, every probability function on SL would satisfy FVP$_n$ with n-ions $B^{(n)}$, contradicting Theorem 59. Therefore, either $\neg \bigvee_{i=1}^{g} \phi_i$ is consistent, but assigned probability zero by w, or for some $\theta \in SL^{(n)}$, $\neg \left(\theta \leftrightarrow \bigvee_{\phi_i \in B_\theta^{(n)}} \phi_i\right)$ is consistent, but assigned probability zero by w. In either case, w fails to satisfy SReg. □

5.3 The Strong Finite Values Property

It was noted with the definition of FVP that, although the image of w restricted to $SL^{(n)}$ must be finite for each n, its size may vary (in general we would expect it to increase) with n. We now investigate the consequences of imposing a fixed, finite bound on the image of $w \restriction SL^{(n)}$, which must hold for all n. We will call this:

The Strong Finite Values Property, SFVP

A probability function w on SL satisfies SFVP if there is some constant $K \in \mathbb{N}^+$ such that for all $n \in \mathbb{N}$

$$|\{w(\theta(a_1, \ldots, a_n)) \mid \theta \in SL^{(n)}\}| \leq K, \tag{5.19}$$

equivalently, since $SL^{(n)} \supset SL^{(m)}$ for all $m \leq n$

$$|\{w(\theta) \mid \theta \in SL\}| \leq K.$$

One might argue that, if FVP is desirable on the grounds of simplicity, then SFVP must be even more appealing. However the fact, for example, that the number of state descriptions for $a_1, \ldots, a_n$ increases without limit as n increases seems to warn against having a minimum positive value for $w(\theta)$ for θ from the whole of SL, as implied by SFVP. The consequences of contravening this warning become clear in Proposition 63. Firstly, however, we state two results which will be needed subsequently.

The following lemma is proved by the same argument as used for its counterpart, Lemma 51.

Lemma 61. *If w is a finite convex sum of probability functions on SL which all satisfy SFVP, then w satisfies SFVP.*

The following theorem of Paris & Vencovská [49], extending results of Krauss [32], is an analogue of de Finetti's Theorem for unary languages, providing a representation theorem for probability functions on polyadic languages which satisfy Ex. It shows that any function satisfying Ex has a representation in terms of some functions ${}^\circ\omega^\Psi$ (the exact definition is not needed here) which, like the $w_{\vec{c}}$ functions of de Finetti's Theorem, satisfy Ex and have the property that where $\theta, \phi \in QFSL$ have no constant symbols in common

$$ {}^\circ\omega^\Psi(\theta \wedge \phi) = {}^\circ\omega^\Psi(\theta) \cdot {}^\circ\omega^\Psi(\phi). \tag{5.20} $$

The proof uses methods from Nonstandard Analysis, in particular Loeb Measure Theory [39] [8]; the set A mentioned in the result is a certain set of state descriptions $\Psi(a_1, \ldots, a_\nu)$ for some nonstandard ν, see [49] for the details.

Theorem 62. *If the probability function w on SL satisfes Ex then w has a representation of the form*

$$ w = \int_A {}^\circ\omega^\Psi \; d\mu(\Psi) $$

for some countably additive measure μ on an algebra of subsets of A, and probability functions ${}^\circ\omega^\Psi$ on SL satisfying (5.20). *Conversely if w has a representation of this form then it satisfes Ex.*

In order to give a classification for SFVP, we introduce the following notation. Let I_n be the set of state descriptions of L for $a_1, \ldots, a_n$ which are invariant up to logical equivalence under any permutation of $a_1, \ldots, a_n$

$$ I_n = \{\Phi(a_1, \ldots, a_n) \mid \Phi(a_1, \ldots, a_n) \equiv \Phi(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \quad \forall \sigma \in \mathsf{S}_n\}, \tag{5.21} $$

(where logically equivalent members are identified). Let r denote the largest arity of any relation in L.

[8] Alternatively see [10].

For $\Theta \in I_r$ we define a unique structure $M_\Theta \in \mathcal{T}L$ as follows. For $\Theta(a_1, \dots, a_r) \in I_r$, let $M_\Theta \models \Theta(a_1, \dots, a_r)$, and note that for $1 \leq n < r$, $\Theta[a_1, \dots, a_n] \in I_n$. For $n \geq r$, where $M_\Theta \models \Upsilon_n(a_1, \dots, a_n) \in I_n$ (so $\Upsilon_r \equiv \Theta$), form an extension $\Upsilon_{n+1}(a_1, \dots, a_{n+1}) \in I_{n+1}$ as follows. Where $b_1, \dots, b_{r_k}$ (not necessarily distinct) from $\{a_1, \dots, a_{n+1}\}^{r_k}$ does not contain a_{n+1} then set $\Upsilon_{n+1} \models R_k(b_1, \dots, b_{r_k})$ just if $\Upsilon_n \models R_k(b_1, \dots, b_{r_k})$. Otherwise, where $b_1, \dots, b_{r_k}$ does contain a_{n+1}, then it also contains at most $r-1$ distinct members of $\{a_1, \dots, a_n\}$, so choose $a_s \in \{a_1, \dots, a_n\}$ not appearing in $b_1, \dots, b_{r_k}$, and set $\Upsilon_{n+1} \models R_k(b_1, \dots, b_{r_k})$ just if $\Upsilon_n \models R_k(\vec{b}')$, where $\vec{b}'$ is $b_1, \dots, b_{r_k}$ with each instance of a_{n+1} replaced by a_s. Since $\Upsilon_n \in I_n$, the same result is obtained regardless of the choice of s, so the extension $\Upsilon_{n+1} \in I_{n+1}$ obtained by this method is unique, and any other extension would not be in I_{n+1}. Set $M_\Theta \models \Upsilon_{n+1}(a_1, \dots, a_{n+1})$ for each $n \geq r$.

Note that, by induction on n, each M_Θ is a model of

$$\bigvee_{\Phi \in I_n} \Phi(a_1, \dots, a_n) \tag{5.22}$$

for each $n \in \mathbb{N}^+$, and that any structure which cannot be constructed by this method must fail to satisfy (5.22) for some n, so that

$$\{M \in \mathcal{T}L \mid M \models \bigvee_{\Phi \in I_n} \Phi(a_1, \dots, a_n), \ \ \forall n \in \mathbb{N}^+\} = \{M_\Theta \mid \Theta \in I_r\}. \tag{5.23}$$

Recall from (1.7) the definition of the probability function V_M for $M \in \mathcal{T}L$, and note that for any M_Θ constructed as above, V_{M_Θ} satisfies Ex.

Proposition 63. *If w is a probability function on SL then the following statements are equivalent:*

1. *w satisfies SFVP.*
2. $w\left(\bigvee_{\Phi(a_1,\dots,a_n) \in I_n} \Phi(a_1, \dots, a_n)\right) = 1$ *for each* $n \in \mathbb{N}^+$.
3. *w is a convex sum of the functions V_{M_Θ} for $\Theta \in I_r$.*
4. *For every $n \in \mathbb{N}$, $\theta \in SL^{(n)}$ and $\sigma \in \mathsf{S}_n$*

$$w(\theta(a_1, \dots, a_n) \leftrightarrow \theta(a_{\sigma(1)}, \dots, a_{\sigma(n)})) = 1.$$

Proof. 1 $\implies$ 2.

Suppose w is a probability function on SL (satisfying Ex). By Theorem 62, w has a representation of the form

$$w = \int_A {}^\circ\omega^\Psi \, d\mu(\Psi)$$

for some countably additive measure μ on an algebra of subsets of A and probability functions ${}^\circ\omega^\Psi$ on SL satisfying (5.20) and Ex.

Suppose $w\left(\bigvee_{\Phi(a_1,\dots,a_m)\in I_m} \Phi(a_1,\dots,a_m)\right) < 1$ for some m. Then there is some state description $\Phi(a_1,\dots,a_m)$ such that $w(\Phi) > 0$ and

$$\Phi(a_1,\dots,a_m) \not\equiv \Phi(a_{\sigma(1)},\dots,a_{\sigma(m)})$$

for some $\sigma \in \mathsf{S}_m$, so that the conjunction $\Phi(a_1,\dots,a_m) \wedge \Phi(a_{\sigma(1)},\dots,a_{\sigma(m)})$ is a contradiction. Since the ${}^\circ\omega^\Psi$ functions satisfy Ex, for any $\Psi \in A$

$${}^\circ\omega^\Psi(\Phi(a_1,\dots,a_m)) = {}^\circ\omega^\Psi(\Phi(a_{\sigma(1)},\dots,a_{\sigma(m)}))$$

so that ${}^\circ\omega^\Psi(\Phi(a_1,\dots,a_m)) \leq \frac{1}{2}$.

Let $A' = \{\Psi \in A \mid 0 < {}^\circ\omega^\Psi(\Phi(a_1,\dots,a_m)) < 1\}$. Then, since the ${}^\circ\omega^\Psi$ satisfy Ex and (5.20),

$$\begin{aligned}
w\left(\bigwedge_{k=0}^{n-1} \Phi(a_{km+1},\dots,a_{km+m})\right) &= \int_A {}^\circ\omega^\Psi\left(\bigwedge_{k=0}^{n-1} \Phi(a_{km+1},\dots,a_{km+m})\right) d\mu(\Psi) \\
&= \int_{A'} {}^\circ\omega^\Psi\left(\bigwedge_{k=0}^{n-1} \Phi(a_{km+1},\dots,a_{km+m})\right) d\mu(\Psi) \\
&= \int_{A'} \prod_{k=0}^{n-1} {}^\circ\omega^\Psi\left(\Phi(a_1,\dots,a_m)\right) d\mu(\Psi) \\
&> \int_{A'} \prod_{k=0}^{n} {}^\circ\omega^\Psi\left(\Phi(a_1,\dots,a_m)\right) d\mu(\Psi) \\
&= w\left(\bigwedge_{k=0}^{n} \Phi(a_{km+1},\dots,a_{km+m})\right) \\
&> 0.
\end{aligned}$$

Therefore, $w\left(\bigwedge_{k=0}^{n} \Phi(a_{km+1},\dots,a_{km+m})\right)$ forms a strictly decreasing sequence bounded below, so that w takes infinitely many values on this sentence as n increases, and therefore fails SFVP. The result follows.

$2 \implies 3.$

Let Q denote the set

$$\{M \in \mathcal{T}L \mid M \models \bigvee_{\Phi \in I_n} \Phi(a_1, \dots, a_n), \quad \forall n \in \mathbb{N}^+\}.$$

Note that all state descriptions $\Phi(a_1)$ mentioning only a_1 belong to I_1. For $n > 1$ let

$$P_n = \{M \in \mathcal{T}L \mid M \models \bigvee_{\Phi \in I_{n-1}} \Phi(a_1, \dots, a_{n-1}) \wedge \neg \bigvee_{\Phi \in I_n} \Phi(a_1, \dots, a_n)\},$$

then Q together with these P_n form a partition of $\mathcal{T}L$.

By Theorem 4,

$$w = \int_{\mathcal{T}L} V_M \; d\nu$$

for some measure ν on the algebra $\mathcal{B}$ of subsets of $\mathcal{T}L$. Suppose $\nu(Q) < 1$, then there is some least value k such that $\nu(P_k) > 0$. Therefore

$$\begin{aligned}
w & \left(\bigvee_{\Phi \in I_k} \Phi(a_1, \dots, a_k) \right) \\
&= \int_{\mathcal{T}L} V_M \left(\bigvee_{\Phi \in I_k} \Phi(a_1, \dots, a_k) \right) \; d\nu \\
&= \int_{\mathcal{T}L - P_k} V_M \left(\bigvee_{\Phi \in I_k} \Phi(a_1, \dots, a_k) \right) \; d\nu + \int_{P_k} V_M \left(\bigvee_{\Phi \in I_k} \Phi(a_1, \dots, a_k) \right) \; d\nu \\
&\leq 1 - \nu(P_k) \\
&< 1
\end{aligned}$$

since $V_M \left(\bigvee_{\Phi \in I_k} \Phi(a_1, \dots, a_k) \right) = 0$ for $M \in P_k$. Therefore, if $w \left(\bigvee_{\Phi \in I_n} \Phi(a_1, \dots, a_n) \right) = 1$ for each $n \in \mathbb{N}^+$ then $\nu(Q)$=1, so by (5.23) and since I_r is finite, the result follows.

$3 \implies 1$

Any w of the form

$$w = \sum_{\Theta \in I_r} \lambda_\Theta V_{M_\Theta}$$

satisfies SFVP by Lemma 51, since each V_{M_Θ} is 2-valued.

$3 \implies 4$

Let $\Theta \in I_r$ and let $M_\Theta \in \mathcal{T}L$ be as defined on p112. Then $M_\Theta \models \Theta(a_{i_1}, \dots, a_{i_r})$ for any distinct $i_1, \dots, i_r$ from $\mathbb{N}^+$, and so for every $n \in \mathbb{N}$, every $\theta(a_1, \dots, a_n) \in SL^{(n)}$ and every $\sigma \in \mathsf{S}_n$, by induction on the quantifier complexity of $\theta(a_1, \dots, a_n)$,

$$M_\Theta \models \theta(a_1, \dots, a_n) \iff M_\Theta \models \theta(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

so that

$$M_\Theta \models \theta(a_1, \dots, a_n) \leftrightarrow \theta(a_{\sigma(1)}, \dots, a_{\sigma(n)}).$$

Therefore, if

$$w = \sum_{\Theta \in I_r} \lambda_\Theta V_{M_\Theta}$$

such that each $\lambda_\Theta \geq 0$ and $\sum_\Theta \lambda_\Theta = 1$, then

$$\begin{aligned} & w(\theta(a_1, \dots, a_n) \leftrightarrow \theta(a_{\sigma(1)}, \dots, a_{\sigma(n)})) \\ & \quad = \sum_{\Theta \in I_r} \lambda_\Theta V_{M_\Theta}(\theta(a_1, \dots, a_n) \leftrightarrow \theta(a_{\sigma(1)}, \dots, a_{\sigma(n)})) \\ & \quad = \sum_{\Theta \in I_r} \lambda_\Theta = 1. \end{aligned}$$

for any $\theta(a_1, \dots, a_n) \in SL$ and any $\sigma \in \mathsf{S}_n$.

$4 \implies 1$

Suppose that w does not satisfy SFVP, so that since $2 \implies 3 \implies 1$

$$w\left(\bigvee_{\Phi(a_1,\dots,a_m) \in I_m} \Phi(a_1, \dots, a_m) \right) < 1$$

for some $m \in \mathbb{N}^+$. It follows that there must be some state description $\Psi(a_1, \dots, a_m) \notin I_m$ such that $w(\Psi(a_1, \dots, a_m)) > 0$. Since $\Psi \notin I_m$, by the definition of I_m there is some $\sigma \in \mathsf{S}_m$ such that

$$\Psi(a_1, \dots, a_m) \not\equiv \Psi(a_{\sigma(1)}, \dots, a_{\sigma(m)})$$

and therefore

$$w(\Psi(a_1, \dots, a_m) \leftrightarrow \Psi(a_{\sigma(1)}, \dots, a_{\sigma(m)})) = 0.$$

Therefore, where w does not satisfy SFVP there is some $\theta(a_1, \dots, a_m) \in SL$ such that $w(\theta(a_1, \dots, a_m) \leftrightarrow \theta(a_{\sigma(1)}, \dots, a_{\sigma(m)})) \neq 1$, and the contrapositive gives the result.

Therefore, by the transitivity of $\implies$, a probability function w satisfies any of the 4 properties stated just if it satisfies all of them. □

So any probability function satisfying SFVP must assign probability 1 to the notion that each distinct n-tuple of individuals in the universe is indiscernible from every other distinct n-tuple of individuals, for every $n \in \mathbb{N}^+$. On unary languages this is equivalent to the assumption that all individuals satisfy the same atom and hence are indistinguishable, as exhibited by, for example, Carnap's c_0^L. On polyadic languages, the assumption that all individuals are indiscernible is strictly weaker than than the assumption that all are indistinguishable, but even so it seems hardly more acceptable. (Indeed it may be thought less acceptable, since indistinguishability may be due to the same individual having multiple 'names' (constant symbols), whereas indiscernibility without indistinguishability does not admit this explanation.) We therefore conclude that SFVP is not a desirable feature of probability functions used to model rational belief.

With regard to FVP, Theorem 50 and the subsequent discussion make a tentative case for FVP as a principle of rational reasoning. It has been shown that it entails an underlying simplicity in the process of assigning probabilities, explicated in terms of *ions* which are used for systematic reasoning in terms of possible worlds. Furthermore, the classification results presented show that FVP is exhibited by several families of probability functions already studied in PIL, including all functions on unary languages, so is not as rare as might have been supposed. Its incompatibility with SReg will count against it with proponents of that principle, but it is in company with at least the principles of Sx and JSP in this respect.

While these preliminary results give a 'flavour' of the Finite Values Property in the context of PIL, we have been unable, so far, to discover any representation result for FVP. It seems that more work is needed before this property, which is rather unlike many purported principles of rational reasoning, can be better understood and thereby judged.

Chapter 6

Conclusions

We have presented here the results of several distinct but related investigations concerning certain principles of rationality for inductive reasoning. Of these, the Elephant Principle, the Perspective Principle, the Abductive Inference Principle and its variations and the Finite Values Property are newly conceived, while the account of the theory of Spectrum Exchangeability gives a new perspective on a young, though not strictly new, principle of Pure Inductive Logic.

Necessarily, those results concerning previously unheard-of principles are preliminary in nature. We have begun, in each case, to investigate the extent to which this new principle might be justified as a requirement of rationality, by the attempt to discover how its adoption along with certain combinations of other more established principles limits an agent's choice of probability function.

We have given, for each new principle, a classification of which members of established families of probability functions studied in inductive logic do and do not satisfy it, in order to elucidate how each fits in to the landscape of Pure Inductive Logic, as so far charted. Specific conclusions and discussion relating to each principle have been included at the end of its own chapter. There remain many questions left unanswered regarding these new principles, but the results presented here provide a basis for any future work to address these.

By contrast, the work to identify the theory of Spectrum Exchangeability builds on established properties and representation results to contribute to a deeper understanding of what it means to adopt this principle, in terms of the underlying assumptions concerning the number of distinguishable individuals in the universe.

It has not been our intention to claim that any of the principles considered here must necessarily be adopted by a rational agent, our aim has been rather to establish previously unknown facts in order to inform any discussion about the extent to which these principles may or may not be justified on grounds of rationality, or the extent to which particular probability functions may be said to provide a good model of rational reasoning. The reader is left to draw his or her own conclusions regarding how the results presented here, within the mathematical framework of Pure Inductive Logic, relate to the wider study of induction and its applications.

Bibliography

[1] Bartha, P., Analogy and Analogical Reasoning, *The Stanford Encyclopedia of Philosophy (Fall 2013 Edition)*, ed. E.Zalta, URL: http://plato.stanford.edu/archives/fall2013/entries/reasoning-analogy/ .

[2] Carnap, R., On Inductive Logic, *Philosophy of Science*, 1945, **12**(2):72–97.

[3] Carnap, R., *Logical Foundations of Probability*, University of Chicago Press, Chicago, Routledge & Kegan Paul, London, 1950.

[4] Carnap, R., *The Continuum of Inductive Methods,* University of Chicago Press, 1952.

[5] Carnap, R., The aim of inductive logic, in *Logic, methodology and philosophy of science* eds. E.Nagel, P.Suppes, A.Tarski, Stanford University Press, Stanford, California, 1962, pp303–318.

[6] Carnap, R., An Axiom System for Inductive Logic, in *The Philosophy of Rudolf Carnap*, ed. Schilpp, P.A., Open Court Publishing Company, La Salle, Illinois, 1963.

[7] Carnap, R., A Basic System of Inductive Logic, Part I, in *Studies in Inductive Logic and Probability, Volume I*, eds. R.Carnap & R.C.Jeffrey, University of California Press, 1971, pp33–165.

[8] Carnap, R., A Basic System of Inductive Logic, Part II, in *Studies in Inductive Logic and Probability, Volume II*, ed. R.C.Jeffrey, University of California Press, 1980, pp7–155.

[9] Carnap, R. & Stegmüller, W., *Induktive Logik und Wahrscheinlichkeit,* Springer Verlag, Wien, 1959.

[10] Cutland, N.J., Loeb measure theory, in *Developments in Nonstandard Mathematics*, eds. N.J.Cutland, F.Oliveira, V.Neves, J.Sousa-Pinto, Pitman Research Notes in Mathematics Series, Vol. 336, Longman Press, 1995, pp151–177.

[11] Fagin, R., Probabilities on finite models, *Journal of Symbolic Logic*, 1976, **41**(1):50–58.

[12] de Finetti, B., Sul significato soggettivo della probabilità, *Fundamenta Mathematicae*, 1931, **17**:298–329.

[13] de Finetti, B., *Theory of Probability, Volume 1*, Wiley, New York, 1974.

[14] de Finetti, B., *Theory of Probability, Volume 2*, Wiley, New York, 1975.

[15] Gaifman, H., Concerning Measures on First Order Calculi, *Israel Journal of Mathematics*, 1964, **2**:1–18.

[16] Gaifman, H., Applications of de Finetti's Theorem to Inductive Logic, in *Studies in Inductive Logic and Probability*, Volume I, eds. R.Carnap & R.C.Jeffrey, University of California Press, 1971, pp235–251.

[17] Gaifman, H. & Snir, M., Probabilities over Rich Languages, *Journal of Symbolic Logic*, 1982, **47**:495–548.

[18] Glebskii, Y.V., Kogan, D.I., Liogon'kii, M.I., Talanov, V.A., Range and degree of realizability of formulas in the restricted predicate calculus, *Cybernetics*, 1969, **5**(2):142–154.

[19] Good, I.J., On the Principle of Total Evidence, *The British Journal for the Philosophy of Science,* 1967, **17**(4):319–321.

[20] Hill, A. & Paris, J.B., An Analogy Principle in Inductive Logic, *Annals of Pure and Applied Logic,* 2013, **164**:1293–1321.

[21] Hill, A. & Paris, J.B., The counterpart principle of analogical support by similarity, *Erkenntnis*, 2013, **79**(6):1169–1184.

[22] Hill, M.J., Paris, J.B. & Wilmers, G.M., Some Observations on Induction in Predicate Probabilistic Reasoning, *Journal of Philosophical Logic*, 2002, **31**:43–75.

[23] Hintikka, J., Distributive Normal Forms in First-Order Logic, *Studies in Logic and the Foundations of Mathematics*, 1965, **40**:48–91.

[24] Hintikka, J., A two dimensional continuum of inductive methods, in *Aspects of inductive logic*, eds. J.Hintikka & P.Suppes, North Holland, Amsterdam, 1966, pp113–132.

[25] Howarth, E. & Paris, J.B., Principles of Remembering and Forgetting, *Logique et Analyse*, 2014, **228**:489–511.

[26] Howarth, E. & Paris, J.B., The Theory of Spectrum Exchangeability, *The Review of Symbolic Logic*, 2015, **8**(1):108–130.

[27] Howarth, E., Paris, J.B. & Vencovská, A., An Examination of the SEP Candidate Analogical Inference Rule within Pure Inductive Logic, submitted to the *Journal of Applied Logic*.

[28] Johnson, W.E., Probability: The Deductive and Inductive Problems, *Mind*, 1932, **41**:409–423.

[29] Kemeny, J., Fair bets and inductive probabilities, *Journal of Symbolic Logic*, 1955, **20**:263–273.

[30] Kemeny, J.G., Carnap's Theory of Probability and Induction, in *The Philosophy of Rudolf Carnap*, ed. P.A.Schilpp, La Salle, Illinois, Open Court, 1963, pp711–738.

[31] Keynes, J.M., A treatise on probability, Macmillan & Co, 1921.

[32] Krauss, P.H., Representation of Symmetric Probability Models, *Journal of Symbolic Logic*, 1969, **34**(2):183–193.

[33] Landes, J., *The Principle of Spectrum Exchangeability within Inductive Logic*, Ph.D. Thesis, University of Manchester, 2009. Available at http://www.maths.manchester.ac.uk/∼jeff/

[34] Landes, J., Paris, J.B. & Vencovská, A., Some Aspects of Polyadic Inductive Logic, *Studia Logica*, 2008, **90**:3–16.

[35] Landes, J., Paris, J.B. & Vencovská, A., Representation Theorems for Probability Functions satisfying Spectrum Exchangeability in Inductive Logic, *International Journal of Approximate Reasoning*, 2009, **51**(1):35–55.

[36] Landes, J., Paris, J.B. & Vencovská, A., A Characterization of the Language Invariant Families satisfying Spectrum Exchangeability in Polyadic Inductive Logic, *Annals of Pure and Applied Logic*, 2010, **161**:800–811.

[37] Landes, J., Paris, J.B. & Vencovská, A., Survey of some recent results on Spectrum Exchangeability in Polyadic Inductive Logic, *Synthese*, 2011, **181**(1):19–47.

[38] Lehman, R.S., On confirmation and rational betting, *Journal of Symbolic Logic*, 1955, **20**:251–262.

[39] Loeb, Conversion from nonstandard to standard measure spaces and applications to probability, *Transactions of the American Mathematical Society*, 1975, **211**:113–122.

[40] Nix, C.J., *Probabilistic Induction in the Predicate Calculus*, Ph.D. Thesis, University of Manchester, UK, 2005. Available at http://www.maths.manchester.ac.uk/∼jeff/

[41] Nix, C.J. & Paris, J.B., A Continuum of Inductive Methods arising from a Generalized Principle of Instantial Relevance, *Journal of Philosophical Logic,* 2006, **35**(1):83–115.

[42] Nix C.J. & Paris, J.B., A Note on Binary Inductive Logic, *Journal of Philosophical Logic*, 2007, **36**(6):735–771.

[43] Paris, J.B., A note on ∗-conditioning, available at http://www.maths.manchester.ac.uk/∼jeff/

[44] Paris, J.B., Pure Inductive Logic, in *The Continuum Companion to Philosophical Logic*, eds. L.Horsten & R.Pettigrew, Continuum International Publishing Group, London, 2011, pp428–449.

[45] Paris, J.B. & Rad, S.R., A Note on the Least Informative Model of a Theory, in *Programs, Proofs, Processes, CiE 2010,* eds. F.Ferreira, B.Löwe, E.Mayordomo, & L.M.Gomes, Springer LNCS 6158, 2010, pp342–351.

[46] Paris, J.B. & Vencovská, A., A General Representation Theorem for Probability Functions Satisfying Spectrum Exchangeability, in *CiE 2009*, eds. K.Ambros-Spies, B.Löwe & W.Merkle, Springer LNCS 5635, pp379–388.

[47] Paris, J.B. & Vencovská, A., The Twin Continua of Inductive Methods, in *Logic Without Borders*, eds. A.Hirvonen, J.Kontinen, R.Kossak, & A.Villaveces, Ontos Mathematical Logic, Frankfurt-Heusenstamm, Germany, 2015, pp355–366.

[48] Paris, J.B. & Vencovská, A., Reichenbach's Axiom – For the record. Available at http://www.maths.manchester.ac.uk/∼jeff/papers/jp140708.pdf

[49] Paris, J.B. & Vencovská, A., *Pure Inductive Logic*, Association of Symbolic Logic Series *Perspectives in Mathematical Logic*, Cambridge University Press, 2015.

[50] Paris, J.B. & Waterhouse, P., Atom Exchangeability and Instantial Relevance, *Journal of Philosophical Logic*, 2009, **38**(3):313–332.

[51] Peirce, C.S., A Theory of Probable Inference, in *Studies in Logic*, ed. C.S.Peirce, Little, Brown, and Company, Boston, 1883.

[52] Peirce, C.S., Lectures on Pragmatism, in *The Collected Papers of Charles Sanders Peirce, Volume V*, eds. C.Hartshorne & P.Weiss, Harvard University Press, 1935.

[53] Ramsey, F.P., *The foundations of mathematics and other logical essays*, Routledge & Kegan Paul, London, 1931.

[54] Shimony, A., Coherence and the axioms of confrmation, *The Journal of Symbolic Logic*, 1955, **20**:1–28.

[55] Trakhtenbrot, B., The Impossibility of an Algorithm for the Decidability Problem on Finite Classes, *Proceedings of the USSR Academy of Sciences* (in Russian), 1950, **70**(4):569–572.

Appendix A

The theory of Johnson's Sufficientness Postulate

Johnson's Sufficientness Postulate appears originally in [28] and is a principle of rationality for inductive logic with unary languages, based on the idea of irrelevance. (We assume throughout this appendix that L is unary). We restate it here for convenience.

Johnson's Sufficientness Postulate, JSP

A probability function w on a unary language L satisfies Johnson's Sufficientness Postulate if

$$w\left(\alpha_j(a_{n+1}) \;\middle|\; \bigwedge_{i=1}^{n} \alpha_{h_i}(a_i)\right)$$

depends only on n and $m_j = |\{i \mid h_i = j\}|$.

This expresses the idea that, in assigning a probability to a particular outcome of an event, only the number of known instances of this outcome and the number of known instances of this event are relevant; all else is irrelevant and should be disregarded.

The following result was proved originally by Johnson in [28] [1].

Theorem 64. *If $q \geq 2$ and w is a probability function on SL, then w satisfies JSP just if $w = c_\lambda^L$ for some $0 \leq \lambda \leq \infty$.*

[1]Proofs have also been given in [9], [30] and [49].

We use this result to determine the theory of JSP, that is, the set of all sentences of L which must be assigned probability 1 by every probability function satisfying JSP:

$$Th(JSP) = \{\theta \in SL \mid w \text{ satisfies JSP } \implies w(\theta) = 1\}.$$

These results are closely related to work by Hintikka [23], [24].

We will need the following well-known result.

Lemma 65. *For any* $0 \leq a < b$,

$$\lim_{n\to\infty} \prod_{j=1}^{n} \frac{j+a}{j+b} = 0.$$

Proof. Let $0 \leq a < b$. Then

$$\begin{aligned} \ln\left(\prod_{j=1}^{n} \frac{j+a}{j+b}\right) &= \sum_{j=1}^{n} \ln\left(\frac{j+a}{j+b}\right) \\ &= \sum_{j=1}^{n} \ln\left(1 - \frac{b-a}{j+b}\right) \end{aligned}$$

where each $\left|\frac{b-a}{j+b}\right| < 1$. By the power series expansion this is equal to

$$\sum_{j=1}^{n}\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p}\left(-\frac{b-a}{j+b}\right)^{p} < \sum_{j=1}^{n}\left(-\frac{b-a}{j+b}\right)$$

since for fixed j, each term in the sum over p is negative. Taking the limit as $n \to \infty$ gives

$$\begin{aligned} \lim_{n\to\infty} \ln\left(\prod_{j=1}^{n} \frac{j+a}{j+b}\right) &< \lim_{n\to\infty} \sum_{j=1}^{n}\left(-\frac{b-a}{j+b}\right) \\ &\leq -(b-a)\lim_{n\to\infty}\left(\sum_{p=2+[b]}^{n+1+[b]} \frac{1}{p}\right) \\ &= -\infty. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \prod_{j=1}^{n} \frac{j+a}{j+b} = \lim_{x\to-\infty} e^{x} = 0.$$

$\square$

The following result concerns Carnap's c_λ^L functions (1.11). A proof is given in [49], for example.

Proposition 66. *Let $\theta_1(x), \dots, \theta_r(x)$ be disjoint, quantifier-free formulae of L. Then for $0 < \lambda$,*

$$c_\lambda^L\left(\theta_j(a_{n+1}) \;\middle|\; \bigwedge_{i=1}^n \theta_{h_i}(a_i)\right) = \frac{m_j + t_j\lambda 2^{-q}}{n+\lambda}$$

where $m_j = |\{i \mid h_i = j\}|$ and $t_j = |\{s \mid \alpha_s(x) \models \theta_j(x)\}|$.

The previous two results go to prove the following:

Proposition 67. *For any $1 \leq k \leq 2^q$ and $\lambda > 0$,*

$$c_\lambda^L(\forall x\, \neg\alpha_k(x)) = 0.$$

Proof. For any $1 \leq k \leq 2^q$, by Proposition 66

$$\begin{aligned} c_\lambda^L\left(\bigwedge_{i=1}^n \neg\alpha_k(a_i)\right) &= c_\lambda^L(\neg\alpha_k(a_1)) \prod_{j=1}^{n-1} c_\lambda^L\left(\neg\alpha_k(a_{j+1}) \mid \bigwedge_{r=1}^j \neg\alpha_k(a_r)\right) \\ &= \frac{2^q-1}{2^q} \prod_{j=1}^{n-1} \frac{j + (2^q-1)\lambda 2^{-q}}{j+\lambda}. \end{aligned}$$

Therefore by Lemma 65,

$$\begin{aligned} c_\lambda^L(\forall x\, \neg\alpha_k(x)) &= \lim_{n\to\infty} c_\lambda^L\left(\bigwedge_{i=1}^n \neg\alpha_k(a_i)\right) \\ &= \frac{2^q-1}{2^q} \lim_{n\to\infty} \prod_{j=1}^{n-1} \frac{j + (2^q-1)\lambda 2^{-q}}{j+\lambda} \\ &= 0. \end{aligned}$$

□

Proposition 68. *Let ζ be the sentence $\bigwedge_{k=1}^{2^q} \exists x\, \alpha_k(x)$. Then for $\lambda > 0$ and $\theta \in SL$,*

$$c_\lambda^L(\theta) = 1 \iff \zeta \models \theta.$$

Proof. From right to left, suppose $\zeta \models \theta$. Then

$$1 = c_\lambda^L(\zeta) \leq c_\lambda^L(\theta) \leq 1$$

by Proposition 67.

In the other direction, suppose $\theta(b_1, \ldots, b_m) \in SL$. Since L is unary, by Proposition 2, θ is logically equivalent to some sentence θ' of the form

$$\bigvee_{k=1}^{l} \left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_{kj}} x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{k_i}}(b_i) \right), \tag{A.1}$$

where each $\vec{\epsilon_k} \in \{0,1\}^n$ and $\exists^1$ stands for $\exists$, while $\exists^0$ stands for $\neg\exists$, and the disjuncts are disjoint and satisfiable. By Proposition 67,

$$c_\lambda^L \left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_j} x\, \alpha_j(x) \right) = \begin{cases} 1 & \epsilon_j = 1,\ j = 1, \ldots, 2^q \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} c_\lambda^L(\theta) &= \sum_{k=1}^{l} c_\lambda^L \left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_{kj}} x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{k_i}}(b_i) \right) \\ &= \sum_{\vec{\epsilon_k} = \mathbf{1}_{2^q}} c_\lambda^L \left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_{kj}} x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{k_i}}(b_i) \right) \\ &= \sum_{\vec{\epsilon_k} = \mathbf{1}_{2^q}} c_\lambda^L \left(\bigwedge_{i=1}^{m} \alpha_{f_{k_i}}(b_i) \right). \end{aligned}$$

This will be equal to 1 if $\{\vec{f_k} \mid \vec{\epsilon_k} = \mathbf{1}_{2^q}\} = (\mathbb{N}_{2^q})^m$, that is, if every possible assignment of atoms to constants $a_1, \ldots, a_m$ occurs in some disjunct k such that $\vec{\epsilon_k} = \mathbf{1}_{2^q}$. Otherwise it will be less than 1, since c_λ^L satisfies Reg, so assigns non-zero probability to any such 'missing' assignment. The condition for this sum to equal 1 occurs just if $\zeta \models \theta'$, and since $\theta \equiv \theta'$ this is equivalent to $\zeta \models \theta$. □

Proposition 69. *Let χ be the sentence $\bigvee_{k=1}^{2^q} \forall x\, \alpha_k(x)$. Then for $\theta \in SL$,*

$$c_0^L(\theta) = 1 \iff \chi \models \theta.$$

Proof. Firstly note that, by (1.12),

$$\begin{aligned} c_0^L(\chi) &= \sum_{j=1}^{2^q} c_0^L(\forall x\, \alpha_j(x)) \\ &= \sum_{j=1}^{2^q} \lim_{n\to\infty} c_0^L\left(\bigwedge_{i\leq n} \alpha_j(a_i)\right) \\ &= \sum_{j=1}^{2^q} \lim_{n\to\infty} 2^{-q} = 1. \end{aligned}$$

Suppose $\chi \models \theta$, then

$$1 = c_0^L(\chi) \leq c_0^L(\theta) \leq 1.$$

In the other direction, suppose $\theta(b_1, \dots, b_m) \in SL$. Since L is unary, by Proposition 2, θ is logically equivalent to some sentence θ' of the form (A.1), where the notation is as described in the previous proof, and the disjuncts are disjoint and satisfiable. Therefore,

$$c_0^L(\theta') = \sum_{k=1}^{l} c_0^L\left(\bigwedge_{j=1}^{2^q} \exists^{\epsilon_{kj}} x\, \alpha_j(x) \wedge \bigwedge_{i=1}^{m} \alpha_{f_{k_i}}(b_i)\right).$$

The kth term of this sum will be non-zero just if $\vec{\epsilon_k}$ consists of a single '1' in position g_k, say, with all other entries zero, with $\vec{f_k} = \langle g_k, g_k, \dots, g_k\rangle$. (Any other configuration means either that the disjunct is inconsistent, or that it logically implies $\neg\chi$ and so receives probability 0 according to c_0^L). In this case, the value of the term is 2^{-q}, so for the total $c_0^L(\theta')$ to be equal to 1, it must be the case that $\{g_k \mid 1 \leq k \leq l\} = \mathbb{N}_{2^q}$, in which case $\chi \models \theta'$. Since $\theta \equiv \theta'$, the result follows. $\square$

We can now identify the theory of JSP as follows.

Theorem 70. *Where $q \geq 2$,*

$$Th(JSP) = \{\theta \in SL \mid \bigwedge_{k=1}^{2^q} \exists x\, \alpha_k(x) \vee \bigvee_{k=1}^{2^q} \forall x\, \alpha_k(x) \models \theta\}.$$

Proof. By Theorem 64, where $q \geq 2$, any probability function which satisfies JSP must be equal to c_λ^L for some $0 \leq \lambda \leq \infty$. It follows from the definitions (1.11) and (1.12) that c_λ^L satisfies Regularity just if $0 < \lambda$. Therefore by Propositions 68 and 69,

$$Th(JSP + Reg) = \{\theta \mid \bigwedge_{k=1}^{2^q} \exists x\, \alpha_k(x) \models \theta\},$$

while

$$Th(JSP + \neg Reg) = \{\theta \mid \bigvee_{k=1}^{2^q} \forall x\, \alpha_k(x) \models \theta\}.$$

The result follows. $\square$

By the previous result, the sentence

$$\forall x_1\, \neg\alpha_1(x_1) \vee \forall x_2\, \neg\alpha_2(x_2) \vee \exists x_3\, \alpha_3(x_3)$$

(where $q \geq 2$) belongs to Th(JSP), though it is not a tautology. Therefore Th(JSP) contains more than just tautologies, from which we obtain:

Corollary 71. *Where w is a probability function on a unary language L with $q \geq 2$, if w satisfies JSP then w does not satisfy Super-Regularity.*

Having identified $Th(JSP)$, it transpires that the power of JSP in narrowing our agent's choice of probability function comes at the price of a non-tautological 'creed', as with the principles of Sx and FVP discussed in earlier chapters. Whether this price is worth paying, in terms of what is 'most rational', is open to debate.