DERIVING INFORMATION FROM INCONSISTENT KNOWLEDGE BASES: A PROBABILISTIC APPROACH

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The core of this thesis is about inferring from inconsistent knowledge bases. To that purpose we first present and study a probabilistic consequence relation, ${}^{\eta}\triangleright_{\zeta}$. The idea behind this consequence relation responds to the situation where the set of premises (or knowledge base) consists of the beliefs of a single rational agent. A sentence in the knowledge base can then be assigned a degree of belief, corresponding to the degree to which our agent believes the sentence to be true (which we identify with subjective probability). What we could do next is to fix a lower bound probability threshold for each sentence in the knowledge base, say η. It might then be argued that we should be willing to accept as a consequence of it any other sentence which as a result has, by probability logic, probability at least some suitable threshold ζ.

Initially we identify our knowledge base with a finite set of propositions (in classical propositional logic). Later we extend results by considering infinite knowledge bases and the possibility of distinct probability thresholds for each sentence in our knowledge base.

We also present the consequence relation ${}^{\eta}\blacktriangleright_{\zeta}$, of the same nature as ${}^{\eta}\triangleright_{\zeta}$, with the difference that η and ζ are now truth-functional thresholds (within the frame of Łukasiewicz and Gödel logics).

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

It is with great frequency that we come across information which is, when formalized in a classical language, inconsistent.[1] In the presence of inconsistency classical entailment *explodes*.

A good example of inconsistent information is our set of beliefs. It is bound to be inconsistent. Moreover in most cases it may be so because we are not aware of such inconsistencies -maybe, if we were, we would *revise* our beliefs. In some cases though that does not seem to be reason why we hold them and the *Sorites paradox* is a good example of what we mean.

The Sorites paradox goes as follows:

> Suppose we have a certain number of objects, say n (if we assume that n is a large number we will *reasonably believe* that we have a *pile* of objects). Next we start removing objects, one at a time. We may believe that removing one object cannot make a difference as to turn a pile into a *non-pile*. Eventually though, by removing objects one at a time, we will come down to just one object, which is not a pile.
>
> Let us formalize this paradox with classical propositional logic. Let p_i stand for the proposition 'i objects are a pile', for all $i \in \{1, ..., n\}$. The following set of sentences gives form to the paradox:
>
> $$\Gamma_n = \{p_n, p_n \rightarrow p_{n-1}, ..., p_2 \rightarrow p_1, \neg p_1\}.$$
>
> The paradox rests upon the fact that it seems reasonable to believe

[1] Unless otherwise stated we will understand inconsistency in a classical way. That is, with respect to a classical language and classical semantics and entailment.

each sentence in Γ_n even though Γ_n is inconsistent.

This paradox, which originates due to the ambiguity of our natural language (due to the *vagueness* in meaning of the word 'pile') seems to indicate that we can *reasonably* hold beliefs that as a whole happen to be inconsistent.

Although of slightly different nature, the next example (known as *Kyburg's lottery paradox*, see [28]) dwells on the same idea.

It goes as follows:

> Suppose we have a lottery with n tickets. One of them will be declared the winner. If n is a large number it seems reasonable to believe that Ticket i will not win the lottery, for $i \in \{1, ..., n\}$ -notice that the probability of Ticket i not winning, provided the lottery is fair, is $\frac{n-1}{n}$ which, for n large, is very close to 1. On the other hand, it is clear that one ticket will be the winner.
>
> Let us proceed as in the previous example and formalize the paradox with propositional logic. Let t_i stand for the proposition 'Ticket i will win the lottery', for $i \in \{1, ..., n\}$. We will have the following set of sentences:
>
> $$\Gamma_n = \{\neg t_1, ..., \neg t_n, t_1 \vee ... \vee t_n\}.$$
>
> The set Γ_n is inconsistent. However, as in the previous example, it seems reasonable to believe each sentence in it.

The idea of *high probability* seems to be behind the formation of such beliefs in this example.

Other examples of inconsistent information may be easily found. Databases or legal systems may be or become easily inconsistent.

Suppose we have a legal system with the following two laws:

- Householders are allowed to vote in local elections.
- Immigrants do not have the right to vote in a local election.

But it may be the case that an immigrant be a householder and so such a legal system is clearly inconsistent. It is true though that this inconsistency is pretty obvious but, nonetheless, it is not hard to imagine situations where legal systems or databases could hold inconsistencies of this nature.

Some theories are clearly inconsistent too. The examples are numerous. We mention just one in the empirical sciences: The *Big Bang* theory and *Classical Thermodynamics*. When brought together and regarded as a unit we get an inconsistent theory. It is well known that one of the consequences of the Big Bang theory is that at some point in time the universe will start contracting. However, according to Classical Thermodynamics, that cannot happen since the entropy tends to increase indefinitely (see for example [14]).

But what to do in the face of such inconsistencies? Revise. The world (or, more generally, what we commonly refer to as *reality*) seems to be consistent –holding the belief that there are contradictory *objects* or *events* would certainly be controversial– and so, in as much as such sets of beliefs, databases, or theories are *representations* of some bits of this reality (and as much as we want them to be so –see [16]), they should be consistent and so revised in order to be left with consistent information. However, it may happen that we do not know how to go about revising, like in the last example, or that it is not efficient or feasible to revise. Some argue (see [48]) that revision theory is sometimes an ongoing process and so we are most of the time reasoning in the presence of inconsistencies and an inference relation able to account for such reasoning could be desirable. That is even clearer when talking about our beliefs in general, not just theories. Even though our beliefs taken as a whole may be inconsistent it is true that we are able to argue in a reasonable way and draw sensible conclusions out of our inconsistent beliefs (we do not *explode* in the presence of such inconsistencies). An inference relation able to account as fairly as possible for our reasoning from possibly inconsistent sets of beliefs would be desirable.[2]

The definition and study of some consequence relations able to model such reasoning in some particular situations is the aim of this thesis. The approach that we propose rests on the concept of *degree of belief*. We identify belief functions with probability functions (based on an identification between *coherent* or *rational* belief functions and probability functions made and justified by De Finetti -see

[2]There literature about inconsistency and the problem of dealing with inconsistent information is abundant. Some survey articles can be found in [2] and [40]. Formal methods for dealing with inconsistency are numerous. Some are described in [18], [38], [39], [41], [42] and [47].

[6] or, alternatively, [19] or [46] for a better insight).

1.1 Notation and some definitions

Throughout, unless otherwise stated, we will work with a finite propositional language $L = \{p_1, ..., p_l\}$. We will denote its corresponding set of sentences by SL (finite boolean combinations of our primitive propositions in L) and its corresponding set of atoms by At^L. By atoms we mean all the sentences of the form

$$\pm p_1 \wedge ... \wedge \pm p_l$$

where $+p_i$ and $-p_i$ stand for p_i and $\neg p_i$ respectively.

Let $\phi \in SL$. By the *Disjunctive Normal Form Theorem* (see for example [1] or [5] for this theorem and its proof) we know that there exists a unique set of atoms $S_\phi \subseteq At^L$ such that $\vdash \phi \leftrightarrow \bigvee S_\phi$, where $\vdash$ is classical entailment (here and throughout). It is clear that $S_\phi = \{\alpha \in At^L | \; \alpha \vdash \phi\}$.

Definition 1 *Let $w : SL \longrightarrow [0, 1]$. We say that w is a probability function on L if the following two conditions hold for all $\theta, \phi \in SL$:*

1. *If $\vdash \theta$ then $w(\theta) = 1$.*

2. *If $\vdash \neg(\theta \wedge \phi)$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$.*

From these two conditions the standard properties of probability functions follow. We cite some without proof (see [37] for a proof and more details).

Let $\theta, \phi \in SL$. The following properties hold:

1. $w(\phi \vee \theta) = w(\phi) + w(\theta) - w(\phi \wedge \theta)$.

2. $w(\neg\phi) = 1 - w(\phi)$.

3. If $\phi \vdash \theta$ then $w(\phi) \leq w(\theta)$.

4. If $\vdash \phi \leftrightarrow \theta$ then $w(\phi) = w(\theta)$.

It is worth observing that a probability function w is determined uniquely by the values it assigns to the atoms via the identities

$$w(\phi) = w(\bigvee S_\phi) = \sum_{\alpha \vdash \phi} w(\alpha)$$

and can thus be regarded as a 2^l-coordinate vector in $\mathbb{D}_{2^l}$, where

$$\mathbb{D}_{2^l} = \{(x_1, ..., x_{2^l}) \,|\, x_i \geq 0, \sum_i x_i = 1\}.$$

Sentences in SL can also be identified with 2^l-coordinate vectors.

Let $\phi \in SL$ and define the 2^l-coordinate vector $\vec{\phi}$ as follows: For each $i \in \{1, ..., 2^l\}$, $\phi^i = 1$ if $\alpha_i \vdash \phi$ and $\phi^i = 0$ otherwise.[3] A finite set of sentences, say $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$ can then be identified with a matrix (which we will denote by $M_{\underline{\Gamma}}$) whose rows are the 2^l-coordinate vectors corresponding to such sentences.

Though the order that we consider for the columns and rows of $M_{\underline{\Gamma}}$ does not really matter to our purposes we will assume throughout, unless otherwise stated, a certain ordering for them. The order of the rows will correspond to the order of Γ (assuming Γ is an ordered set of the form specified above, $\{\phi_1, ..., \phi_k\}$). The order of the columns will match that of the atoms of L, say $\leqslant$, which we define in a quite intuitive way.

First, for $\alpha \in At^L$, we define $|\alpha| = |\{p \in L|\, \alpha \vdash p\}|$. Now let $i, j \in \{1, ..., 2^l\}$, $i \neq j$.

1. If $|\alpha_i| < |\alpha_j|$ then $\alpha_i \leqslant \alpha_j$.

2. Assume now that $|\alpha_i| = |\alpha_j|$. Let us take the first propositional variable in L (which for this purpose we will assume ordered, $\{p_1, ..., p_l\}$), say p_r, such that $\alpha_i \vdash p_r$ and $\alpha_j \vdash \neg p_r$ or $\alpha_i \vdash \neg p_r$ and $\alpha_j \vdash p_r$. In the former we will have that $\alpha_i \leqslant \alpha_j$ and in the latter $\alpha_j \leqslant \alpha_i$.

That $\leqslant$ is a total ordering on At^L is clear.

Notice that in $M_{\underline{\Gamma}}$ we will likely have columns which are identical -which correspond to distinct atoms in At^L logically implying exactly the same sentences in Γ. To simplify we will sometimes consider, instead of At^L, the set of consistent sentences of the form

$$\pm\phi_1 \wedge ... \wedge \pm\phi_k$$

which we will denote by $\beta_1, ..., \beta_m$ and set $\mathcal{B} = \{\beta_1, ..., \beta_m\}$. We will consider $\mathcal{B}$ to be an ordered set.

[3] Here the superscript denotes the coordinate of a vector. In some other sections the notation will be slightly different and coordinates will be denoted by subscripts. We also assume that the atoms of L are ordered and the subscript i in α_i refers to that ordering.

We can define a total ordering $\leqslant$ on $\mathcal{B}$ as we did above for At^L.

For $\beta \in \mathcal{B}$, we define $|\beta| = |\{\phi \in \Gamma|\, \beta \vdash \phi\}|$. Now let $i, j \in \{1, ..., m\}$, $i \neq j$.

1. If $|\beta_i| < |\beta_j|$ then $\beta_i \leqslant \beta_j$.
2. Assume now that $|\beta_i| = |\beta_j|$. Let us take the first sentence in Γ, say ϕ_r, such that $\beta_i \vdash \phi_r$ and $\beta_j \vdash \neg\phi_r$ or $\beta_i \vdash \neg\phi_r$ and $\beta_j \vdash \phi_r$. In the former we will have that $\beta_i \leqslant \beta_j$ and in the latter $\beta_j \leqslant \beta_i$.

By using the sentences in $\mathcal{B}$ each sentence ϕ_i can be identified with an m-coordinate vector $\vec{\phi_i}$: For $j \in \{1, ..., m\}$, $\phi_i^j = 1$ if $\beta_j \vdash \phi_i$ and $\phi_i^j = 0$ otherwise. The matrix of Γ (denoted M_Γ) will then be a $k \times m$ matrix. In some sections, in contexts where Γ is taken to be a set of premises or knowledge base and θ a possible consequence of Γ under some inference relation, it will be useful to appeal to what we call the matrix for Γ and θ (denoted $M_{\Gamma,\theta}$), which is given by the matrix of Γ just defined by adding an additional row, that corresponding to $\vec{\theta}$, defined as follows: For $j \in \{1, ..., m\}$, $\theta^j = 1$ if $\beta_j \vdash \theta$ and $\theta^j = 0$ otherwise.

Let us consider an example:

Let $L = \{p, q, r\}$,

$$\Gamma = \{(p \vee q) \wedge r, (q \vee r) \wedge p, (r \vee p) \wedge q\} = \{\phi_1, \phi_2, \phi_3\}$$

and $\theta = p$.

Clearly $\mathcal{B} = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$, with

$$\begin{aligned}
&\beta_1 = \phi_1 \wedge \phi_2 \wedge \phi_3,\\
&\beta_2 = \phi_1 \wedge \phi_2 \wedge \neg\phi_3,\\
&\beta_3 = \phi_1 \wedge \neg\phi_2 \wedge \phi_3,\\
&\beta_4 = \neg\phi_1 \wedge \phi_2 \wedge \phi_3,\\
&\beta_5 = \neg\phi_1 \wedge \neg\phi_2 \wedge \neg\phi_3.
\end{aligned}$$

We will thus have the following matrix for Γ and θ:

$$M_{\Gamma,\theta} = \left(\begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

The matrix of Γ would be the corresponding submatrix of $M_{\Gamma,\theta}$:

$$M_\Gamma = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$M_{\underline{\Gamma}}$ would have the following form:

$$M_{\underline{\Gamma}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We can trivially extend $M_{\underline{\Gamma}}$ to the matrix $M_{\underline{\Gamma},\theta}$ as we did above with M_Γ and $M_{\Gamma,\theta}$.

In some further applications we will not consider columns which contain only 0's for the sake of simplicity.

Chapter 2

Measuring inconsistency

We start by considering some inconsistency measures.

2.1 η-coherence

In [45] Schotch and Jennings talk about the idea of *level of coherence* of sets of sentences and, to make the idea precise, they define what they call the *coherence function*, **c**, which in our settings can be defined as follows:

Definition 2 *Let* $\boldsymbol{c} : \mathcal{P}(SL) \longrightarrow \mathbb{N} \cup \{\omega\}$ [1] *and* $\Gamma \subseteq SL$.

> *If* $\bot \notin \Gamma$, $\boldsymbol{c}(\Gamma) = m \iff m$ *is the least natural number such that there exist sets* $\Delta_1, ..., \Delta_m$ *(with* $\Delta_i \nvdash \bot$ *for all* $i \in \{1, ..., m\}$*) and* $\bigcup_{i=1}^{m} \Delta_i = \Gamma$.[2]
>
> *If* $\bot \in \Gamma$ *we adopt the convention* $\boldsymbol{c}(\Gamma) = \omega$.

Although probably suitable to measure inconsistency in some contexts this function does not seem to be able to capture certain differences in *degree of consistency* that to us seem quite intuitive.

Let us consider Kyburg's lottery paradox as presented in the introduction. Recall that the set of sentences considered in this example to formalize the paradox was $\Gamma_n = \{\neg t_1, ..., \neg t_n, t_1 \vee ... \vee t_n\}$ for $n \in \mathbb{N}$.

Clearly $\mathbf{c}(\Gamma_n) = 2$ –since any proper subset of Γ_n is consistent (for example, set $\Delta_1 = \{\neg t_1, ..., \neg t_n\}$ and $\Delta_2 = \{t_1 \vee ... \vee t_n\}$). Notice that this is so for any

[1] $\mathcal{P}(SL)$ is the power set of SL and ω is the first transfinite cardinal.

[2] Here and throughout $\bot$ stands for classical contradiction.

value of n. $\mathbf{c}(\Gamma_n)$ is independent of n. But this is in some way counterintuitive and we are tempted to argue that the larger the n the *less* inconsistent Γ_n seems to be.

Based on this approach we can give an alternative measure of consistency.

For the next two definitions let $\Gamma \subseteq SL$ and $\eta \in [0,1]$.

Definition 3 *We say that Γ is η-coherent if and only if there exists a collection $\mathcal{A}$ of q copies of some consistent subsets of Γ and every sentence in Γ occurs at least in p copies, for some $\frac{p}{q} \geq \eta$.*

Let us consider again the set of sentences $\Gamma_n = \{\neg t_1, ..., \neg t_n, t_1 \vee ... \vee t_n\}$ in Kyburg's lottery paradox. Every proper subset of Γ_n is consistent. We can consider a collection $\mathcal{A}$ consisting of a copy of each maximal consistent subset of Γ_n. We will have $n+1$ such subsets, each containing n sentences. Thus Γ_n will be $\frac{n}{n+1}$-coherent, which somehow reflects our intuitive idea that the larger the n the less inconsistent Γ_n seems to be.

2.2 η-consistency

Next we recall the definition of η-consistency given in [25] and [26].

Let $\Gamma = \{\phi_1, ..., \phi_n\} \subseteq SL$ and $\eta \in [0,1]$.

Definition 4 *We say that Γ is η-consistent if and only if there exists a probability function w such that $w(\phi) \geq \eta$ for all $\phi \in \Gamma$ and maximally η-consistent if and only if Γ is η-consistent and there is no $\lambda > \eta$ for which Γ is λ-consistent.*

Normally we will use the abbreviation $w(\Gamma) \geq \eta$ to mean that $w(\phi) \geq \eta$ for all $\phi \in \Gamma$ and the abbreviation $w(\Gamma) = \eta$ to mean that $w(\phi) \geq \eta$ for all $\phi \in \Gamma$ and $w(\phi) = \eta$ for some $\phi \in \Gamma$.

We will denote the maximal consistency of Γ by $mc(\Gamma)$.

It is important to remember that $mc(\Gamma)$ is always rational and always attained[3] by a certain probability function (see [25] or [26] for these and other properties of η-consistency).

[3] That is to say, there exists a probability function w on L such that $w(\Gamma) = mc(\Gamma)$.

Lemma 5 *There exists a probability function w that assigns only rational values to the atoms in At^L such that $w(\Gamma) = mc(\Gamma)$.*

Proof. Let $mc(\Gamma) = \lambda$.

Recall from Section 1.1 that we can identify any probability function on L with a vector $\vec{x}$ in $\mathbb{D}_{2^l}$ and a sentence $\phi \in \Gamma$ with a vector $\vec{\phi}$ of $0's$ and $1's$ of the same dimension. Thus the statement (which we will denote by Θ)

'*There exists a probability function w such that $w(\Gamma) = \lambda$*'

can be expressed in the language of the structure $\langle \mathbb{R}, +, <, =, 0, 1, \lambda \rangle$. Furthermore, Θ is true in $\langle \mathbb{R}, +, <, =, 0, 1, \lambda \rangle$.

$\langle \mathbb{Q}, +, <, =, 0, 1, \lambda \rangle$ is an elementary substructure of $\langle \mathbb{R}, +, <, =, 0, 1, \lambda \rangle$ and thus Θ has to be true in $\langle \mathbb{Q}, +, <, =, 0, 1, \lambda \rangle$ too. Therefore, there has to exist a probability function w that assigns only rational values to the atoms of L such that $w(\Gamma) = mc(\Gamma) = \lambda$. ■

Again let $\mathcal{B} = \{\beta_1, ..., \beta_m\}$ be the set of sentences of the form

$$\pm\phi_1 \wedge ... \wedge \pm\phi_n$$

that are consistent.

Proposition 6 *Γ is η-coherent if and only if Γ is η-consistent.*

Proof. Let us assume that Γ is η-coherent. Thus there exists a collection $\mathcal{A}$ with q copies of consistent subsets of Γ in which every sentence occurs at least p times, for $\eta \leq \frac{p}{q}$.

Assume that $\mathcal{A} = \{\Delta_1, ..., \Delta_q\}$. Since the subsets Δ_i are consistent we can take $\alpha_{k_i} \in At^L$ such that $\alpha_{k_i} \vdash \Delta_i$. This way we get a collection of q copies of atoms in At^L, $\{\alpha_{k_1}, ..., \alpha_{k_q}\}$. Now let w be a probability function that assigns to each atom $\alpha \in At^L$ probability $\frac{r}{q}$, where r is the number of copies of α in the collection $\{\alpha_{k_1}, ..., \alpha_{k_q}\}$. Thus Γ will be η-consistent since $w(\phi) \geq \frac{p}{q}$ for all $\phi \in \Gamma$.

In the other direction let us assume that Γ is η-consistent, $\eta \leq \frac{p}{q} = mc(\Gamma)$. Let us consider a set of atoms $\{\alpha_{k_1}, ..., \alpha_{k_r}\} \subseteq At^L$ that yields the maximal consistency of Γ (that is, there exists a probability function w such that $w(\bigvee_{i=1}^{r} \alpha_{k_i}) = 1$ and $w(\Gamma) = mc(\Gamma) = \frac{p}{q}$). Assume further that w is such that $w(\alpha_{k_i}) = \frac{p_i}{q_i}$ for some positive integers p_i and q_i, for each $i \in \{1, ..., r\}$ (we know that such a probability function needs to exist by Lemma 5). Let Q be the least common multiple of the q_i and p_i^* be such that $\frac{p_i}{q_i} = \frac{p_i^*}{Q}$ (and $\frac{p}{q} = \frac{P}{Q}$). Let us set now

$$\mathcal{A} = \{\Delta_1^1, ..., \Delta_{p_1^*}^1, ..., \Delta_1^r, ..., \Delta_{p_r^*}^r\},$$

where $\Delta_j^i = \{\phi \in \Gamma \mid \alpha_{k_i} \vdash \phi\}$ for each $j \in \{1, ..., p_i^*\}$. Notice that in such collection there are at least P copies of each sentence. Thus Γ is coherent to degree $\frac{p}{q}$ and therefore η-coherent. ■

Corollary 7 *There exists $\eta \in \mathbb{Q} \cap [0, 1]$ for which Γ is maximally η-coherent (that is to say, Γ is η-coherent and there is no $\lambda > \eta$ for which Γ is λ-coherent).*

We will denote the maximal coherence of Γ by $MC(\Gamma)$. We will then have that $MC(\Gamma) = mc(\Gamma)$.

Chapter 3

The consequence relation ${}^{\eta}\triangleright_{\zeta}$

In this chapter we study different aspects of the consequence relation ${}^{\eta}\triangleright_{\zeta}$ which, in a more restricted form, was first presented in [34] (for $\eta = \zeta$) and later extended in [43] (for η, ζ rational).

The idea behind this consequence relation responds to the situation where the set of premises (or knowledge base) consists of the assertions held by a single rational agent, such as ourselves. A sentence in our knowledge base can then be assigned a degree of belief, corresponding to the degree to which we believe the sentence is true (which we identify with subjective probability –see [6], [19] or [46] for a justification of this identification and a better insight into the topic). What we could do next is to fix a lower bound probability threshold for each sentence in our knowledge base, say η. It might then be argued that we should be willing to accept as consequences of it any other sentences which as a result have (by probability logic, see for example [10] or [37]) probability at least some suitable threshold ζ. The consequence relation ${}^{\eta}\triangleright_{\zeta}$ responds to this idea.

The natural choice for ζ is η itself –that is, that given a probability threshold for the premises the consequences be at least as probable as the premises (that was the approach followed in [34]). However, in some situations, the fact that $\zeta > \eta$ or $\eta < \zeta$ may be well justified.

3.1 ${}^{\eta}\triangleright_{\zeta}$: Definition and some properties

Throughout this section let $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$, $\theta \in SL$ and $\eta, \zeta \in [0, 1]$.

Definition 8 *We say that* Γ *(η, ζ)-implies* θ *(denoted* $\Gamma^{\eta}\triangleright_{\zeta}\theta$*) if and only if, for* *all probability functions w, if $w(\Gamma) \geq \eta$ then $w(\theta) \geq \zeta$.*

The next proposition gives us some information about how the relation between Γ and θ under ${}^{\eta}\!\rhd_{\zeta}$ is preserved when varying the values η, ζ.

Proposition 9 *If $\Gamma^{\eta} \rhd_{\zeta} \theta$ then $\Gamma^{\eta^{+}} \rhd_{\zeta^{-}} \theta$, where $\zeta^{-} \leq \zeta$ and $\eta \leq \eta^{+}$.*

Proof. It follows directly from the definition of ${}^{\eta}\!\rhd_{\zeta}$. ■

Next we state some properties of ${}^{\eta}\!\rhd_{\zeta}$ regarding extreme values for η, ζ and classical entailment.

Proposition 10 *We have what follows:*

- *(i)* $\Gamma^{\eta} \rhd_{0} \theta$ *for all* $\eta \in [0, 1]$.
- *(ii)* *For* $\eta > mc(\Gamma)$ *we have that* $\Gamma^{\eta} \rhd_{1} \theta$.
- *(iii)* *For* $\zeta > 0$, $\Gamma^{1} \rhd_{\zeta} \theta \iff \Gamma \vdash \theta$.
- *(iv)* *For* $\zeta > 0$, $\Gamma^{0} \rhd_{\zeta} \theta \iff \vdash \theta$.

Proof. Parts (i) and (ii) follow immediately from the definition of ${}^{\eta}\!\rhd_{\zeta}$.

Let us prove (iii) by *reductio ad absurdum.* Let us assume that $\Gamma^{1} \rhd_{\zeta} \theta$, for $\zeta > 0$, and that $\Gamma \nvdash \theta$. Thus there exists $\alpha \in At^{L}$ such that $\alpha \vdash \phi$ for all $\phi \in \Gamma$ and $\alpha \nvdash \theta$. Then we can define a probability function w that assigns probability 1 to this atom and null probability to the other atoms, which contradicts the fact that $\Gamma^{1} \rhd_{\zeta} \theta$ for $\zeta > 0$. Conversely suppose that $\Gamma \vdash \theta$ and that w is a probability function on L for which $w(\Gamma) = 1$ (the result follows trivially if there is no such probability function). Thus, if $w(\alpha) > 0$ then $\alpha \vdash \phi$ for all $\phi \in \Gamma$. But since $\Gamma \vdash \theta$ we will have that $\alpha \vdash \theta$ too, giving $w(\theta) = 1$ and consequently $\Gamma^{1} \rhd_{\zeta} \theta$ for any $\zeta \in [0, 1]$. This completes the proof of (iii).

To prove (iv) let us proceed again by *reductio ad absurdum* and assume that $\Gamma^{0} \rhd_{\zeta} \theta$, for $\zeta > 0$, and that $\nvdash \theta$. Thus there has to exist an atom $\alpha \in At^{L}$ such that $\alpha \nvdash \theta$. Then we can define a probability function w that assigns probability 1 to this atom and null probability to the other atoms. That would imply that $\Gamma^{0} \not\rhd_{\zeta} \theta$, contradicting the assumption above. In the other direction, if $\vdash \theta$ then all probability functions w are such that $w(\theta) = 1$ and therefore $\Gamma^{0} \rhd_{\zeta} \theta$ for any $\zeta \in [0, 1]$. This completes the proof of (iv). ■

Proposition 11 $\Gamma^{\eta} \rhd_{\eta} \theta$ *for all* $\eta \in (0, 1]$ *if and only if there exists* $\phi \in \Gamma$ *such that* $\phi \vdash \theta$.

Proof. Let us proceed by *reductio at absurdum* to prove the right implication. Let us assume that $\Gamma^{\eta} \triangleright_{\eta} \theta$ for all $\eta \in (0,1]$ and that for all $i \in \{1, ..., k\}$ we have that $\phi_i \nvdash \theta$. Then for all $i \in \{1, ..., k\}$ we can find an atom $\alpha_{j_i} \in \{\alpha_1, ..., \alpha_{2^l}\}$ such that $\alpha_{j_i} \vdash \phi_i$ and $\alpha_{j_i} \nvdash \theta$. Let us take one such atom for each ϕ_i and denote by S the set defined by them. We can define a probability function w that assigns probability $\frac{1}{|S|}$ to every atom in S and null probability to any other atom. Clearly w will be such that $w(\Gamma) \geq \frac{1}{|S|}$ and $w(\theta) = 0$, contradicting the assumption above.

Conversely suppose that $\phi_i \vdash \theta$ for some $i \in \{1, ..., k\}$. Thus, $w(\phi_i) \leq w(\theta)$ for any probability function w on L (see Section 1.1). Therefore, for all probability functions w, if $w(\Gamma) \geq \eta$ then $w(\theta) \geq \eta$, for all $\eta \in (0,1]$. ■

We state now two properties of ${}^{\eta}\triangleright_{\zeta}$, those corresponding to *left weakening* (monotonicity) and *right weakening*.

Proposition 12 *We have what follows:*

(i) *If* $\Gamma^{\eta} \triangleright_{\zeta} \theta$ *then* $\Gamma \cup \{\psi\}^{\eta} \triangleright_{\zeta} \theta$.

(ii) *If* $\Gamma^{\eta} \triangleright_{\zeta} \theta$ *and* $\theta \vdash \psi$ *then* $\Gamma^{\eta} \triangleright_{\zeta} \psi$.

Proof. The proof follows directly from the definition of ${}^{\eta}\triangleright_{\zeta}$ and the properties of probability functions presented in Section 1.1. ■

The next proposition gives a closure property of the pairs (η, ζ) for which $\Gamma^{\eta} \triangleright_{\zeta} \theta$.

Proposition 13 *Let* $\{\eta_n\}$ *be an increasing sequence with limit* η *and* $\{\zeta_n\}$ *a sequence with limit* ζ. *If* $\Gamma^{\eta_n} \triangleright_{\zeta_n} \theta$ *for all* $n \in \mathbb{N}$ *then* $\Gamma^{\eta} \triangleright_{\zeta} \theta$.

Proof. Let us proceed by *reductio ad absurdum* assuming that $\Gamma^{\eta} \not\triangleright_{\zeta} \theta$. Then there exists a probability function w such that $w(\Gamma) \geq \eta$ and $w(\theta) < \zeta$. But then, for some $n \in \mathbb{N}$, $w(\theta) < \zeta_n$ and, since $\eta_n \leq \eta$, $\Gamma^{\eta_n} \not\triangleright_{\zeta_n} \theta$. This contradicts our assumption. ■

We next prove that ${}^{\eta}\triangleright_{\zeta}$ is *language invariant*.

By *language invariant* we mean that, given two finite propositional languages L_1 and L_2 with $\Gamma \subseteq SL_1 \cap SL_2$ and $\theta \in SL_1 \cap SL_2$, $w_1(\theta) \geq \zeta$ for any probability function w_1 on L_1 such that $w_1(\Gamma) \geq \eta$ if and only if $w_2(\theta) \geq \zeta$ for any probability function w_2 on L_2 such that $w_2(\Gamma) \geq \eta$ (in other words, $\Gamma^{\eta} \triangleright_{\zeta} \theta$ in the context of the language L_1 if and only if $\Gamma^{\eta} \triangleright_{\zeta} \theta$ in the context of the language L_2).

Proposition 14 *The relation ${}^{\eta}\!\triangleright_{\zeta}$ is language invariant.*

Proof. Suppose that $\Gamma^{\eta}\triangleright_{\zeta}\theta$ in the context of the language L. It is enough to show that if L^* is the language obtained from L by adding a single new propositional variable p then $\Gamma^{\eta}\triangleright_{\zeta}\theta$ in the context of L^* if and only if $\Gamma^{\eta}\triangleright_{\zeta}\theta$ in the context of L.

Let us first suppose that w^* is a probability function on L^* such that $w^*(\Gamma)\geq\eta$ but $w^*(\theta)<\zeta$. Let w be the restriction of w^* to L. Then w will agree with w^* on Γ and θ and so $\Gamma^{\eta}\not\triangleright_{\zeta}\theta$ in the context of L too.

Conversely suppose that w is a probability function on L such that $w(\Gamma)\geq\eta$ but $w(\theta)<\zeta$. Notice that the atoms of L^* are of the form $\alpha\wedge\pm p$, where $+p$ and $-p$ stand for p and $\neg p$ respectively and α is an atom of L. We can define w^* on L^* as follows:

$$w^*(\alpha\wedge p)=w(\alpha),$$

$$w^*(\alpha\wedge\neg p)=0.$$

Then, for $\phi\in SL$,

$$w(\phi)=\sum_{\alpha\vdash\phi}w(\alpha)=\sum_{\alpha\vdash\phi}w^*(\alpha\wedge p)+w^*(\alpha\wedge\neg p)=\sum_{\beta\vdash\phi}w^*(\beta)=w^*(\phi)$$

where the β's range over the atoms of L^* since, for $\phi\in SL$,

$$\alpha\vdash\phi\iff\alpha\wedge p\vdash\phi\iff\alpha\wedge\neg p\vdash\phi.$$

Hence $\Gamma^{\eta}\not\triangleright_{\zeta}\theta$ in the context of the language L^*. ■

3.2 An equivalent of ${}^{\eta}\!\triangleright_{\zeta}$ within propositional logic

In this section we derive an equivalent to ${}^{\eta}\!\triangleright_{\zeta}$ in terms of propositional logic.

The first part of the derivation we present here follows the pattern of that in [34] for the consequence relation ${}^{\eta}\!\triangleright_{\eta}$. It first appeared in [43] for the consequence relation ${}^{\eta}\!\triangleright_{\zeta}$, where η and ζ were rational values, and we reproduce it here (with some important modifications).

We start by considering the case of η and ζ rational, say $\eta=\frac{c}{d}$ and $\zeta=\frac{e}{f}$, with $c, d, e, f \in \mathbb{N}$. We can assume that $\eta, \zeta > 0$ since, if either of these values is 0, we trivially have an equivalent propositional version by Proposition 10.

So let us assume that

$$\theta_1, ..., \theta_n \; {}^{\frac{c}{d}}\!\triangleright_{\frac{e}{f}} \; \phi \tag{3.1}$$

and for the present that ϕ is not a tautology.

Consider $\mathcal{B} = \{\beta_1, ..., \beta_m\}$.

Let $\vec{\theta_i}$ be that m-vector with j^{th} coordinate 1 if $\beta_j \vdash \theta_i$ and 0 otherwise (i.e. in case $\beta_j \vdash \neg\theta_i$) and let $\vec{\phi}$ be the m-vector with j^{th} coordinate 1 if $\beta_j \vdash \phi$ and 0 otherwise.

Condition (3.1) is equivalent to

$$\text{For all } \vec{x} \in \mathbb{D}_m, \text{ if } \vec{\theta_i} \cdot \vec{x} \geq \frac{c}{d} \text{ for } i \in \{1, ..., n\} \text{ then } \vec{\phi} \cdot \vec{x} \geq \frac{e}{f} \tag{3.2}$$

where

$$\mathbb{D}_m = \{(x_1, ..., x_m) \,|\, x_i \geq 0, \sum_i x_i = 1\}.$$

This follows since for any probability function w,

$$(w(\beta_1), ..., w(\beta_m)) \in \mathbb{D}_m$$

and

$$w(\theta_i) = \sum_{\beta_j \vdash \theta_i} w(\beta_j) = \vec{\theta_i} \cdot (w(\beta_1), ..., w(\beta_m)).$$

Let $\vec{1}$ be the m-vector with 1's at each coordinate and let

$$\underline{\vec{\theta_i}} = \vec{\theta_i} - \frac{c}{d}\vec{1}, \qquad \underline{\vec{\phi}} = \vec{\phi} - \frac{e}{f}\vec{1}.$$

Then (3.2) can be restated as

$$\text{For all } \vec{x} \in \mathbb{D}_m, \text{ if } \underline{\vec{\theta_i}} \cdot \vec{x} \geq 0 \text{ for } i \in \{1, ..., n\} \text{ then } \underline{\vec{\phi}} \cdot \vec{x} \geq 0. \tag{3.3}$$

This is equivalent to the assertion that $\underline{\vec{\phi}}$ is in the cone in $\mathbb{Q}^m$ given by

$$\left\{ \sum_{i=1}^{n} a_i \underline{\vec{\theta_i}} + \sum_{j=1}^{m} b_j \vec{u_j} \,|\, 0 \leq a_i, b_j \in \mathbb{Q} \right\}$$

where $\vec{u_j}$ is the m-vector with j^{th} coordinate 1 and all other coordinates 0.[1] In other words, it is equivalent to (3.3) that there are some positive $a_i \in \mathbb{Q}$ such that

$$\underline{\vec{\phi}} \geq \sum_{i=1}^{n} a_i \underline{\vec{\theta_i}}. \tag{3.4}$$

Written in terms of a common denominator M let $a_i = \frac{N_i}{M}$ where $M, N_i \in \mathbb{N}$. Then (3.4) becomes

$$M(df\vec{\phi} - de\vec{1}) \geq \sum_{i=1}^{n} N_i(df\vec{\theta_i} - cf\vec{1}).$$

Equivalently

$$[Md(f-e) + cf\sum_{i=1}^{n} N_i]\vec{1} \geq Mdf(\vec{1} - \vec{\phi}) + \sum_{i=1}^{n} dfN_i\vec{\theta_i}. \tag{3.5}$$

Conversely if (3.5) holds for some natural numbers $M > 0$, $N_1, ..., N_n \geq 0$ then we can reverse this chain to get back (3.1).

Now let $\chi_1, ..., \chi_N \in \{\theta_1, ..., \theta_n\}$ be such that among these $\chi_1, ..., \chi_N$ the sentence θ_i appears exactly dfN_i times for each $i \in \{1, ..., n\}$ (so $N = df\sum_i N_i$). Then for $\beta_r \nvdash \phi$ it follows from (3.5) that the r^{th} coordinate of $\vec{\chi_j}$ is non-zero for at most

$$-deM + cf\sum_i N_i = \frac{cN - d^2eM}{d}$$

many j (notice that because ϕ is not a tautology there is at least one such r). Hence

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S| > \frac{cN - d^2eM}{d}}} \bigwedge_{j \in S} \chi_j \vdash \phi. \tag{3.6}$$

Notice that, by the choice of N, $\frac{cN-d^2eM}{d}$ is an integer and indeed non-negative

[1] See [34] for a more detailed explanation of this equivalence and [27] (page 50, Theorem 2) for the result in linear algebra in which such equivalence is justified.

since ϕ is not a tautology. Similarly if $\beta_r \vdash \phi$ then it follows from (3.5) that the r^{th} coordinate of $\vec{\chi_j}$ is non-zero for at most

$$Md(f-e) + cf\sum_{i=1}^{n} N_i = \frac{cN + d^2M(f-e)}{d}$$

many j. Hence

$$\bigvee_{\substack{S\subseteq\{1,\ldots,N\} \\ |S|>\frac{cN+d^2M(f-e)}{d}}} \bigwedge_{j\in S} \chi_j \vdash \bot. \tag{3.7}$$

Now let

$$\begin{aligned} Z &= 1 + \frac{cN + d^2M(f-e)}{d} \\ T &= 1 + \frac{cN - d^2eM}{d} \end{aligned}$$

so $1 \leq T < Z$ and

$$Td(f-e) = fcN - edZ + df.$$

From (3.6), (3.7) we have that

$$\bigvee_{\substack{S\subseteq\{1,\ldots,N\} \\ |S|=Z}} \bigwedge_{j\in S} \chi_j \vdash \bot, \tag{3.8}$$

$$\bigvee_{\substack{S\subseteq\{1,\ldots,N\} \\ |S|=T}} \bigwedge_{j\in S} \chi_j \vdash \phi, \tag{3.9}$$

$$Td(f-e) = fcN - edZ + df \text{ and } T < Z. \tag{3.10}$$

Conversely suppose that for some $T, Z \in \mathbb{N}$ and $\chi_1, ..., \chi_N$ (not necessarily those above) (3.8), (3.9) and

$$Td(f-e) \leq fcN - edZ + df \text{ and } 1 \leq T < Z \tag{3.11}$$

hold. Then for any atom $\alpha \in At^L$, if $\alpha \vdash \neg\phi$ then for at most $T-1$ many j can we have that $\alpha \vdash \chi_j$. Similarly if $\alpha \vdash \phi$ then there can be at most $Z-1$ such j. Hence, using the earlier vector notation but now with the genuine atoms in At^L

replacing the β's in $\mathcal{B}$ we have that

$$\sum_{j=1}^{N} \vec{\chi_j} \leq (T-1)\vec{1} + (Z-T)\vec{\phi}. \tag{3.12}$$

Now suppose $\vec{x} \in \mathbb{D}_{2^l}$ and $\vec{\chi_j} \cdot \vec{x} \geq \frac{c}{d}$ for $j \in \{1, ..., N\}$. Then dotting each side of (3.12) with $\vec{x}$ we obtain

$$(Z-T)\vec{\phi} \cdot \vec{x} \geq \frac{c}{d}N - T + 1.$$

But from (3.11) we have that

$$\frac{\frac{c}{d}N - T + 1}{Z - T} \geq \frac{e}{f}$$

so $\vec{\phi} \cdot \vec{x} \geq \frac{e}{f}$.

To sum up, if (3.8), (3.9) and (3.11) hold then

$$\chi_1, ..., \chi_N \ {}^{\frac{c}{d}}\rhd_{\frac{e}{f}} \ \phi$$

and by Proposition 12 (i) (if necessary) we have

$$\theta_1, ..., \theta_n \ {}^{\frac{c}{d}}\rhd_{\frac{e}{f}} \ \phi.$$

Conversely if

$$\theta_1, ..., \theta_n \ {}^{\frac{c}{d}}\rhd_{\frac{e}{f}} \ \phi$$

then there are sentences $\chi_1, ..., \chi_N \in \Gamma$ (possibly with repeats) such that for some Z and T conditions (3.8), (3.9) and (3.11) hold. [Indeed we can even have equality in the first inequality in (3.11) though for practical purposes it is very convenient to adopt the weaker version.]

Taking $\eta = \frac{c}{d}$ and $\zeta = \frac{e}{f}$ we now obtain the following propositional equivalent of ${}^{\eta}\rhd_{\zeta}$. We will complete the proof of this theorem for possibly irrational η, ζ at the end of the next section (which will assume this theorem but only in the proved rational case).

Theorem 15 *Let $\eta, \zeta \in (0, 1]$. Then for $\theta_1, ..., \theta_n, \phi \in SL$,*

$\theta_1, ..., \theta_n \ {}^{\eta}\rhd_{\zeta} \ \phi \iff$

$$\exists \chi_1, ..., \chi_N \in \{\theta_1, ..., \theta_n\} \ \ \textit{(possibly with repeats) and} \ \ T, Z \in \mathbb{N} \ \ \textit{such that}$$

$$T(1-\zeta) \leq \eta N - \zeta Z + 1, \quad T < Z \ \textit{and}$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=Z}} \bigwedge_{j \in S} \chi_j \vdash \bot,$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=T}} \bigwedge_{j \in S} \chi_j \vdash \phi.$$

Proof. The proof follows from the above discussion except for the case when ϕ is a tautology.

Let us then assume that ϕ is a tautology. Then both sides hold since we can take $T = N = 0$ and $Z = 1$. ■

Theorem 15 allows us to work with ${}^{\eta}\triangleright_{\zeta}$ entirely within the framework of the propositional calculus.

To give an idea of how this works in practice let us consider an example.

> Let $\Gamma = \{p \wedge q, p \wedge \neg q \wedge r, \neg p \wedge q \wedge r\}$ and $\phi = r$.
>
> To see that indeed $\Gamma^{\frac{1}{3}} \triangleright_{\frac{2}{3}} \phi$ it is enough here to take $\chi_1 = p \wedge \neg q \wedge r$ and $\chi_2 = \neg p \wedge q \wedge r$ (so $N = 2$). Then, for $Z = 2$ and $T = 1$, the conditions (3.8), (3.9) and (3.11) in the above discussion hold.

3.3 $F_{\Gamma,\theta}$: Definition and some properties

We have seen in the previous section that if $\Gamma^{\eta} \triangleright_{\zeta} \theta$ then $\Gamma^{\eta} \triangleright_{\zeta^-} \theta$ for any $\zeta^- < \zeta$. In this sense the supremum of ζ for which $\Gamma^{\eta} \triangleright_{\zeta} \theta$ at each value η is of special interest.

Let us define the map $F_{\Gamma,\theta}$ as follows, for all $\eta \in [0,1]$:

$$F_{\Gamma,\theta}(\eta) = sup\,\{\, \zeta \mid \Gamma^{\eta} \triangleright_{\zeta} \theta \,\}.$$

We now state some basic properties of $F_{\Gamma,\theta}$. The first one is rather trivial.

Proposition 16 $F_{\Gamma,\theta}$ *is increasing.*

Proof. It follows directly from the definition of $^{\eta}\triangleright_{\zeta}$. ■

The next proposition states that, given Γ η-consistent, the value $F_{\Gamma,\theta}(\eta)$ is actually attainable by a certain probability function. The proof of this proposition makes use of the fact that we can represent sentences and probability functions as 2^l-coordinate vectors (see Section 1.1).

For the next propositions let $\Gamma = \{\phi_1, ..., \phi_m\} \subseteq SL$ and $\theta \in SL$.

Proposition 17 *Let Γ be η-consistent, $\eta \in [0,1]$. There exists a probability function w such that $w(\Gamma) \geq \eta$ and $w(\theta) = F_{\Gamma,\theta}(\eta)$.*

Proof. Let $M_{\underline{\Gamma}}$ be the matrix representing the set of sentences Γ with respect to the atoms of L and $\vec{\theta}$ the sentence θ.

We can define a decreasing sequence $\{\zeta_n\}$ whose limit is ζ such that for all $n \in \mathbb{N}$ there exists a probability function w_n with $w_n(\theta) = \zeta_n$ and $w_n(\Gamma) \geq \eta$. By using the same notation as above we can represent $\{w_n\}$ by a sequence of vectors $\{\vec{x_n}\}$ such that $\vec{\theta} \cdot \vec{x_n} = \zeta_n$ for all $n \in \mathbb{N}$ and $(M_{\underline{\Gamma}}(\vec{x_n})^T)^j \geq \eta$ for all $j \in \{1, ..., m\}$.

We need to prove now that there exists a probability function $\vec{x} \in \mathbb{D}_{2^l}$ such that $\vec{\theta} \cdot \vec{x} = \zeta$ and $(M_{\underline{\Gamma}}(\vec{x})^T)^j \geq \eta$ for all $j \in \{1, ..., m\}$.

We can take a convergent subsequence $\{\vec{x}^1_{n_r}\}$ in the first coordinates of $\{\vec{x}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0,1]$ by compactness –the first components $(\vec{x}_n)^1$ of the vectors of the sequence $\{\vec{x}_n\}$ are all in the compact space $[0,1]$. Next we can pick a convergent subsequence $\{\vec{x}^2_{n_r}\}$ in the second coordinates of $\{\vec{x}^1_{n_r}\}$. As before, such subsequence needs to exist by compactness. We can proceed in the same way for the other coordinates.

The final subsequence, $\{\vec{x}^{2^l}_{n_r}\}$, will have as a limit the probability function $\vec{x} \in \mathbb{D}_{2^l}$ such that $(M_{\underline{\Gamma}}(\vec{x})^T)^j \geq \eta$ for all $j \in \{1, ..., m\}$ and $\vec{\theta} \cdot \vec{x} = \zeta$. ■

Proposition 18 *$F_{\Gamma,\theta}$ is convex on $[0, mc(\Gamma)]$.*

Proof. Assume that $0 \leq \eta_1 < \eta_2 \leq mc(\Gamma)$. By definition $F_{\Gamma,\theta}$ will be convex on $[0, mc(\Gamma)]$ if and only if, for all $\lambda \in [0,1]$,

$$F_{\Gamma,\theta}(\lambda\eta_1 + (1-\lambda)\eta_2) \leq \lambda F_{\Gamma,\theta}(\eta_1) + (1-\lambda) F_{\Gamma,\theta}(\eta_2).$$

By Proposition 17 we know that there exists a probability function w_1 such that $w_1(\Gamma) \geq \eta_1$ and $w_1(\theta) = F_{\Gamma,\theta}(\eta_1)$. In the same way we know that there exists a probability function w_2 such that $w_2(\Gamma) \geq \eta_2$ and $w_2(\theta) = F_{\Gamma,\theta}(\eta_2)$.

Thus we can define a new probability function w as follows:

$$w(\phi) = \lambda_1 w_1(\phi) + (1 - \lambda_2) w_2(\phi)$$

for $\phi \in SL$.

Then $w(\Gamma) \geq \lambda_1\eta_1 + (1-\lambda_2)\eta_2$ and $w(\theta) = \lambda_1 F_{\Gamma,\theta}(\eta_1) + (1-\lambda_2)F_{\Gamma,\theta}(\eta_2)$.

It follows that

$$F_{\Gamma,\theta}(\lambda_1\eta_1 + (1-\lambda_2)\eta_2) \leq \lambda_1 F_{\Gamma,\theta}(\eta_1) + (1-\lambda_2)F_{\Gamma,\theta}(\eta_2).$$

Therefore $F_{\Gamma,\theta}$ is convex on $[0, mc(\Gamma)]$. ■

Proposition 19 *$F_{\Gamma,\theta}$ is continuous on $[0, mc(\Gamma)]$.*

Proof. Let us prove it by *reductio ad absurdum.*

Suppose $F_{\Gamma,\theta}$ is not continuous from the right at $\eta \in [0, mc(\Gamma))$.

That means that there exists $\epsilon > 0$ such that, for all $x \in (\eta, mc(\Gamma)]$,

$$|F_{\Gamma,\theta}(x) - F_{\Gamma,\theta}(\eta)| > \epsilon$$

Let us consider $x = \lambda\, mc(\Gamma) + (1-\lambda)\eta$, where

$$\lambda = \frac{\epsilon^-}{F_{\Gamma,\theta}(mc(\Gamma)) - F_{\Gamma,\theta}(\eta)}$$

with $0 < \epsilon^- < \epsilon$. Then

$$F_{\Gamma,\theta}(x) \leq \lambda F_{\Gamma,\theta}(mc(\Gamma)) + (1-\lambda)F_{\Gamma,\theta}(\eta)$$

since $F_{\Gamma,\theta}$ is convex.

Therefore

$$|F_{\Gamma,\theta}(x) - F_{\Gamma,\theta}(\eta)| \leq |\lambda F_{\Gamma,\theta}(mc(\Gamma)) + (1-\lambda)F_{\Gamma,\theta}(\eta) - F_{\Gamma,\theta}(\eta)| = \epsilon^- < \epsilon$$

which contradicts the assumption we started with.

To prove continuity from the left at $\eta \in (0, mc(\Gamma)]$ let us consider again $M_{\underline{\Gamma}}$, the matrix of Γ with respect to the atoms of L, and the vector $\vec{\theta}$.

Let us assume $F_{\Gamma,\theta}$ is not continuous from the left at η.

Let $\zeta = \sup\{F_{\Gamma,\theta}(\gamma) \mid \gamma < \eta\}$. We can define an increasing sequence $\{\eta_n\}$ with limit η and a sequence $\{\zeta_n\}$ with limit ζ such that for all $n \in \mathbb{N}$ there exists a probability function w_n with $w_n(\Gamma) = \eta_n$ and $w_n(\theta) = \zeta_n$. We can represent $\{w_n\}$ by a sequence of vectors $\{\vec{x_n}\}$ such that $(M_{\underline{\Gamma}}(\vec{x_n})^T)^j \geq \eta_n$ for all $j \in \{1, ..., m\}$ (with $(M_{\underline{\Gamma}}(\vec{x_n})^T)^j \geq \eta_n$ for some j) and $\vec{\theta} \cdot \vec{x_n} = \zeta_n$ for all n.

We can take a convergent subsequence $\{\vec{x}^1_{n_k}\}$ in the first coordinates of $\{\vec{x}_n\}$. We know such a convergent subsequence exists by compactness –the first components $(\vec{x}_n)^1$ of the vectors of the sequence $\{\vec{x}_n\}$ are all in the compact space $[0, 1]$. Next we can pick a convergent subsequence $\{\vec{x}^2_{n_k}\}$ in the second coordinates of $\{\vec{x}^1_{n_k}\}$. As before, such subsequence needs to exist by compactness. We can proceed in the same way for the other coordinates. The final subsequence, $\{\vec{x}^{2^l}_{n_k}\}$, will have as a limit a probability function $\vec{x}$ such that $(M_{\underline{\Gamma}}(\vec{x})^T)^j \geq \eta$ for all $j \in \{1, ..., m\}$ and $\vec{\theta} \cdot \vec{x} = \lim_{n\to\infty} F_{\Gamma,\theta}(\eta_n) = \zeta$ since $F_{\Gamma,\theta}$ is increasing. Therefore $F_{\Gamma,\theta}$ needs to be continuous from the left at η. ■

Proposition 20 *$F_{\Gamma,\theta}$ is made up of a finite number of line segments with rational coefficients.*

Proof. Let $\mathcal{R} = \langle \mathbb{R}, +, <, =, 0, 1\rangle$. The set

$$\{(x, y) \in \mathbb{R}^2 |\ y = F_{\Gamma,\theta}(x)\}$$

is $\mathcal{R}$-definable and, since $\mathcal{R}$ is an elementary extension of the structure

$$\mathcal{Q} = \langle \mathbb{Q}, +, <, =, 0, 1\rangle,$$

it is $\mathcal{Q}$-definable too.

The theory of $\mathcal{R}$ has quantifier elimination (see for example [29]). Therefore the set

$$\{(x, y) \in \mathbb{R}^2 |\ y = F_{\Gamma,\theta}(x)\}$$

is given by a finite boolean combination (which reduces to a finite union of intersections by the complement and distributive laws for sets) of sets of the form

$$\{(x, y) \in \mathbb{R}^2 |\ my < nx + k\}$$

and

$$\{(x, y) \in \mathbb{R}^2 |\ my = nx + k\},$$

for $n, m, k \in \mathbb{Z}$.

Notice that each intersection of sets of such form is convex so, since $F_{\Gamma,\theta}$ is a function, such intersection has to be a polynomial of the above mentioned form, with coefficients in $\mathbb{Q}$. ■

Next we state some further conditions $F_{\Gamma,\theta}$ needs to satisfy.

Proposition 21 *$F_{\Gamma,\theta}(0) \in \{0,1\}$, $F_{\Gamma,\theta}(1) \in \{0,1\}$ and on the interval $(mc(\Gamma), 1]$ $F_{\Gamma,\theta}$ has constant value 1.*

Proof. Notice that $F_{\Gamma,\theta}(0) = 1$ if and only if θ is a tautology. Otherwise $F_{\Gamma,\theta}(0) = 0$.

The fact that on the interval $(mc(\Gamma), 1]$ $F_{\Gamma,\theta}$ has constant value 1 follows directly from the definition of $^{\eta}\triangleright_{\zeta}$.

If Γ is not 1-consistent then clearly $F_{\Gamma,\theta}(1) = 1$. Notice that Γ is 1-consistent if and only if Γ is consistent. If Γ is consistent then $F_{\Gamma,\theta}(1) = 1$ only if $\Gamma \vdash \theta$ by Proposition 10 (iii), otherwise $F_{\Gamma,\theta}(1) = 0$. ■

Proposition 22 *The line segments $y = q_1 x + q_2$ that constitute $F_{\Gamma,\theta}$ are such that $q_1 = q_2 = 0$ or $q_1 = 0$ and $q_2 = 1$ or $q_1 \geq 1 - q_2 \geq 1$.*

Proof. Let us assume that $q_1 > 0$ (otherwise $q_2 \in \{0, 1\}$ by what has already been proved) and pick an interior rational point (η, ζ) on this line segment (so $\eta, \zeta < 1$). $F_{\Gamma,\theta}(x)$ will be of the form $q_1 x + q_2$ for all x in a neighborhood of η, say $(\eta - \epsilon, \eta + \epsilon)$ (for some $\epsilon > 0$).

By Theorem 15 for rational values there exist N, Z and T for this pair and the corresponding Γ and θ that give the propositional equivalent of $\Gamma\,^{\eta}\triangleright_{\zeta}\,\theta$. Notice that $T \geq 1$ (otherwise $\vdash \theta$ and $q_1 = 0$, $q_2 = 1$) and $T \leq N$ (otherwise $q_1 = q_2 = 0$). So

$$(1 - \zeta)T \leq \eta N - \zeta Z + 1$$

and

$$\zeta \leq \frac{\eta N - T + 1}{Z - T}. \tag{3.13}$$

Clearly we must have equality in (3.13) since otherwise we could increase ζ to some ζ^+ and these values of N, Z and T would give a propositional equivalent for $\Gamma\,^{\eta}\triangleright_{\zeta^+}\,\theta$, contradicting the fact that $F_{\Gamma,\theta}(\eta) = \zeta$.

Notice also that the line

$$y = \frac{xN - T + 1}{Z - T}$$

must be the same as the line $y = q_1 x + q_2$ since otherwise the values N, Z and T considered above would give a propositional equivalent for $\Gamma^{\eta'} \triangleright_{\zeta'} \theta$, with $\eta' \in (\eta - \epsilon, \eta + \epsilon)$ ($\eta' \in \mathbb{Q}$, $\eta' \neq \eta$) and $\zeta' > q_1\eta' + q_2$, contradicting the fact that $F_{\Gamma,\theta}(x) = q_1 x + q_2$ on the interval $(\eta - \epsilon, \eta + \epsilon)$.

Therefore $q_1 = \frac{N}{Z-T}$ and $q_2 = \frac{1-T}{Z-T}$. Notice that $Z \leq N + 1$ since otherwise we could replace Z by $N+1$ without changing the required conditions and that would contradict the fact that $F_{\Gamma,\theta}(\eta) = \zeta$. The required inequalities $q_1 \geq 1 - q_2 \geq 1$ follow. ■

3.4 A representation theorem for $F_{\Gamma,\theta}$

Next we show that any function satisfying the properties stated in Propositions 16, 18, 19, 20, 21 and 22 above is of the form $F_{\Gamma,\theta}$ for some Γ and θ in a finite language L. The next lemma is key to proving this.

Lemma 23 *Given $\Gamma_1, \Gamma_2, \theta_1, \theta_2$ there are Γ and θ such that*

$$F_{\Gamma,\theta}(x) = max\{F_{\Gamma_1,\theta_1}(x), F_{\Gamma_2,\theta_2}(x)\}.$$

Proof. We may assume that $\Gamma_1 \subseteq SL_1$, $\Gamma_2 \subseteq SL_2$ and $\theta_1 \in SL_1$, $\theta_2 \in SL_2$, for $L_1 = \{p_1, ..., p_n\}$ and $L_2 = \{q_1, ..., q_m\}$ disjoint languages with atoms $At^{L_1} = \{\alpha_1, ..., \alpha_{2^l}\}$ and $At^{L_2} = \{\beta_1, ..., \beta_{2^m}\}$ respectively. Let $L = L_1 \cup L_2$ and set $\Gamma = \Gamma_1 \cup \Gamma_2 \subseteq SL$ and $\theta = \theta_1 \vee \theta_2 \in SL$.

First note that by the language invariance of ${}^{\eta}\triangleright_{\zeta}$ if w is a probability function on L such that $w(\Gamma) \geq \eta$ then $w(\theta_1) \geq F_{\Gamma_1,\theta_1}(\eta)$ and $w(\theta_2) \geq F_{\Gamma_2,\theta_2}(\eta)$, so certainly

$$w(\theta) \geq \max\{F_{\Gamma_1,\theta_1}(\eta), F_{\Gamma_2,\theta_2}(\eta)\}.$$

Thus it only remains to show that there is some probability function w which takes exactly this value.

Without loss of generality assume that

$$F_{\Gamma_1,\theta_1}(\eta) \geq F_{\Gamma_2,\theta_2}(\eta).$$

Let w_1 be a probability function on L_1 such that $w_1(\Gamma_1) \geq \eta$, $w_1(\theta_1) = F_{\Gamma_1,\theta_1}(\eta)$ and w_2 a probability function on L_2 such that $w_2(\Gamma_2) \geq \eta$, $w_2(\theta_2) = F_{\Gamma_2,\theta_2}(\eta)$.

We define a finite sequence of probability functions $\{w^r\}$ on L such that for each r

$$\begin{aligned} w^r(\alpha_i) &= w_1(\alpha_i) \text{ for } i \in \{1, ..., 2^l\}, \\ w^r(\beta_j) &= w_2(\beta_j) \text{ for } j \in \{1, ..., 2^m\}, \end{aligned} \tag{3.14}$$

so in consequence

$$w^r(\theta_1) = F_{\Gamma_1,\theta_1}(\eta) \geq w^r(\theta_2) = F_{\Gamma_2,\theta_2}(\eta)$$

and such that for the final w^r in this sequence

$$w^r(\theta) = w^r(\theta_1),$$

equivalently

$$w^r(\alpha_i \wedge \beta_j) = 0 \text{ whenever } \alpha_i \nvdash \theta_1,\ \beta_j \vdash \theta_2. \tag{3.15}$$

To start with set

$$w^0(\alpha_i \wedge \beta_j) = w_1(\alpha_i) \cdot w_2(\beta_j).$$

Now suppose we have successfully constructed w^r. If (3.15) holds for this w^r then we are done. Otherwise take the atoms $\alpha_i \wedge \beta_j$ with $w^r(\alpha_i \wedge \beta_j) > 0$, $\beta_j \vdash \theta_2$, $\alpha_i \nvdash \theta_1$. In this case we can find an atom $\alpha_p \wedge \beta_q$ with $w^r(\alpha_p \wedge \beta_q) > 0$, $\alpha_p \vdash \theta_1$, $\beta_q \nvdash \theta_2$. Such an atom of L must exist since if not then

$$w^r(\theta_2) = \sum_t w^r(\alpha_{i_t} \wedge \beta_{jt}) + \sum_s w^r(\alpha_{i_s} \wedge \beta_{j_s})$$

and

$$w^r(\theta_1) = \sum_t w^r(\alpha_{i_t} \wedge \beta_{jt})$$

for t, s such that $\beta_{jt} \vdash \theta_2$, $\alpha_{i_t} \vdash \theta_1$, $\beta_{j_s} \vdash \theta_2$, $\alpha_{i_s} \nvdash \theta_1$. But then

$$F_{\Gamma_2,\theta_2}(\eta) = w^r(\theta_2) > w^r(\theta_1) = F_{\Gamma_1,\theta_1}(\eta),$$

contradiction.

Now define w^{r+1} as follows, for i, j, p and q as above:

$$\begin{aligned}
w^{r+1}(\alpha_i \wedge \beta_j) &= w^r(\alpha_i \wedge \beta_j) - \min\{w^r(\alpha_i \wedge \beta_j), w^r(\alpha_p \wedge \beta_q)\}, \\
w^{r+1}(\alpha_i \wedge \beta_q) &= w^r(\alpha_i \wedge \beta_q) + \min\{w^r(\alpha_i \wedge \beta_j), w^r(\alpha_p \wedge \beta_q)\}, \\
w^{r+1}(\alpha_p \wedge \beta_j) &= w^r(\alpha_p \wedge \beta_j) + \min\{w^r(\alpha_i \wedge \beta_j), w^r(\alpha_p \wedge \beta_q)\}, \\
w^{r+1}(\alpha_p \wedge \beta_q) &= w^r(\alpha_p \wedge \beta_q) - \min\{w^r(\alpha_i \wedge \beta_j), w^r(\alpha_p \wedge \beta_q)\},
\end{aligned}$$

and w^{r+1} agreeing with w^r on all other atoms of L. Then again we have (3.14) holding for w^{r+1} in place of w^r and compared with w^r the probability function w^{r+1} gives non-zero probability to strictly fewer atoms $\alpha_i \wedge \beta_j$ with either $\beta_j \vdash \theta_2$ and $\alpha_i \nvdash \theta_1$ or with $\alpha_i \vdash \theta_1$ and $\beta_j \nvdash \theta_2$. Clearly then this process eventually terminates at the required probability function. ■

We now state and prove the representation theorem we mentioned at the outset.

Theorem 24 *Let $r \in [0,1] \cap \mathbb{Q}$ and let F be a function such that*

(i) F is increasing, with $F(0) \in \{0,1\}$ and $F(1) \in \{0,1\}$.

(ii) On $[0,r]$ F is continuous and convex and made up of a finite number of line segments of the form $q_1x + q_2$ with $q_1, q_2 \in \mathbb{Q}$ and $q_1 \geq 1 - q_2 \geq 1$.

(iii) $F(x) = 1$ for all $x \in (r, 1]$.

Then there are Γ and θ such that $F = F_{\Gamma,\theta}$ on $[0,1]$.

Proof. In view of Lemma 23, it is enough to show

(A) If $r \in [0,1] \cap \mathbb{Q}$ then there are Γ and θ such that

$$F_{\Gamma,\theta}(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq r, \\ 1 & \text{for } r < x \leq 1. \end{cases}$$

(Notice that if $r = 1$ the result is trivial. We can take θ to be a contradiction and Γ any consistent set of sentences).

(B) If $q_1, q_2 \in \mathbb{Q}$, $q_1 \geq 1 - q_2 \geq 1$ then there are Γ and θ such that

$$F_{\Gamma,\theta}(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{-q_2}{q_1}, \\ q_1x + q_2 & \text{for } \frac{-q_2}{q_1} \leq x \leq \frac{1-q_2}{q_1}, \\ 1 & \text{for } \frac{1-q_2}{q_1} \leq x \leq 1. \end{cases}$$

We can proceed in several ways to define suitable Γ and θ for the graphs described above, both for (A) and for (B). Here we adopt what I call the *matrix* approach -we first set a suitable matrix for the Γ and θ we are looking for (as seen in Section 1.1)[2] and then from it we define Γ and θ by considering suitable disjunctions of atoms (to view a slightly different approach see [35]).

To show (A), if $r = 0$ just take θ to be a contradiction and $\Gamma = \{\theta\}$.

Suppose that $r = \frac{s}{t} > 0$. Let us set a matrix of $0's$ and $1's$ with $t+1$ rows and t columns. The row $t+1$ will consist of t $0's$. For the other rows we will have exactly s $1's$ distributed as follows: For the i^{th} row we will have s consecutive $1's$ starting at the i^{th} coordinate (when we reach the last column we go back to the first one and carry on until we complete a sequence of s consecutive $1's$).

This way we get the matrix $M_{\Gamma,\theta}$.

We will associate an atom to each column. We will thus need at least t distinct atoms, $\alpha_1, ..., \alpha_t \in At^L$ (L has to be large enough: $2^{|L|} \geq t$).

For the first t rows we will define a sentence as follows: For the i^{th} row we will set ϕ_i to be the disjunction of the atoms for the columns that at the i^{th} row have coordinate 1. We will take Γ to be the set given by these sentences. The row $t+1$ corresponds to the sentence θ. Since this row only contains $0's$ we will take θ to be a contradiction.

Let w be a probability function on L that gives Γ its maximum consistency. We can identify w with a vector $\vec{x} \in \mathbb{D}_{2^{|L|}}$. Any permutation of the first t coordinates of $\vec{x}$ of the form $x_i \longrightarrow x_{i+1}$ for $i \in \{1, ..., t-1\}$ and $x_t \longrightarrow x_1$ will give us a new probability function that gives Γ its maximum consistency and thus so will the average over these permutations. Hence we see that Γ attains its maximum consistency of $\frac{s}{t}$ for the probability function which gives each α_i, for $i \in \{1, ..., t\}$, probability $\frac{1}{t}$.

To show (B) let us first suppose that $q_1 = 1$ and $q_2 = 0$. In this case we can set θ to be a tautology.

Let us suppose now that $\frac{-q_2}{q_1} = \frac{r}{t}$ and $\frac{1-q_2}{q_1} = \frac{s}{t}$ where, by the conditions on q_1 and q_2, $0 \leq \frac{r}{t} \leq \frac{s}{t} \leq 1$.

We will proceed as above by setting a matrix of $0's$ and $1's$ with $t+1$ rows and $2t$ columns.

[2] Though the resulting ordering of the columns of the matrix for Γ and θ will be distinct to the one mentioned in Section 1.1.

The row $t+1$ will consist of $0's$ for the first t columns and $1's$ from the column $t+1$ onwards. For the other rows we will have exactly r $1's$ for the first t columns and exactly s $1's$ for last t columns distributed as follows: For the i^{th} row we will have r consecutive $1's$ starting at the i^{th} coordinate (when we reach the column t we go back to the first one and carry on until we complete a sequence of s consecutive $1's$) and s consecutive $1's$ starting at the column $t+i$ (when we reach the column $2t$ we go back to the column $t+1$ and carry on until we complete a sequence of r consecutive $1's$).

This way we get the matrix for Γ and θ, $M_{\Gamma,\theta}$.

We associate an atom to each column (we need at least $2t$ atoms, $\alpha_1, ..., \alpha_{2t} \in At^L$) and define the sentences $\phi_1, ..., \phi_t$ and θ by considering suitable disjunctions of such atoms as we did in the previous case.

Then if $w(\alpha_i) = \frac{1}{t}$, for $i \in \{1, ..., t\}$, $w(\Gamma) = \frac{r}{t}$ and $w(\theta) = 0$, so $F_{\Gamma,\theta}(\frac{r}{t}) = 0$. Now let $\frac{r}{t} \leq g < \frac{s}{t}$, $F_{\Gamma,\theta}(g) = h < 1$ and let w be the probability function attaining such supremum. As above we can assume that $w(\alpha_i)$ has constant value, a say, for $i \in \{1, ..., t\}$ and constant value, b say, for $i \in \{t+1, ..., 2t\}$ and that all the probability is assigned to $\alpha_1, ..., \alpha_{2t}$. Then $h = tb$, $g = sb + ra$ and $ta + tb = 1$ so $h = \frac{tg-r}{s-r} = q_1 g + q_2$. From this and the properties of $F_{\Gamma,\theta}$ part (B) follows. ■

Let us look at an example to see how this works in practice.

Suppose that we want to find Γ and θ for which $F_{\Gamma,\theta}$ is as follows:

$$F_{\Gamma,\theta}(x) = \begin{cases} 0 & 0 \leq x \leq \frac{2}{5} \\ 5x - 2 & if\ \frac{2}{5} < x < \frac{3}{5} \\ 1 & if\ \frac{3}{5} \leq x \leq 1. \end{cases}$$

We can set a suitable matrix for Γ and θ as explained above:

$$M_{\Gamma,\theta} = \left(\begin{array}{ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

Let us take ten atoms $\alpha_1, ..., \alpha_{10}$ in a sufficiently large language L, one for each column of the matrix.

By following the above procedure we will get the following set of sentences $\Gamma = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$:

$$\begin{aligned}
\phi_1 &= \alpha_1 \vee \alpha_2 \vee \alpha_6 \vee \alpha_7 \vee \alpha_8, \\
\phi_2 &= \alpha_2 \vee \alpha_3 \vee \alpha_7 \vee \alpha_8 \vee \alpha_9, \\
\phi_3 &= \alpha_3 \vee \alpha_4 \vee \alpha_8 \vee \alpha_9 \vee \alpha_{10}, \\
\phi_4 &= \alpha_4 \vee \alpha_5 \vee \alpha_9 \vee \alpha_{10} \vee \alpha_6, \\
\phi_5 &= \alpha_5 \vee \alpha_1 \vee \alpha_{10} \vee \alpha_6 \vee \alpha_7.
\end{aligned}$$

On the other hand θ will be given by the following disjunction:

$$\theta = \alpha_6 \vee \alpha_7 \vee \alpha_8 \vee \alpha_9 \vee \alpha_{10}.$$

It is easy to see that $F_{\Gamma,\theta}$ indeed corresponds to the graph defined above.

3.5 Theorem 15 revisited

At this point we finally return to complete the proof of Theorem 15 in the case when one or both values for η and ζ are irrational.

Proof. (of Theorem 15 continued). We first consider the case where η is irrational and ζ rational. In this case if $\Gamma^{\eta} \triangleright_{\zeta} \phi$ then by Proposition 20 $F_{\Gamma,\theta}(x) = q_1 x + q_2$ for some $q_1, q_2 \in \mathbb{Q}$ for all x in some open non-empty neighborhood $(\eta - \epsilon, \eta + \epsilon)$. Since $q_1\eta + q_2$ is irrational ($q_1 \neq 0$ otherwise $\zeta = 0$) it must be that $q_1\eta + q_2 > \zeta$ so there are $r_1, r_2 \in \mathbb{Q}$ (with $r_1 \in (\eta - \epsilon, \eta + \epsilon)$) such that $r_1 < \eta$, $r_2 > \zeta$ and $q_1 r_1 + q_2 \geq r_2$. We then have that $F_{\Gamma,\theta}(r_1) \geq r_2$ so there is some $\chi_1, ..., \chi_N \in \Gamma$ and Z, T such that $T(1 - r_2) \leq r_1 N - r_2 Z + 1$, $T < Z \leq N + 1$ and

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S| = Z}} \bigwedge_{j \in S} \chi_j \vdash \bot, \tag{3.16}$$

$$\bigvee_{\substack{S\subseteq\{1,\ldots,N\}\\|S|=T}} \bigwedge_{j\in S} \chi_j \vdash \phi. \tag{3.17}$$

But then

$$T(1-\zeta) \leq \eta N - \zeta Z + 1 \tag{3.18}$$

as required.

Conversely if we have $\chi_1, ..., \chi_N \in \Gamma$ and Z, T satisfying (3.16),(3.17) and (3.18) then the inequality in (3.18) must be strict, since $\zeta \in \mathbb{Q}$ so there must be $r_1 < \eta$, $r_2 > \zeta$ such that

$$T(1-r_2) \leq r_1 N - r_2 Z + 1.$$

Thus by the two rational case already proved $\Gamma^{r_1} \triangleright_{r_2} \phi$ and, by Proposition 9, $\Gamma^{\eta} \triangleright_{\zeta} \phi$.

The case where $\eta \in \mathbb{Q}$, $\zeta \notin \mathbb{Q}$ is proved similarly.

Suppose that both η and ζ are irrational. If $\Gamma^{\eta} \triangleright_{\zeta} \phi$ and $F_{\Gamma,\theta}(\eta) = q_1\eta + q_2 > \zeta$ then we can proceed as in the previous case. So suppose $F_{\Gamma,\theta}(\eta) = q_1\eta + q_2 = \zeta$. In this case by Proposition 20 $F_{\Gamma,\theta}(x) = q_1 x + q_2$ for x in some non-empty open neighborhood $(\eta - \epsilon, \eta + \epsilon)$. Pick r_1 in this interval and set $r_2 = q_1 r_1 + q_2$. Then by the two rational case there are some $\chi_1, \ldots, \chi_N \in \Gamma$, Z and T such that $T < Z$ and $T(1-r_2) \leq r_1 N - r_2 Z + 1$. Notice that we must have equality here, otherwise we could increase r_2 with r_1 fixed and so (using the two rational case) show that $F_{\Gamma,\theta}(r_1) > r_2$. It must be the case that the two lines $T(1-y) = xN - yZ + 1$ and $y = q_1 x + q_2$ are the same, otherwise the former would go above the latter at a rational point in the interval $(\eta - \epsilon, \eta + \epsilon)$, contradicting the proved completeness result in the rational case.

This provides the required equivalent to $\Gamma^{\eta} \triangleright_{\zeta} \phi$.

Finally in the other direction in the case $\eta, \zeta \notin \mathbb{Q}$, suppose that we have the required T, Z and $\chi_1, ..., \chi_N$ satisfying (3.17),(3.16) and (3.18). Then for rational r_1 close to η and $r_2 \leq \frac{r_1 N - T + 1}{Z - T}$, r_2 close to ζ these same $\chi_1, ..., \chi_N$, Z and T give $\Gamma^{r_1} \triangleright_{r_2} \phi$. Since r_1 and r_2 can be made arbitrarily close to η and ζ respectively we can conclude by Proposition 13 that $\Gamma^{\eta} \triangleright_{\zeta} \phi$, as required. ■

3.6 A brief note on the values of N, Z and T

In Theorem 15 we established an equivalence between ${}^{\eta}\triangleright_{\zeta}$ and some condition only dependent of propositional logic. That condition was formulated in terms of three values: N, Z and T.

Let us recall the theorem:

Let $\eta, \zeta \in (0,1]$. Then for $\phi_1, ..., \phi_n, \theta \in SL$,

$\phi_1, ..., \phi_n \, {}^{\eta} \triangleright_{\zeta} \, \theta \iff$

$\exists \chi_1, ..., \chi_N \in \{\phi_1, ..., \phi_n\}$ (possibly with repeats) and $T, Z \in \mathbb{N}$ such that

$$T(1-\zeta) \leq \eta N - \zeta Z + 1, \quad T < Z \text{ and}$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=Z}} \bigwedge_{j \in S} \chi_j \vdash \bot,$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=T}} \bigwedge_{j \in S} \chi_j \vdash \theta.$$

We have that N, Z and T need to be positive integers, that $T < Z$ and that Z need not be greater than $N+1$. The question rests on N. How big can N be? Can we set an upper bound on N in terms of the size of Γ?

As above, consider $\Gamma = \{\phi_1, ..., \phi_n\} \subseteq SL$ η-consistent and $\theta \in SL$. First notice that the maximum number of distinct sentences in the set $\mathcal{B}$ is 2^n and thus the maximum number of sets of sentences of this form is 2^{2^n}. Therefore the number of distinct matrices $M_{\Gamma,\theta}$ that we can have for $|\Gamma| = n$ is $2^{2^{n+1}}$.

We can find values N, Z and T for which the above propositional equivalence for $\Gamma^{\eta} \triangleright_{F_{\Gamma,\theta}(\eta)} \theta$ holds (notice that such values are valid in general for $\Gamma^{\eta} \triangleright_{\zeta} \theta$, with $0 \leq \zeta \leq F_{\Gamma,\theta}(\eta)$). Since such values depend only on the sentences in $\mathcal{B}$ and θ –that is to say, on the matrix $M_{\Gamma,\theta}$– we will have that we can effectively fix an upper bound for N (since the number of distinct matrices $M_{\Gamma,\theta}$ that we can have is finite).

We do not know about any sharp bound for N though. The next example shows that any bound for N that we could determine would be pretty large.

Consider a collection of prime numbers $p_1, ..., p_k$.

Next, for each number p_i, consider a set of $p_i + 1$ distinct atoms,

$$\alpha^i_1, ..., \alpha^i_{p_i+1} \subset At^L,$$

where L is large enough to allow for such a number of atoms.

Let $\Delta_i = \{\phi^i_1, ...\phi^i_{p_i+1}\}$ be the collection of sentences given by all the distinct disjunctions of p_i of the atoms $\alpha^i_1, ..., \alpha^i_{p_i+1}$, for each $i \in \{1, ..., k\}$.

Set $\Gamma = \bigcup_{i=1}^k \Delta_i$.

Let w be a probability function for which $w(\Gamma) = mc(\Gamma)$. Then, by symmetry, we will have that

$$w(\alpha^i_1) = w(\alpha^i_2) = ... = w(\alpha^i_{p_i+1}) = a_i,$$

for some values a_i, and

$$a_1 p_1 = a_2 p_2 = ... = a_k p_k$$

for all $i \in \{1, ..., k\}$. (Notice that $a_1 p_1 = mc(\Gamma)$).

By the properties of probability functions we will also have that

$$\sum_{i=1}^{k} a_i(p_i + 1) = 1.$$

From all these identities it follows that

$$mc(\Gamma) = \frac{\prod_{i=1}^k p_i}{\sum_{i=1}^k \prod_{j \neq i} p_j(p_i + 1)}.$$

Let us define the sentence θ as follows:

$$\theta = \bigvee_{i=1}^{k} \bigvee_{j=1}^{p_i+1} \alpha^i_j.$$

Assume that there exists $\alpha \in At^L$ such that $\alpha \nvDash \theta$ (that is to say, θ is not a tautology).

Then the graph of $F_{\Gamma,\theta}$ on $[0, mc(\Gamma)]$ will then consist of a line segment joining the origin $(0,0)$ and $(mc(\Gamma), 1)$ with slope

$$\frac{\sum_{i=1}^{k}\prod_{j\neq i} p_j(p_i+1)}{\prod_{i=1}^{k} p_i}.$$

For any pair (η,ζ) on this line segment we will have that $\Gamma^{\eta}\triangleright_{\zeta}\theta$ and the values N, Z and T for which the condition above holds will have to satisfy the equality

$$\frac{N}{Z-T} = \frac{\sum_{i=1}^{k}\prod_{j\neq i} p_j(p_i+1)}{\prod_{i=1}^{k} p_i}.$$

For the choices of $p_1, ..., p_k$ such fraction will be irreducible and thus $N \geq \sum_{i=1}^{k}\prod_{j\neq i} p_j(p_i+1)$. That this value is large is clear. Consider for example $p_1 = 29$, $p_2 = 31$ and $p_3 = 37$. We have

$$\frac{p_2p_3(p_1+1)+p_1p_3(p_2+1)+p_1p_2(p_3+1)}{p_1p_2p_3} = \frac{102908}{33263}$$

which is a proper fraction. Thus $N \geq 102908$ for this example when working with pairs (η,ζ) on the line segment joining $(0,0)$ and $(mc(\Gamma), 1)$, and that for a relatively small Γ ($|\Gamma| = 100$).

3.7 An Example

In this section we give a real world example of how the consequence relation ${}^{\eta}\triangleright_{\zeta}$ works in practice.

A dedicated naturalist has acquired, and remembered, the following facts concerning the world's largest amphibian:

- It can kill a chicken and comes from Japan.
- It is not the Japanese salamander but it can kill a chicken.
- It is a salamander and if it is not a chicken killer then it must be the Japanese salamander.

Taking p to stand for '*it can kill a chicken*', q to stand for '*it is Japanese*' and r to stand for '*it is a salamander*' these can be formalized as $p \wedge q$, $\neg(q \wedge r) \wedge p$ and $r \wedge (\neg p \rightarrow (r \wedge q))$.

Denoting this set by Γ we find that $mc(\Gamma) = \frac{2}{3}$, being given by the probability function which gives each of the atoms $p \wedge q \wedge r$, $p \wedge q \wedge \neg r$ and $p \wedge \neg q \wedge r$ probability $\frac{1}{3}$. From this it follows that

$$\Gamma^{\frac{2}{3}} \triangleright_{\frac{2}{3}} p \wedge r.$$

In other words if our naturalist sets his primary threshold at $\frac{2}{3}$ then on the basis of just this knowledge Γ he should be willing to accept '*it is a chicken killing salamander*' at this same threshold.

On the other hand if the naturalist felt that his recall was so faulty that a higher secondary threshold was required before actually making any public assertion based on it then setting the threshold at its highest possible value of 1 would give

$$\Gamma^{\frac{2}{3}} \triangleright_1 p \wedge (q \vee r).$$

In other words, with this more rigorous demand in place, the naturalist should still be happy to assert that the world's largest amphibian is '*a chicken killer and either Japanese or a salamander*'.

In the other direction lowering the secondary threshold sufficiently would in this case enable the naturalist to make stronger assertions, but at the same time risk being unacceptably inconsistent. For example

$$\Gamma^{\frac{2}{3}} \triangleright_{\frac{1}{3}} r,\ \neg r$$

so at the threshold $\frac{1}{3}$ he would be directly asserting both the statement that it is a salamander and the statement that it is not a salamander.

Chapter 4

Inference from inconsistent premises: Other approaches

In this chapter we review some relevant inference relations in the literature and compare them with ${}^{\eta}\!\triangleright_{\zeta}$ on distinct grounds. We also define a consequence relation based on the notion of η-coherence presented in Chapter 2.

4.1 ${}^{\eta}\!\triangleright_{\zeta}$ and consistency

In this section we compare ${}^{\eta}\!\triangleright_{\zeta}$ in terms of consistency with other consequence relations. Even though the aim of ${}^{\eta}\!\triangleright_{\zeta}$ is not to yield a consistent set of consequences (in our approach this is in general rather irrelevant) it may have some interest in some situations.

First we introduce some other inference relations from the literature. We proceed as in [4] and [3] to a large extent.

We start with some definitions.

Throughout let $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$ and $\theta \in SL$.

Definition 25 *We say that $\Delta \subseteq \Gamma$ is maximally consistent if and only if Δ is consistent and, for all $\phi \in \Gamma - \Delta$, $\Delta \cup \{\phi\} \vdash \bot$.*

Definition 26 *We say that $\Delta \subseteq \Gamma$ is minimally inconsistent if and only if Δ is inconsistent and, for all $\phi \in \Delta$, $\Delta - \{\phi\} \nvdash \bot$.*

We will denote the collection of maximally consistent subsets of Γ by $\mathcal{MC}(\Gamma)$ and the collection of minimally inconsistent subsets by $\mathcal{MI}(\Gamma)$.

The approach made by Rescher and Manor in [42] is among the most popular when dealing with inconsistent information. They define two consequence relations for this purpose.

Definition 27 *θ is said to be a universal (or inevitable) consequence of Γ (denoted by $\Gamma \vdash_\forall \theta$) if and only if, for all $\Delta \in \mathcal{MC}(\Gamma)$, $\Delta \vdash \theta$.*

Definition 28 *θ is said to be an existential consequence of Γ (denoted by $\Gamma \vdash_\exists \theta$) if and only if there exists $\Delta \in \mathcal{MC}(\Gamma)$ for which $\Delta \vdash \theta$.*

We will denote the set of consequences of Γ under $\vdash_\forall$ by $C_\forall(\Gamma)$ and those under $\vdash_\exists$ by $C_\exists(\Gamma)$.

Throughout $C(\Gamma)$ will denote the set of consequences of Γ under classical entailment.

Given the difficulty of finding all the maximal consistent subsets of a set of sentences (in general the cardinality of $\mathcal{MC}(\Gamma)$ increases exponentially with respect to the cardinality of Γ) some authors have proposed selecting a subset of $\mathcal{MC}(\Gamma)$ (see [4]), denoted in this reference by $Lex(\Gamma)$ and defined as follows: For $\Delta \in \mathcal{MC}(\Gamma)$, $\Delta \in Lex(\Gamma)$ if and only if, for all $\Delta' \in \mathcal{MC}(\Gamma)$, $|\Delta| \geq |\Delta'|$.

Definition 29 *θ is said to be a Lex-consequence of Γ (denoted by $\Gamma \vdash_{Lex} \theta$) if and only if, for all $\Delta \in Lex(\Gamma)$, $\Delta \vdash \theta$.*

We will denote the set of consequences of Γ under $\vdash_{Lex}$ by $C_{Lex}(\Gamma)$.

For the next approach we appeal to minimally inconsistent subsets.

Define

$$Inc(\Gamma) = \{\phi \in SL |\ \exists\Delta \in \mathcal{MI}(\Gamma),\ \phi \in \Delta\}$$

and

$$Free(\Gamma) = \Gamma - Inc(\Gamma).$$

Definition 30 *θ is said to be a $Free$-consequence of Γ (denoted by $\Gamma \vdash_{Free} \theta$) if and only if $Free(\Gamma) \vdash \theta$.*

The set of consequences of Γ under $\vdash_{Free}$ will be denoted by $C_{Free}(\Gamma)$.

It is interesting to observe that, for all $\Gamma \subseteq SL$,

$$C_{Free}(\Gamma) \subseteq C_\forall(\Gamma) \subseteq C_{Lex}(\Gamma) \subseteq C_\exists(\Gamma).[1]$$

[1]See [4] for a proof and more details.

We now move back to our probability approach.

Throughout let $C_{\eta,\zeta}(\Gamma) = \{\theta \in SL|\ \Gamma^{\eta} \rhd_{\zeta} \theta\}$.

The next proposition shows that the consequence relation ${}^{\eta}\!\rhd_{\zeta}$ does not do completely away with inconsistency. For certain values of η ${}^{\eta}\!\rhd_{\zeta}$ *explodes* like classical entailment.

Proposition 31 *Assume that* Γ *is inconsistent and let* $1 \geq \eta > mc(\Gamma)$*. Then* $C_{\eta,\zeta}(\Gamma) = C(\Gamma) = SL$*.*

Proof. It follows trivially from the definition of ${}^{\eta}\!\rhd_{\zeta}$. ■

The next two propositions state some closure properties of $C_{\eta,\zeta}(\Gamma)$.

Proposition 32 *Let* $\lambda < \eta$*. We have that* $C_{\lambda,\zeta}(\Gamma) \subseteq C_{\eta,\zeta}(\Gamma)$

Proof. The result follows trivially from observing that the set of probability functions w such that $w(\Gamma) \geq \eta$ is a subset of that of probability functions w for which $w(\Gamma) \geq \lambda$.

Let $\theta \in C_{\lambda,\zeta}(\Gamma)$. Thus, for all w, if $w(\Gamma) \geq \lambda$ then $w(\theta) \geq \zeta$. But then, since $\eta > \lambda$, $\theta \in C_{\eta,\zeta}(\Gamma)$ too. ■

Proposition 33 *Let* $\zeta < \mu$*. We have that* $C_{\eta,\mu}(\Gamma) \subseteq C_{\eta,\zeta}(\Gamma)$*.*

Proof. Let $\theta \in C_{\eta,\mu}(\Gamma)$. Then, for all w, if $w(\Gamma) \geq \eta$ then $w(\theta) \geq \mu > \zeta$. Thus $\theta \in C_{\eta,\zeta}(\Gamma)$ too. ■

Throughout let us assume that $\mathcal{B} = \{\beta_1, ..., \beta_m\}$.

Proposition 34 *Assume that* $0 \leq \eta \leq mc(\Gamma)$ *and* $0 < \zeta \leq 1$*. We have what follows:*

$$C_{\eta,\zeta}(\Gamma) = \{\theta \in SL|\ \exists \mathcal{C} \subseteq \mathcal{B}, \bigvee \mathcal{C} \vdash \theta \ \ and \ \ F_{\Gamma,\bigvee \mathcal{C}}(\eta) \geq \zeta\}.$$

Proof. It is clear that

$$\{\theta \in SL|\ \exists \mathcal{C} \subseteq \mathcal{B}, \bigvee \mathcal{C} \vdash \theta \ \ and \ \ F_{\Gamma,\bigvee \mathcal{C}}(\eta) \geq \zeta\} \subseteq C_{\eta,\zeta}(\Gamma).$$

Let us assume now that $\theta \in \mathcal{C}_{\eta,\zeta}(\Gamma)$ and that $\eta > 0$ (notice that for $\eta = 0$ the set $\mathcal{C}_{\eta,\zeta}(\Gamma)$ becomes the set of tautologies of SL and thus the result is trivial). Recall from Theorem 15 that

$\Gamma \; {}^{\eta}\!\rhd_{\zeta}\, \theta \iff$

$\exists \chi_1, ..., \chi_N \in \Gamma$ (possibly with repeats) and $T, Z \in \mathbb{N}$ such that

$$T(1-\zeta) \leq \eta N - \zeta Z + 1, \quad T < Z \text{ and}$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=Z}} \bigwedge_{j \in S} \chi_j \vdash \perp,$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=T}} \bigwedge_{j \in S} \chi_j \vdash \theta.$$

Notice though that the sentence

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=T}} \bigwedge_{j \in S} \chi_j$$

is of the form (or logically equivalent to one of the form) $\bigvee \mathcal{C}$, for some $\mathcal{C} \subseteq \mathcal{B}$. This proves that

$$C_{\eta,\zeta}(\Gamma) \subseteq \{\theta \in SL | \; \exists \mathcal{C} \subseteq \mathcal{B}, \bigvee \mathcal{C} \vdash \theta \; \textit{and} \; F_{\Gamma, \bigvee \mathcal{C}}(\eta) \geq \zeta\}$$

and completes the proof. ■

Proposition 35 $C_{\eta,\zeta}(\Gamma)$ *is consistent if and only if there exists a nonempty set* $\mathcal{C} \subseteq \mathcal{B}$ *such that* $\bigvee \mathcal{C} \vdash \theta$ *for all* $\theta \in C_{\eta,\zeta}(\Gamma)$.

Proof. The implication from right to left follows trivially.

In the other direction let us proceed by *reductio ad absurdum* by assuming that $C_{\eta,\zeta}(\Gamma)$ is consistent and that there is not any nonempty subset $\mathcal{C} \subseteq \mathcal{B}$ for which $\bigvee \mathcal{C} \vdash \theta$ for all $\theta \in C_{\eta,\zeta}(\Gamma)$. Thus, by Proposition 34 there would exist subsets $\mathcal{C}_1 \subseteq \mathcal{B}$ and $\mathcal{C}_2 \subseteq \mathcal{B}$, with $\mathcal{C}_1 \bigcap \mathcal{C}_2 = \emptyset$, with $\bigvee \mathcal{C}_1 \in C_{\eta,\zeta}(\Gamma)$ and $\bigvee \mathcal{C}_2 \in C_{\eta,\zeta}(\Gamma)$, contradicting the fact that $C_{\eta,\zeta}(\Gamma)$ is consistent. ■

As an example of the above condition for consistency let us consider the case when $\eta = 0$ and $\zeta > 0$. We have that $C_{0,\zeta}(\Gamma) = \{\theta \in SL | \; \vdash \theta\}$. Notice that the condition for consistency stated in Proposition 35 is satisfied (take any $\mathcal{C} \subseteq \mathcal{B}$).

Corollary 36 *$C_{\eta,\zeta}(\Gamma)$ is consistent if and only if*

$$\bigcap\{\mathcal{C} \subseteq \mathcal{B} |\ F_{\Gamma,\bigvee \mathcal{C}}(\eta) \geq \zeta\}$$

is not empty.

Proposition 37 *We can always find η and ζ for which $\mathcal{C}_{\eta,\zeta}(\Gamma)$ is consistent.*

Proof. Take $\eta = mc(\Gamma)$ and $\zeta = 1$. Notice that Proposition 35 is satisfied here by setting

$$\mathcal{C} = \{\beta \in \mathcal{B} |\ \exists w,\ w(\beta) > 0 \, and \, w(\Gamma) = \eta\}.$$

■

Proposition 38 *Let $\zeta^* = min\{F_{\Gamma,\neg\beta}(\eta) |\ \beta \in \mathcal{B}\}$. The set $C_{\eta,\zeta}(\Gamma)$ is consistent if and only if $\zeta > \zeta^*$.*

Proof. Let us proceed by *reductio ad absurdum* by assuming that $C_{\eta,\zeta}(\Gamma)$ is consistent and that $\zeta \leq \zeta^*$. Then

$$\{\bigvee(\mathcal{B} - \{\beta_i\}) |\ 1 \leq i \leq m\} \subseteq C_{\eta,\zeta}(\Gamma).$$

But $\{\bigvee(\mathcal{B} - \{\beta_i\}) |\ 1 \leq i \leq m\}$ is inconsistent and, therefore, so is $C_{\eta,\zeta}(\Gamma)$, which contradicts the assumption we started with.

Let us now assume that $\zeta > \zeta^*$ and that $\theta \in C_{\eta,\zeta}(\Gamma)$. Let β be the sentence in $\mathcal{B}$ such that $\zeta^* = F_{\Gamma,\neg\beta}(\eta)$. Thus it has to be the case that $\beta \vdash \theta$. To see this notice that if $\beta \nvdash \theta$ then, by Proposition 34, there would exist $\mathcal{C} \subseteq \mathcal{B} - \{\beta\}$ such that $\bigvee \mathcal{C} \vdash \theta$ and $F_{\Gamma,\bigvee \mathcal{C}}(\eta) \geq \zeta > \zeta^*$. But that contradicts the assumption that $\zeta^* = F_{\Gamma,\neg\beta}(\eta)$. This completes the proof. ■

In what follows we will focus our attention on the consequence relation ${}^{mc(\Gamma)}\rhd_1$ and $C_{mc(\Gamma),1}(\Gamma)$ –which is consistent and most of the times can be worked out easily from any set of sentences Γ, as we shall see in the comparisons that follow.

Notice that, for Γ consistent, we have $mc(\Gamma) = 1$ and thus

$$\Gamma^1 \rhd_1 \theta \iff \Gamma \vdash \theta.$$

We can compare ${}^{mc(\Gamma)}\rhd_1$ in terms of consistency with $\vdash_{Free}$, $\vdash_\forall$, $\vdash_\exists$ and $\vdash_{Lex}$.

We first show that ${}^{mc(\Gamma)}\triangleright_1$ is not generally comparable with $\vdash_\forall$. That is to say, it is not the case that, for $\Gamma \subseteq SL$, $C_{mc(\Gamma),1}(\Gamma) \subseteq C_\forall(\Gamma)$ or $C_\forall(\Gamma) \subseteq C_{mc(\Gamma),1}(\Gamma)$. To see this consider

$$\Gamma = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$$

with

$$\begin{aligned}
\phi_1 &= \alpha_2 \vee \alpha_4, \\
\phi_2 &= \alpha_3 \vee \alpha_5, \\
\phi_3 &= \alpha_1 \vee \alpha_2 \vee \alpha_4, \\
\phi_4 &= \alpha_1 \vee \alpha_3 \vee \alpha_5, \\
\phi_5 &= \alpha_3 \vee \alpha_4 \vee \alpha_5, \\
\phi_6 &= \alpha_1 \vee \alpha_2 \vee \alpha_4 \vee \alpha_5
\end{aligned}$$

and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in At^L$.

Notice that $mc(\Gamma) = \frac{1}{2}$ and that the maximal consistent subsets of Γ are $\{\phi_3, \phi_4, \phi_6\}$, $\{\phi_1, \phi_3, \phi_5, \phi_6\}$ and $\{\phi_2, \phi_4, \phi_5, \phi_6\}$.

We then have what follows:

$$C_{mc(\Gamma),1}(\Gamma) = \{\theta \in SL|\ \alpha_2 \vee \alpha_3 \vee \alpha_4 \vee \alpha_5 \vdash \theta\}$$

and

$$C_\forall(\Gamma) = \{\theta \in SL|\ \alpha_1 \vee \alpha_4 \vee \alpha_5 \vdash \theta\}.$$

In this case then $C_\forall(\Gamma) \nsubseteq C_{mc(\Gamma),1}(\Gamma)$ and $C_{mc(\Gamma),1}(\Gamma) \nsubseteq C_\forall(\Gamma)$.

Thus we have that both $\vdash_\forall$ and ${}^{mc(\Gamma)}\triangleright_1$ yield consistent sets of inferences but they are not comparable in the terms presented above.

Notice that for the above example we have four minimally inconsistent subsets of Γ: $\{\phi_1, \phi_2\}$, $\{\phi_1, \phi_4\}$, $\{\phi_2, \phi_3\}$ and $\{\phi_3, \phi_4, \phi_5\}$. The sentence ϕ_6 is not in any of these subsets. Thus we have that

$$C_{Free}(\Gamma) = \{\theta \in SL|\ \alpha_1 \vee \alpha_2 \vee \alpha_4 \vee \alpha_5 \vdash \theta\}.$$

Therefore, $C_{Free}(\Gamma)$ and $C_{mc(\Gamma),1}(\Gamma)$ are not comparable either.

We may wonder if such a comparison is possible between $\vdash_{Lex}$ and ${}^{mc(\Gamma)}\triangleright_1$. The answer is negative. It is not in general the case that, given $\Gamma \subseteq SL$, $C_{mc(\Gamma),1}(\Gamma) \subseteq C_{Lex}(\Gamma)$ or $C_{Lex}(\Gamma) \subseteq C_{mc(\Gamma),1}(\Gamma)$.

To show this let us set

$$\Gamma = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$$

with

$$\phi_1 = \alpha_2,$$
$$\phi_2 = \alpha_3,$$
$$\phi_3 = \alpha_4,$$
$$\phi_4 = \alpha_1 \vee \alpha_2,$$
$$\phi_5 = \alpha_1 \vee \alpha_3,$$
$$\phi_6 = \alpha_1 \vee \alpha_4$$

and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in At^L$.

Notice that $mc(\Gamma) = \frac{1}{3}$ and the largest consistent subset of Γ is $\{\phi_4, \phi_5, \phi_6\}$. We then have what follows:

$$C_{mc(\Gamma),1}(\Gamma) = \{\theta \in SL |\ \alpha_2 \vee \alpha_3 \vee \alpha_4 \vdash \theta\}$$

and

$$C_{Lex}(\Gamma) = \{\theta \in SL |\ \alpha_1 \vdash \theta\}.$$

Clearly, $C_{Lex}(\Gamma) \nsubseteq C_{mc(\Gamma),1}(\Gamma)$ and $C_{mc(\Gamma),1}(\Gamma) \nsubseteq C_{Lex}(\Gamma)$.

Proposition 39 *Let* $\Gamma \subseteq SL$. $C_{mc(\Gamma),1}(\Gamma) \subseteq C_{\exists}(\Gamma)$.

Proof. Let

$$\mathcal{C} = \{\beta \in \mathcal{B} |\ \exists w,\ w(\beta) > 0 \, and \, w(\Gamma) = \eta\}.$$

As seen previously

$$C_{mc(\Gamma),1} = \{\theta \mid \bigvee \mathcal{C} \vdash \theta\}.$$

Consider $\beta \in \mathcal{C}$ and assume that there is no $\Delta \in \mathcal{MC}(\Gamma)$ such that $\Delta \vdash \beta$. Let $\Phi = \{\phi_{i_1}, ..., \phi_{i_r}\}$ be the set of sentences in Γ logically implied by β. We can extend Φ to a maximal consistent subset of Γ, say Φ^*. Clearly the conjunction given by the sentences in Φ^* and the negation of the sentences in $\Gamma - \Phi^*$ will be a sentence implied by Φ^* in $\mathcal{C}$. ■

4.2 $\vdash_{MC}$ and others

Schotch and Jennings define an inference relation in terms of *coherence* in connection with the function $\mathbf{c}$ defined in Chapter 2.

Let $\Gamma \subseteq SL$ and $\theta \in SL$.

Definition 40 *We say that a collection $\mathcal{C}_m = \{\Delta_1, ..., \Delta_m\}$ of consistent subsets of Γ is an m-cover of Γ if and only if $\cup_{i=1}^{m} \Delta_i = \Gamma$.*

For the next definition let us assume that $\mathbf{c}(\Gamma) = m$.

Definition 41 *We say that Γ forces θ (denoted $\Gamma[\vdash \theta$) if and only if for every m-cover $\mathcal{C}_m = \{\Delta_1, ..., \Delta_m\}$ of Γ there is some $i \in \{1, ..., m\}$ such that $\Delta_i \vdash \theta$.*

We will denote the set of consequences of Γ under the relation $[\vdash$ by $C_{SJ}(\Gamma)$.

The relation $[\vdash$ possesses some desirable classical properties (the classical structural rules):

1. *Reflexivity:* If $\phi \in \Gamma$ then $\Gamma[\vdash \phi$.
2. *Monotonicity*:* For $\mathbf{c}(\Gamma \cup \Delta) = \mathbf{c}(\Gamma)$, if $\Gamma[\vdash \phi$ then $\Gamma \cup \Delta[\vdash \phi$.
3. *Transitivity:* If $\Gamma \cup \{\phi\}[\vdash \theta$ and $\Gamma[\vdash \phi$ then $\Gamma[\vdash \theta$.

According to Schotch and Jennings this fact makes $[\vdash$ the most suitable consequence relation among its competitors. Consistency of $C_{SJ}(\Gamma)$ was not pursued and, in general, $C_{SJ}(\Gamma)$ is not consistent. In fact, $C_{SJ}(\Gamma)$ will be consistent only if Γ is since, by *reflexivity*, we will have that $\Gamma \subset C_{SJ}(\Gamma)$.

We can define a similar consequence relation by appealing to the notion of η-coherence presented in Chapter 2 for which the classical structural rules (with some restrictions) will also hold.

For the next definition, lemma and propositions let $MC(\Gamma) = \frac{p}{q}$. [2]

Definition 42 $\Gamma \vdash_{MC} \theta$ *if and only if for each collection of copies of consistent subsets of Γ of the form $\mathcal{A} = \{\Delta_1, ..., \Delta_{tq}\}$ yielding $MC(\Gamma)$ there exist tp copies classically entailing θ, where t is a positive integer.*

[2]Recall that $MC(\Gamma)$ is the maximal coherence of Γ, which is a rational value, and that $MC(\Gamma) = mc(\Gamma)$, see Chapter 2.
For simplicity we can assume that $\frac{p}{q}$ is irreducible.

Lemma 43 *If* $\Gamma \vdash_{MC} \theta$ *then* $MC(\Gamma) = MC(\Gamma \cup \{\theta\})$.

Proof. Let us assume that $\Gamma \vdash_{MC} \theta$. That $MC(\Gamma) \geq MC(\Gamma \cup \{\theta\})$ is clear.

Let $\mathcal{A} = \{\Delta_1, ..., \Delta_{tq}\}$ be a collection of copies of consistent subsets of Γ yielding $MC(\Gamma)$, where t is a positive integer. We can define $\mathcal{A}^* = \{\Delta_1^*, ..., \Delta_{tq}^*\}$, a collection of copies of consistent subsets of $\Gamma \cup \{\theta\}$, where $\Delta_i^* = \Delta_i \cup \{\theta\}$ if $\Delta_i \vdash \theta$ and $\Delta_i^* = \Delta_i$ if $\Delta_i \nvdash \theta$. Notice that, since $\Gamma \vdash_{MC} \theta$, θ will belong to more than tp copies of subsets of $\Gamma \cup \{\theta\}$ in $\mathcal{A}^*$. Thus we can conclude that $MC(\Gamma \cup \{\theta\}) = MC(\Gamma)$. ∎

The next proposition states that the classical structural rules (with some restrictions) hold for $\vdash_{MC}$.

Proposition 44 *The following rules are sound for* $\vdash_{MC}$*:*

1. *Reflexivity: If* $\phi \in \Gamma$ *then* $\Gamma \vdash_{MC} \phi$.
2. *Monotonicity*: For* $MC(\Gamma \cup \Delta) = MC(\Gamma)$*, if* $\Gamma \vdash_{MC} \phi$ *then* $\Gamma \cup \Delta \vdash_{MC} \phi$.
3. *Transitivity: If* $\Gamma \cup \{\phi\} \vdash_{MC} \theta$ *and* $\Gamma \vdash_{MC} \phi$ *then* $\Gamma \vdash_{MC} \theta$.

Proof. That $\vdash_{MC}$ is reflexive and monotone in the terms stated above is clear. Let us prove transitivity.

Let us assume that $\Gamma \cup \{\phi\} \vdash_{MC} \theta$ and $\Gamma \vdash_{MC} \phi$. By Lemma 43 we know that $MC(\Gamma) = MC(\Gamma \cup \{\phi\})$. Let us proceed by *reductio ad absurdum* by assuming that $\Gamma \nvdash_{MC} \theta$. In this case there has to exist a collection of copies of consistent subsets of Γ of the form $\mathcal{A} = \{\Delta_1, ..., \Delta_{tq}\}$, for some positive integer t, yielding $MC(\Gamma)$ and containing less than tp copies of subsets classically implying θ. Notice that, since $\Gamma \vdash_{MC} \phi$, $\mathcal{A}$ will contain at least tp copies of subsets of Γ classically entailing ϕ. Let us now define $\mathcal{A}^* = \{\Delta_1^*, ..., \Delta_{tq}^*\}$, a collection of copies of consistent subsets of $\Gamma \cup \{\phi\}$, where $\Delta_i^* = \Delta_i \cup \{\phi\}$ if $\Delta_i \vdash \phi$ and $\Delta_i^* = \Delta_i$ if, on the contrary, $\Delta_i \nvdash \phi$. $\mathcal{A}^*$ yields $MC(\Gamma \cup \{\phi\})$ but it contains less than tp copies of subsets of $\Gamma \cup \{\phi\}$ classically entailing θ, contradicting the fact that $\Gamma \cup \{\phi\} \vdash_{MC} \theta$. Therefore it has to be the case that if $\Gamma \cup \{\phi\} \vdash_{MC} \theta$ and $\Gamma \vdash_{MC} \phi$ then $\Gamma \vdash_{MC} \theta$ ∎

Proposition 45 $\Gamma \vdash_{MC} \theta$ *if and only if* $\Gamma^{mc(\Gamma)} \rhd_{mc(\Gamma)} \theta$.

Proof. Let us proceed by *reductio ad absurdum* to prove the right implication. Let us assume that $\Gamma \vdash_{MC} \theta$ and that $\Gamma^{mc(\Gamma)} \not\triangleright_{mc(\Gamma)} \theta$. Thus there exists a probability function w on L such that $w(\Gamma) = mc(\Gamma)$ and $w(\theta) < mc(\Gamma)$. Suppose that $C = \{\alpha_{k_1}, ..., \alpha_{k_r}\} \subseteq At^L$ is the set of atoms that are given probability greater than zero by w and that $C^* \subset C$ is the set of atoms among them that logically imply θ. Let us suppose that, for each $i \in \{1, ..., r\}$, $w(\alpha_{k_i}) = \frac{p_i}{q_i} > 0$ for some positive integers p_i and q_i.[3] Let Q be the least common multiple of the q_i and p_i^* be such that $\frac{p_i}{q_i} = \frac{p_i^*}{Q}$ (and $\frac{p}{q} = \frac{P}{Q}$). Let us set now $\mathcal{A} = \{\Delta_1^1, ...\Delta_{p_1^*}^1, ..., \Delta_1^r, ..., \Delta_{p_r^*}^r\}$, where $\Delta_j^i = \{\phi \in \Gamma |\ \alpha_{k_i} \vdash \phi\}$ for each $j \in \{1, ..., p_i^*\}$. Thus in such collection there have to be at least P copies of each sentence of Γ and thus $\mathcal{A}$ yields $MC(\Gamma)$. On the other hand, $\mathcal{A}$ will contain less than P copies of subsets of Γ classically entailing θ, which contradicts the assumption we started with. Therefore, if $\Gamma \vdash_{MC} \theta$ then $\Gamma^{mc(\Gamma)} \triangleright_{mc(\Gamma)} \theta$.

Let us proceed again by *reductio ad absurdum* to prove the left implication. So let us assume that $\Gamma^{mc(\Gamma)} \triangleright_{mc(\Gamma)} \theta$ but that $\Gamma \nvdash_{MC} \theta$. In this case there has to exist a collection of consistent subsets of Γ of the form $\mathcal{A} = \{\Delta_1, ..., \Delta_{tq}\}$ for some positive integer t that yields $MC(\Gamma)$ but that contains less than tp copies of subsets classically implying θ. Let $\mathcal{A}^* \subset \mathcal{A}$ be the collection of subsets in $\mathcal{A}$ that classically imply θ. Since the subsets Δ_i are consistent we can find a collection of copies of atoms in At^L, $\{\alpha_{k_1}, ..., \alpha_{k_{tq}}\}$ (possibly with repeats), such that $\alpha_{k_i} \vdash \bigwedge \Delta_i$. For $\Delta \notin \mathcal{A}^*$ we will choose $\alpha \in At^L$ such that $\alpha \nvdash \theta$. If $\Delta \in \mathcal{A}^*$ then any atom α classically implying Δ will do. Now let w be a probability function that assigns to each atom $\alpha \in At^L$ probability $\frac{r}{tq}$, where r is the number of copies of α in the collection $\{\alpha_{k_1}, ..., \alpha_{k_{tq}}\}$. Thus $w(\Gamma) = \frac{p}{q}$ and $w(\theta) < \frac{p}{q}$, since the number of copies of atoms in $\{\alpha_{k_1}, ..., \alpha_{k_{tq}}\}$ that classically imply θ is less than tp. This contradicts our initial assumption. Therefore, if $\Gamma^{mc(\Gamma)} \triangleright_{mc(\Gamma)} \theta$ then $\Gamma \vdash_{MC} \theta$. ∎

[3]We can assume that the probability given to the atoms $\alpha_{k_1}, ..., \alpha_{k_r}$ by w is rational by *substructureness*. As in Lemma 5, if the statement

'There exists a probability function w such that $w(\Gamma) = \lambda$ and $w(\theta) < \lambda$'

is true in the structure $\langle \mathbb{R}, +, <, =, 0, 1, \lambda \rangle$ then so is in $\langle \mathbb{Q}, +, <, =, 0, 1, \lambda \rangle$ by *substructureness*.

4.3 Dempster-Shafer belief functions

In this section we adopt Dempster-Shafer belief functions to measure degrees of belief and redefine our consequence relation ${}^{\eta}\!\!\rhd_{\zeta}$ in terms of such belief functions.[4]

For what follows let $\mathcal{P}^*(At^L) = \mathcal{P}(At^L) - \{\emptyset\}$, where $\mathcal{P}(At^L)$ is the power set of At^L.

Definition 46 *A probability mass assignment is a map* $\mu : \mathcal{P}^*(At^L) \longrightarrow [0,1]$.

Definition 47 *We say that a map from SL to $[0,1]$ is a Dempster-Shafer belief function (DS-belief function for short) if there exists a probability mass assignment μ for which*

$$\sum_{S \in \mathcal{P}^*(At^L)} \mu(S) = 1$$

and, for $\phi \in SL$,

$$w(\phi) = \sum_{\emptyset \neq S \subseteq S_\phi} \mu(S)$$

We will denote this function by w^μ.

Notice that a probability function can be regarded as a DS-belief function. To see this assume that w is a probability function. We can easily define a DS-belief function w^μ from w such that $w^\mu(\phi) = w(\phi)$ for all $\phi \in SL$ by setting $\mu(\{\alpha\}) = w(\alpha)$ for all $\alpha \in At^L$.

In this sense we will say that probability functions *are* DS-belief functions.

For what follows let $\Gamma \subseteq SL$ and $\eta \in [0,1]$.

Proposition 48 *Γ is η-consistent if and only if there exists a DS-belief function w^μ such that $w^\mu(\Gamma) \geq \eta$.*

Proof. Let us first assume that Γ is η-consistent. Then there exists a probability function w such that $w(\Gamma) \geq \eta$. We can define a DS-belief function w^μ as mentioned above: $\mu(\{\alpha\}) = w(\alpha)$ for all $\alpha \in At^L$. Thus $w^\mu(\Gamma) \geq \eta$.

[4]See [7] for a good insight into Dempster-Shafer belief functions.

Let us assume now that w^μ is a DS-belief function such that $w^\mu(\Gamma) \geq \eta$. Suppose that $\mu(S) > 0$ for all $S \in \{S_1, ..., S_k\}$ and that $\mu(S) = 0$ for all $S \notin \{S_1, ..., S_k\}$ (where $S_i \subseteq At^L$ for all $i \in \{1, ..., k\}$). For each $i \in \{1, ..., k\}$ we can choose an atom in S_i and define a probability function w such that

$$w(\alpha) = \sum \{\mu(S_i) \,|\, \alpha \textit{ is the chosen atom of } S_i\}.$$

Then we will have that $w(\Gamma) \geq \eta$. ■

Corollary 49 *Γ is maximally η-consistent if and only if there exists a DS-belief function w^μ such that $w^\mu(\Gamma) \geq \eta$ and there is no other DS-belief function $w^{\mu'}$ such that $w^{\mu'}(\Gamma) > \eta$.*

Definition 50 *We say that Γ (η, ζ)-implies θ (denoted $\Gamma^\eta \Vdash_\zeta \theta$) if and only if, for every DS-belief function w^μ, if $w^\mu(\Gamma) \geq \eta$ then $w^\mu(\theta) \geq \zeta$.*

We now prove the equivalence between ${}^\eta\triangleright_\zeta$ and ${}^\eta \Vdash_\zeta$.

Proposition 51 *$\Gamma^\eta \Vdash_\zeta \theta$ if and only if $\Gamma^\eta \triangleright_\zeta \theta$.*

Proof. Let us proceed by *reductio ad absurdum* by assuming that $\Gamma^\eta \Vdash_\zeta \theta$ and that $\Gamma^\eta \not\triangleright_\zeta \theta$. Then there exists a probability function w such that $w(\Gamma) \geq \eta$ and $w(\theta) < \zeta$. But this leads to a contradiction since, as we have seen above, probability functions *are* DS-belief functions.

Let us assume now that $\Gamma^\eta \triangleright_\zeta \theta$ and that there exists a DS-belief function w^μ such that $w^\mu(\Gamma) \geq \eta$ and $w^\mu(\theta) < \zeta$. Suppose that $\mu(S) > 0$ for all $S \in \{S_1, ..., S_k\}$ and that $\mu(S) = 0$ for $S \notin \{S_1, ..., S_k\}$ (where $S_i \subseteq At^L$ for all $i \in \{1, ..., k\}$). Let us choose an atom α for each S_i (any atom in S_i if $\bigvee S_i \vdash \theta$ and an atom not implying θ in S_i if $\bigvee S_i \nvdash \theta$. If S_i is singleton we then take the only atom it contains) and define a probability function w as follows:

$$w(\alpha) = \sum \{\mu(S_i) \,|\, \alpha \textit{ is the chosen atom of } S_i\}.$$

Thus w is such that $w(\Gamma) \geq \eta$ and $w(\theta) < \zeta$, which contradicts the initial assumption. ■

4.4 Possibility theory

Let us consider now possibility functions (see [8] or [9] for an introduction to possibility theory).

Definition 52 *A map $w : SL \longrightarrow [0,1]$ is said to be a possibility function if and only if for all $\phi, \theta \in SL$ the following conditions hold:*

1. *If $\vdash \theta$ then $w(\theta) = 1$ and $w(\neg\theta) = 0$.*
2. *If $\vdash \phi \leftrightarrow \theta$ then $w(\phi) = w(\theta)$.*
3. *$w(\phi \vee \theta) = max(w(\phi), w(\theta))$.*

It follows from the definition that possibility functions can be characterized by the values they assign to the atoms of L. Thus they can be identified with 2^l-coordinate vectors in C^L:

$$C^L = \{(x_1, ..., x_{2^l}) \in \mathbb{R}^{2^l} \,|\, x_i \geq 0, max_i\{x_i\} = 1\}.$$

Throughout let $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$, $\theta \in SL$ and $\eta, \zeta \in [0,1]$.

Definition 53 *We say that Γ (η, ζ)-entails θ (denoted $\Gamma^\eta \models_\zeta \theta$) if and only if, for all possibility functions w, if $w(\Gamma) \geq \eta$ then $w(\theta) \geq \zeta$.*

Definition 54 *Let $\mathcal{P}_{\Gamma,\theta} : [0,1] \longrightarrow [0,1]$ be defined by*

$$\mathcal{P}_{\Gamma,\theta}(\eta) = sup\{\zeta \,|\, \Gamma^\eta \models_\zeta \theta\}.$$

Proposition 55 *The function $\mathcal{P}_{\Gamma,\theta}$ is of one of the following forms:*

1. *$\mathcal{P}_{\Gamma,\theta}(x) = 0$ for all $x \in [0,1]$.*
2. *$\mathcal{P}_{\Gamma,\theta}(x) = 1$ for all $x \in [0,1]$.*
3. *$\mathcal{P}_{\Gamma,\theta}(x) = x$ for all $x \in [0,1]$.*

Proof. If θ is a tautology then clearly $\mathcal{P}_{\Gamma,\theta}(x) = 1$ for all $x \in [0,1]$. Thus, let us assume that θ is not a tautology. We will distinguish two cases:

1. There is no sentence in Γ which classically implies θ.

   This means that for each $\phi_i \in \Gamma$, $i \in \{1, ..., k\}$, there exists an atom $\alpha_{\phi_i} \in S_{\phi_i}$ such that $\alpha_{\phi_i} \nvdash \theta$. We can define a possibility function w that assigns value 1 to every such atom. This way, $w(\phi) = 1$ for all $\phi \in \Gamma$ and $w(\theta) = 0$. Thus, $\mathcal{P}_{\Gamma,\theta}(x) = 0$ for all $x \in [0, 1]$.

2. There exists $\phi \in \Gamma$ such that $\phi \vdash \theta$.

   Let w be a possibility function for which $w(\Gamma) \geq \eta$. Then clearly $w(\theta) \geq \eta$. Further, we can define a possibility function w^* such that $w^*(\Gamma) \geq \eta$ and $w^*(\theta) = \eta$ by setting $w^*(\alpha) = \eta$ for all $\alpha \in S_\theta$ and $w^*(\alpha) = 1$ for all $\alpha \in S_{\neg\theta}$. Thus, $\mathcal{P}_{\Gamma,\theta}(x) = x$ for all $x \in [0, 1]$.

$\blacksquare$

We can easily find $\Gamma \subseteq SL$ and $\theta \in SL$ for which the function $\mathcal{P}_{\Gamma,\theta}$ is of any of the three forms above (the proof of the preceding proposition shows how to find them).

Corollary 56 *Let $\eta \in (0, 1]$.*

1. *If $\Gamma^\eta \models_\zeta \theta$ for $0 < \zeta < \eta$ then $\Gamma^\eta \models_\eta \theta$.*
2. *If $\Gamma^\eta \models_\zeta \theta$ for $0 < \eta < \zeta$ then $\Gamma^\zeta \models_\zeta \theta$.*

Corollary 57 *Let $\eta \in (0, 1]$. $\Gamma^\eta \models_\eta \theta$ if and only if there exists $\phi \in \Gamma$ such that $\phi \vdash \theta$.*

Chapter 5

Multiple thresholds

In Chapter 3 we introduced the consequence relation ${}^{\eta}\triangleright_{\zeta}$, where η was regarded as a lower bound probability threshold for each sentence in the set of premises. Each sentence was, in that sense, given the same preference and that seemed to be well justified. However, at least in some situations, it seems reasonable to treat sentences differently and assign them distinct levels of preference, which in our settings translates into the possibility that distinct sentences in our knowledge base have distinct probability thresholds. This is precisely what we explore in this chapter.[1]

The notation will be slightly different to that of the previous chapters so we start by introducing some notation.

5.1 Notation and remarks

In this section our primitive sentences will be denoted by expressions of the form f_{θ}^{η} –with the intended meaning 'the probability of θ is at least η' where $\theta \in SL$ and $\eta \in [0,1]$. Let $\mathcal{FL}$ be the set of such sentences (that is to say, $\mathcal{FL} = \{\, f_{\theta}^{\eta} \,|\, \theta \in SL \, and \, \eta \in [0,1] \}$) and $\mathcal{SFL}$ the set of boolean combinations of our primitive sentences in $\mathcal{FL}$ (for simplicity we will just consider the connectives $\wedge$, $\vee$ and $\neg$). We will use letters $g, h \ldots$ (possibly with subscripts) for elements of $\mathcal{SFL}$.

Let w be a probability function and $g \in \mathcal{SFL}$. We use the expression $w \triangleright g$

[1] There are a number of formal proof theories in some way similar to the one that we present in this chapter (see for example [10], [11], [12], [17], [20], [22], [23], [31], [32], [33] or [36]). The approach in this chapter is particularly close to that in [17] and [22].

to mean that w *satisfies* g. We define satisfiability for sentences in $\mathcal{SFL}$ in the obvious way:

$$\begin{aligned} w \rhd f_\phi^\eta &\iff w(\phi) \geq \eta \\ w \rhd \neg g &\iff w \not\rhd g \\ w \rhd g \wedge h &\iff w \rhd g \text{ and } w \rhd h \\ w \rhd g \vee h &\iff w \rhd g \text{ or } w \rhd h. \end{aligned}$$

Throughout let $\Gamma = \{g_1, ..., g_n\} \subset \mathcal{SFL}$ and $h \in \mathcal{SFL}$.

Definition 58 *We say that Γ implies h (denoted $\Gamma \rhd h$) if and only if, for all probability functions w on L, if $w \rhd g$ for all $g \in \Gamma$ then $w \rhd h$.*

Sometimes we will use the abbreviation $w \rhd \Gamma$ to mean that $w \rhd g$ for all $g \in \Gamma$.

5.2 The consequence relation $\rhd$

Throughout this section let $\Phi = \{f_{\phi_1}^{\eta_1}, ..., f_{\phi_r}^{\eta_r}\} \subset \mathcal{FL}$ and $f_\theta^\zeta \in \mathcal{FL}$.

The next proposition states some properties of the consequence relation $\rhd$ for simple sets of premises like Φ.

Proposition 59 *We have what follows:*

(1) $\emptyset \rhd f_\theta^0$ for any $\theta \in SL$.

(2) If $\theta \vdash \phi$ then $f_\theta^\eta \rhd f_\phi^\eta$.

(3) $\emptyset \rhd f_\theta^1 \iff \vdash \theta$.

(4) If $0 \leq \zeta \leq \eta \leq 1$ then $f_\theta^\eta \rhd f_\theta^\zeta$.

(5) If $\Phi \rhd f_\theta^\zeta$ then $\Phi \cup \Phi' \rhd f_\theta^\zeta$.

(6) If $\Phi_i \rhd f_{\phi_i}^{\eta_i}$ for each $f_{\phi_i}^{\eta_i} \in \Phi$ and $\Phi \rhd f_\phi^\zeta$ then $\bigcup_i \Phi_i \rhd f_\phi^\zeta$.

Proof. All these properties follow immediately from the definition of $\rhd$. ■

Next we state a closure property of the probability thresholds in $\rhd$.

Proposition 60 *Suppose that* $\{\eta_i^n\}$ *is an increasing sequence with limit* η_i *for* $i \in \{1, ..., r\}$, $\{\zeta^n\}$ *a sequence with limit* ζ *and*

$$f_{\phi_1}^{\eta_1^n}, ..., f_{\phi_r}^{\eta_r^n} \triangleright f_{\phi}^{\zeta^n}$$

for all n. *Then*

$$f_{\phi_1}^{\eta_1}, ..., f_{\phi_r}^{\eta_r} \triangleright f_{\phi}^{\zeta}.$$

Proof. Let us proceed by *reductio ad absurdum* and assume that

$$f_{\phi_1}^{\eta_1}, ..., f_{\phi_r}^{\eta_r} \not\triangleright f_{\phi}^{\zeta}.$$

Thus there exists a probability function w such that $w(\phi_i) \geq \eta_i$ for all $i \in \{1, ..., r\}$ and $w(\phi) < \zeta$. But then, for some n, $w(\phi) < \zeta^n$ and, since $\eta_i^n \leq \eta_i$,

$$f_{\phi_1}^{\eta_1^n}, \ldots, f_{\phi_r}^{\eta_r^n} \not\triangleright f_{\phi}^{\zeta^n},$$

which contradicts the assumption above. ■

For what follows let $\vec{\phi} = (\phi_1, \ldots, \phi_r) \in SL^r$ and $\vec{\eta} = (\eta_1, \ldots, \eta_r) \in [0, 1]^r$.

Definition 61 *We say that* $\vec{\phi}$ *is* $\vec{\eta}$*-consistent if there exists a probability function* w *for which* $w(\phi_i) \geq \eta_i$ *for all* $i \in \{1, ..., r\}$.

Let $\mathcal{C}(\vec{\phi})$ denote the set of $\vec{\eta} \in [0, 1]^r$ for which $\vec{\phi}$ is $\vec{\eta}$-consistent.

Clearly if $\vec{\eta} \in \mathcal{C}(\vec{\phi})$ and $\lambda_i \leq \eta_i$ for $i \in \{1, ..., r\}$ then $\vec{\lambda} \in \mathcal{C}(\vec{\phi})$.

A somewhat more interesting closure property is given by the next proposition.

Proposition 62 $\mathcal{C}(\vec{\phi})$ *is a closed subset of* $[0, 1]^r$.

Proof. Let $M_{\underline{\vec{\phi}}}$ be the matrix representing the vector $\vec{\phi}$ with respect to the atoms in At^L.[2]

Let $\{\vec{\eta}^n\}$ be a sequence in $\mathcal{C}(\vec{\phi})$ that converges to $\vec{\eta}$. Then there must be probability functions w^n such that $w^n(\phi_i) \geq \eta_i^n$ for all $i \in \{1, ..., r\}$ and all n. Let $\vec{x}^n \in \mathbb{D}_{2^l}$ be the vector representation of w^n and consider the sequence $\{\vec{x}^n\}$.

We need to prove now that there exists a probability function $\vec{x} \in \mathbb{D}_{2^l}$ such that $\left(M_{\underline{\vec{\phi}}}(\vec{x})^T\right)^i \geq \eta_i$ for all $i \in \{1, ..., r\}$.

[2]Such matrix is analogous to those defined in the introduction for sets of sentences in SL.

We can take a convergent subsequence $\{\vec{x}^1_{n_k}\}$ of the first coordinates of $\{\vec{x}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0,1]$ by compactness. We can proceed in the same way for the other coordinates.

The final subsequence, $\{\vec{x}^{2^l}_{n_k}\}$, will have as a limit a probability function $\vec{x} \in \mathbb{D}_{2^l}$ such that $\left(M_{\underline{\vec{\phi}}}(\vec{x})^T\right)^i \geq \eta_i$ for all $i \in \{1, ..., r\}$. ■

5.3 The function $F_{\vec{\phi},\theta}$

We define the function $F_{\vec{\phi},\theta}$ as follows:

$$F_{\vec{\phi},\theta}(\vec{\eta}) = sup\{\, \zeta \mid f^{\eta_1}_{\phi_1}, ..., f^{\eta_r}_{\phi_r} \rhd f^{\zeta}_{\theta} \,\}.$$

Proposition 63 *$F_{\vec{\phi},\theta}$ is increasing on $[0,1]^r$.*

Proof. It follows immediately from the definition of $\rhd$. ■

The next proposition shows that $F_{\vec{\phi},\theta}$ measures the guaranteed minimum probability that θ can assume given the stated, attainable, lower bounds on the probabilities of the ϕ_i.

Proposition 64 *If $\vec{\phi}$ is $\vec{\lambda}$-consistent then there is a probability function w such that $w(\phi_j) \geq \lambda_j$ for all $j \in \{1, ..., r\}$ and $w(\theta) = F_{\vec{\phi},\theta}(\vec{\lambda})$.*

Proof. Let $M_{\underline{\vec{\phi}}}$ be the matrix representing the vector $\vec{\phi}$ with respect to the atoms of L and $\vec{\theta}$ the sentence θ.

We can define a decreasing sequence $\{\zeta_n\}$ whose limit is ζ such that for all $n \in \mathbb{N}$ there exists a probability function w_n with $w_n(\theta) = \zeta_n$ and $w_n(\phi_i) \geq \lambda_i$ for all $i \in \{1, ..., r\}$. We can represent $\{w_n\}$ by a sequence of vectors $\{\vec{x_n}\}$ such that $\vec{\theta} \cdot \vec{x_n} = \zeta_n$ for all $n \in \mathbb{N}$ and $(M_{\underline{\vec{\phi}}}(\vec{x_n})^T)^i \geq \lambda_i$ for all $i \in \{1, ..., r\}$.

We need to prove now that there exists a probability function $\vec{x} \in \mathbb{D}_{2^l}$ such that $\vec{\theta} \cdot \vec{x} = \zeta$ and $\left(M_{\underline{\vec{\phi}}}(\vec{x})^T\right)^i \geq \lambda_i$ for all $i \in \{1, ..., r\}$.

We can take a convergent subsequence $\{\vec{x}^1_{n_k}\}$ in the first coordinates of $\{\vec{x}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0,1]$ by compactness. We can proceed in the same way for the other coordinates.

The final subsequence, $\{\vec{x}^{2^l}_{n_k}\}$, will have as a limit a probability function $\vec{x} \in \mathbb{D}_{2^l}$ such that $\left(M_{\underline{\vec{\phi}}}(\vec{x})^T\right)^i \geq \lambda_i$ for all $i \in \{1, ..., r\}$ and $\vec{\theta} \cdot \vec{x} = \zeta$. ■

It is clear that as $\vec{\lambda}^n$ moves upwards to a limit $\vec{\lambda}$ at which $\vec{\phi}$ is not consistent then the value of $F_{\vec{\phi},\theta}(\vec{\lambda}^n)$ suddenly jumps to 1.

Proposition 65 *$F_{\vec{\phi},\theta}$ is continuous on $\mathcal{C}(\vec{\phi})$.*

Proof.[3] Let $\{\vec{\eta}^n\} \subseteq C(\vec{\phi})$ converge to $\vec{\eta}$. We show that $\{F_{\vec{\phi},\theta}(\vec{\eta}^n)\}$ converges to $F_{\vec{\phi},\theta}(\vec{\eta})$.

Let $w^n(\phi_i) \geq \eta^n_i$ for all $i \in \{1, ..., r\}$ and

$$w^n(\theta) = F_{\vec{\phi},\theta}(\vec{\eta}^n).$$

By taking a subsequence if necessary we may assume that $\{w^n\}$ converges to w^∞ so it is enough to show that

$$w^\infty(\theta) = F_{\vec{\phi},\theta}(\vec{\eta}).$$

Suppose not. In that case

$$w^\infty(\theta) > F_{\vec{\phi},\theta}(\vec{\eta}) = w(\theta)$$

where w is chosen such that $w(\phi_i) \geq \eta_i$ for all $i \in \{1, ..., r\}$ and $w(\theta) = F_{\vec{\phi},\theta}(\vec{\eta})$.

Let $\epsilon > 0$ be sufficiently small that for $v = (1-\epsilon)w + \epsilon w^\infty$, $v(\theta) < w^n(\theta) + \epsilon$ for all n eventually. Then for any atom α, if $v(\alpha) = 0$ then $w(\alpha) = w^\infty(\alpha) = 0$.

Now let n be large and consider $u = v + w^n - w^\infty$. This is a probability function since for any α if $v(\alpha) = 0$ then $w^n(\alpha) - w^\infty(\alpha) = w^n(\alpha) \geq 0$ whilst if $v(\alpha) > 0$ then, since n is large,

$$|w^n(\alpha) - w^\infty(\alpha)| < v(\alpha)$$

so again $u(\alpha) \geq 0$.

Also if one of $w^\infty(\phi_i), w(\phi_i)$ is strictly larger than η_i for $i \in \{1, ..., r\}$ then for some $\delta > 0$ we will have that $v(\phi_i) > \eta_i + \delta$ so since n is large

$$u(\phi_i) = v(\phi_i) + w^n(\phi_i) - w^\infty(\phi_i) \geq \eta_i + \frac{\delta}{2} \geq \eta^n_i.$$

[3]This proof is due to Jeff Paris.

On the other hand if $w^\infty(\phi_i) = w(\phi_i) = \eta_i$ then

$$u(\phi_i) = v(\phi_i) + w^n(\phi_i) - w^\infty(\phi_i) = w^n(\phi_i) \geq \eta_i^n.$$

But this contradicts the definition of w^n since

$$u(\theta) = (1-\epsilon)w(\theta) + \epsilon w^\infty(\theta) + (w^n(\theta) - w^\infty(\theta))$$

which is less than $w^n(\theta)$. ∎

Together with the previous propositions our next result provides a clear picture of the shape of the graph of $F_{\vec{\phi},\theta}$.

Proposition 66 *On $\mathcal{C}(\vec{\phi})$ the function $F_{\vec{\phi},\theta}$ is made up of a finite collection of linear polynomials of the form $q_0 + \sum_{i=1}^r q_i x_i$ for $q_i \in \mathbb{Q}$. That is to say, for $\vec{x} \in \mathcal{C}(\vec{\phi})$,*

$$F_{\vec{\phi},\theta}(\vec{x}) = \begin{cases} q_0^1 + q_1^1 x_1 + \ldots + q_r^1 x_r & if \;\; \vec{x} \in C_1 \\ \ldots & \\ q_0^n + q_1^n x_1 + \ldots + q_r^n x_r & if \;\; \vec{x} \in C_n \end{cases}$$

where the C_i are closed convex sets with union $\mathcal{C}(\vec{\phi})$ each defined as the solutions of a finite set of linear inequalities

$$a_0 + a_1x_1 + a_2x_2 + \ldots + a_rx_r \geq b_0 + b_1x_1 + b_2x_2 + \ldots + b_rx_r$$

with coefficients in $\mathbb{Q}$.

Proof. Let $\mathcal{R} = \langle \mathbb{R}, +, <, =, 0, 1\rangle$. The set

$$\{(x_1, \ldots, x_r, y) \in \mathbb{R}^{r+1} \mid y = F_{\vec{\phi},\theta}(x_1, \ldots, x_r)\}$$

is $\mathcal{R}$-definable, so, since $\mathcal{R}$ is an elementary extension of the structure

$$\mathcal{Q} = \langle \mathbb{Q}, +, <, =, 0, 1\rangle,$$

it is $\mathcal{Q}$-definable too.

The theory of $\mathcal{R}$ has quantifier elimination (see for example [29]). Therefore the set

$$\{(x_1, \ldots, x_r, y) \in \mathbb{R}^{r+1} \mid y = F_{\vec{\phi},\theta}(x_1, \ldots, x_r)\}$$

is given by a finite boolean combination (which reduces to a finite union of intersections by the complement and distributive laws for sets) of sets of the form

$$\{(x_1, \dots, x_r, y) \in \mathbb{R}^{r+1} \mid my < \sum_{i=1}^{r} n_i x_i + k\}$$

and

$$\{\,(x_1, \dots, x_r, y) \in \mathbb{R}^{r+1} \mid my = \sum_{i=1}^{r} n_i x_i + k\},$$

for some $n_i, m, k \in \mathbb{Z}$.

Notice that each intersection of sets of such form is convex so, since $F_{\vec{\phi},\theta}$ is a function, such an intersection has to be a polynomial of the above mentioned form (with coefficients in $\mathbb{Q}$) and sets of constraints C_i (with coefficients in $\mathbb{Q}$ too) in the form of finite conjunctions of linear equalities

$$c_0 + c_1 x_1 + \ldots + c_r x_r = 0 \tag{5.1}$$

and strict inequalities

$$a_0 + a_1 x_1 + \ldots + a_r x_r > b_0 + b_1 x_1 + \ldots + b_r x_r.$$

Clearly we may assume the C_i are satisfiable in which case, being defined by linear constraints, their closure is given by the corresponding equalities (5.1) and inequalities

$$a_0 + a_1 x_1 + \ldots + a_r x_r \geq b_0 + b_1 x_1 + \ldots + b_r x_r.$$

Furthermore since $F_{\vec{\phi},\theta}$ is continuous it still takes the required linear polynomial form on these closures. ■

We now give the main result of this section, an equivalent of the consequence relation $\triangleright$ (similar in flavour to that given in [22]) which captures it entirely within classical propositional calculus.

Theorem 67 *For $\phi_1, \dots, \phi_r, \theta \in SL$ and $\eta_1, ..., \eta_r, \zeta \in (0, 1]$,*

$$f^{\eta_1}_{\phi_1}, ..., f^{\eta_r}_{\phi_r} \triangleright f^{\zeta}_{\theta}$$

if and only if there exist a multiset Ψ consisting of some N^j copies of ϕ_j for

$j \in \{1, ..., r\}$ and $Z, T \in \mathbb{N}$ with $T < Z$ such that,

$$T(1-\zeta) \leq \sum_{j=1}^{r} \eta_j N^j - \zeta Z + 1,$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=Z}} \bigwedge_{j \in S} \chi_j \vdash \bot$$

and

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=T}} \bigwedge_{j \in S} \chi_j \vdash \theta.$$

Proof. The derivation follows a similar pattern to that of ${}^{\eta}\triangleright_{\zeta}$ in Chapter 3.

Let us take $\vec{\phi} = (\phi_1, \ldots, \phi_r) \in SL^r$ and $\theta \in SL$. For the time being we assume that θ is not a tautology.

We first consider the case where $\eta_1, ..., \eta_r, \zeta \in (0,1] \cap \mathbb{Q}$, say $0 < \eta_i = \frac{p_i}{q_i}$, for $i \in \{1, ..., r\}$, $\zeta = \frac{c}{d}$, for $p_i, q_i, c, d \in \mathbb{N}$, and assume that

$$f^{\eta_1}_{\phi_1}, \ldots, f^{\eta_r}_{\phi_r} \triangleright f^{\zeta}_{\theta}. \tag{5.2}$$

Let $\mathcal{B} = \{\beta_1, ..., \beta_m\}$.

Let $\vec{\phi_j}$ be the m-coordinate vector with i^{th} coordinate 1 if $\beta_i \vdash \phi_j$ and 0 otherwise and $\vec{\theta}$ the m-coordinate vector with i^{th} coordinate 1 if $\beta_i \vdash \theta$ and 0 otherwise.

We have what follows:

For all $\vec{x} \in \mathbb{D}_m$, if $\vec{\phi_j} \cdot \vec{x} \geq \frac{p_j}{q_j}$ for all $j \in \{1, ..., r\}$ then $\vec{\theta} \cdot \vec{x} \geq \frac{c}{d}$.

Now let $\frac{\vec{p_i}}{q_i}$ and $\frac{\vec{c}}{d}$ be m-coordinate vectors with each coordinate $\frac{p_i}{q_i}$ for the first and $\frac{c}{d}$ for the latter and define

$$\begin{aligned} \underline{\vec{\phi}}_j &= \vec{\phi}_j - \frac{\vec{p_j}}{q_j}, \\ \underline{\vec{\theta}} &= \vec{\theta} - \frac{\vec{c}}{d}, \end{aligned}$$

for $j \in \{1, ..., r\}$.

Thus we have:

$$\text{For all } \vec{x} \in \mathbb{D}_m, \text{ if } \underline{\vec{\phi}}_j \cdot \vec{x} \geq 0 \text{ for all } j \in \{1, ..., r\} \text{ then } \underline{\vec{\theta}} \cdot \vec{x} \geq 0.$$

From this it follows that $\underline{\vec{\theta}}$ has to be in the cone in $\mathbb{Q}^m$ given by

$$\left\{ \sum_{j=1}^{r} a_j \underline{\vec{\phi}}_j + \sum_{i=1}^{m} b_i \vec{e_i} \mid 0 \leq a_j, b_i \in \mathbb{Q} \right\}$$

where $\vec{e_i}$ is the m-vector with i^{th} coordinate 1 and 0's elsewhere, say

$$\underline{\vec{\theta}} = \sum_{j=1}^{r} a_j \underline{\vec{\phi}}_j + \sum_{i=1}^{m} b_i \vec{e_i}$$

with $0 \leq a_j, b_i \in \mathbb{Q}$.

Let $a_j = \frac{u_j}{v_j}$ and define $M = \prod_j v_j$, $N_j = M a_j$, $Q = \prod_j q_j$ and $Q_j = \frac{Q}{q_j}$. Removing the rightmost summation and multiplying both sides by MQd gives the inequality

$$MQ(d\vec{\theta} - c\vec{1}) \geq \sum_{j=1}^{r} dN_j Q_j (q_j \vec{\phi}_j - \vec{1} p_j). \tag{5.3}$$

Notice that if (5.3) holds for some natural numbers M and N_j then (5.2) holds too.

From (5.3) we obtain

$$\left[d \sum_{j=1}^{r} N_j Q_j p_j - MQc \right] \vec{1} \geq dQ(\sum_{i=1}^{r} N_j \vec{\phi}_j - M\vec{\theta}).$$

Equivalently

$$\left[d \sum_{i=1}^{r} N_j Q_j p_j + MQ(d-c) \right] \vec{1} \geq dQ(\sum_{j=1}^{r} N_j \vec{\phi}_j + M(\vec{1} - \vec{\theta})).$$

Let $\chi_1, \ldots, \chi_N$ consist of $N^j = dQN_j$ copies of each ϕ_j for $j \in \{1, ..., r\}$, so $N = \sum_j N^j = \sum_j dQN_j$.

For $\beta_k \nvdash \theta$ (notice that there is at least one such k since θ is not a tautology) we have the following inequality for the k^{th} coordinate:

$$\sum_{j=1}^{r} \frac{p_j}{q_j} N^j - cMQ \geq dQ \sum_{j=1}^{r} N_j \phi_j^k$$

where ϕ_j^k is the k^{th} coordinate of $\vec{\phi}_j$. Thus we have that:

$$\bigvee_S \bigwedge_{i \in S} \chi_i \vdash \theta$$

where the disjunction is over $S \subseteq \{1, ..., N\}$ with

$$|S| > \sum_{j=1}^{r} \frac{p_j}{q_j} N^j - cMQ$$

(notice that $\sum_{j=1}^{r} \frac{p_j}{q_j} N^j - cMQ$ will have to be an integer greater than or equal to zero).

Similarly, for $\beta_k \vdash \theta$ we will have the following inequality:

$$\sum_{j=1}^{r} \frac{p_j}{q_j} N^j + (d-c)MQ \geq dQ \sum_{j=1}^{r} N_j \phi_j^k,$$

so

$$\bigvee_S \bigwedge_{i \in S} \chi_i \vdash \perp$$

where the disjunction is over $S \subseteq \{1, ..., N\}$ with

$$|S| > \sum_{j=1}^{r} \frac{p_j}{q_j} N^j + (d-c)MQ.$$

Now let

$$Z = 1 + \sum_{j=1}^{r} \frac{p_j}{q_j} N^j + (d-c)MQ$$

$$T = 1 + \sum_{j=1}^{r} \frac{p_j}{q_j} N^j - cMQ$$

giving

$$T(1-\zeta) = \sum_{j=1}^{r} \eta_j N^j + 1 - \zeta Z$$

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S| = Z}} \bigwedge_{i \in S} \chi_i \vdash \perp \tag{5.4}$$

and

$$\bigvee_{\substack{S\subseteq\{1,...,N\}\\|S|=T}} \bigwedge_{i\in S} \chi_i \vdash \theta. \tag{5.5}$$

Conversely suppose that for some $Z, T \in \mathbb{N}$ and $\chi_1, ..., \chi_N$ consisting of N^j copies of ϕ_j for $j \in \{1, ..., r\}$ the above conditions (5.4), (5.5) hold and

$$T(1-\zeta) \leq \sum_{j=1}^{r} \eta_j N^j - \zeta Z + 1 \tag{5.6}$$

with $1 \leq T < Z$.

Then for any atom $\alpha \in At^L$, if $\alpha \vdash \neg\theta$ then for at most $T-1$ j's can we have that $\alpha \vdash \chi_j$. If $\alpha \vdash \theta$ then for at most $Z-1$ j's can we have that $\alpha \vdash \psi_j$. Hence, adopting the previous vector notation (now with real atoms instead of the β's) we have that

$$\sum_{i=1}^{N} \vec{\chi}_i \leq (T-1)\vec{1} + (Z-T)\vec{\theta}.$$

Now let us suppose that $\vec{x} \in \mathbb{D}_{2^l}$ and that $\vec{\phi}_j \cdot \vec{x} \geq \eta_j$ for $j \in \{1, ..., r\}$. Then

$$\vec{\theta} \cdot \vec{x} \geq \frac{\sum_{j=1}^{r} \eta_j N^j - T + 1}{Z - T}.$$

Notice that the right expression is at least ζ if

$$T(1-\zeta) \leq \sum_{j=1}^{r} \eta_j N^j + 1 - \zeta Z$$

which it is. Thus $\vec{\theta} \cdot \vec{x} \geq \zeta$ and the result follows.

Summarizing, if (5.4), (5.5) and (5.6) hold then

$$f^{\eta_1}_{\chi_1}, ..., f^{\eta_1}_{\chi_{N^1}}, ..., f^{\eta_r}_{\chi_{\sum_{j=1}^{r-1} N^j+1}}, ..., f^{\eta_r}_{\chi_{N^r}} \triangleright f^{\zeta}_{\theta}$$

and by Proposition 59 (5) (if necessary) we have

$$f^{\eta_1}_{\phi_1}, ..., f^{\eta_r}_{\phi_r} \triangleright f^{\zeta}_{\theta}.$$

Conversely if

$$f_{\phi_1}^{\eta_1}, ..., f_{\phi_r}^{\eta_r} \triangleright f_\theta^\zeta$$

then there are sentences

$$\chi_1, ..., \chi_N \in \{\phi_1, ..., \phi_r\}$$

(possibly with repeats) such that for some Z and T conditions (5.4), (5.5) and (5.6) hold.

Let us suppose now that θ is a tautology.

From left to right we can see that the result follows by taking values $T = N = 0$ and $Z = 1$. From right to left, if the right-hand side of the equivalence holds with $T = 0$ then θ must be a tautology.

This completes the proof in the case where the η_i and ζ are rational. Suppose now that not all the η_i and/or ζ are rational. Notice that we only need to consider the case when $\zeta = F_{\vec{\phi},\theta}(\vec{\eta})$ since, if $F_{\vec{\phi},\theta}(\vec{\eta}) = \zeta' > \zeta$ and we have a collection of sentences

$$\chi_1, ..., \chi_{N^1}, ..., \chi_{\sum_{j=1}^{r-1} N^j+1}, ..., \chi_N \in \{\phi_1, ..., \phi_r\}$$

(possibly with repeats) and Z, T for which conditions (5.4), (5.5) hold and

$$\zeta' \leq \frac{\sum_{j=1}^r \eta_j N^j + 1 - T}{Z - T}$$

with $T < Z$ then, clearly, such a collection of sentences and Z and T will be such that

$$\zeta < \frac{\sum_{j=1}^r \eta_j N^j + 1 - T}{Z - T}.$$

So suppose first that

$$f_{\phi_1}^{\eta_1}, ..., f_{\phi_r}^{\eta_r} \triangleright f_\theta^\zeta.$$

Referring back to Proposition 66 we may assume that $\vec{\eta} \in C_1$ in the notation of that proposition. Let $\vec{s}$ be the mean of the (finitely many) extreme points of C_1. Notice that such extreme points are all rational by Proposition 66 and that $\vec{s} \in C_1$.

By Proposition 66 $F_{\vec{\phi},\theta}(\vec{s})$ is rational so by the discussion above we know that there exist sentences

$$\chi_1, ..., \chi_N \in \{\phi_1, ..., \phi_r\},$$

and Z, T with $T < Z$ for which conditions (5.4), (5.5) hold and

$$F_{\vec{\phi},\theta}(\vec{s}) \leq \frac{\sum_{j=1}^r s_j N^j + 1 - T}{Z - T}.$$

In fact we must have equality here since otherwise we could increase the left hand side whilst keeping $\vec{s}$ fixed, contradicting the maximality of $F_{\vec{\phi},\theta}(\vec{s})$. Hence

$$F_{\vec{\phi},\theta}(\vec{x}) = \frac{\sum_{j=1}^r x_j N^j + 1 - T}{Z - T} \tag{5.7}$$

when $\vec{x} = \vec{s}$.

By Proposition 66,

$$F_{\vec{\phi},\theta}(\vec{x}) = q_0 + \sum_{j=1}^r x_j q^j \tag{5.8}$$

for $\vec{x} \in C_1$ and in fact the right hand side polynomials in (5.7), (5.8) must actually be identically equal on C_1. Otherwise, because $\vec{s}$ could not then be the only (necessarily then extreme) point of C_1, there would be a point $\vec{u} \in C_1$ close to $\vec{s}$ at which

$$F_{\vec{\phi},\theta}(\vec{u}) = q_0 + \sum_{j=1}^r u_j q^j < \frac{\sum_{j=1}^r u_j N^j + 1 - T}{Z - T}$$

contradicting the already proved rational case.

Therefore, given the above equalities, we must have that

$$\zeta = \frac{\sum_{j=1}^r \eta_j N^j + 1 - T}{Z - T}.$$

In the other direction, suppose that we have sentences

$$\chi_1, ..., \chi_N \in \{\phi_1, ..., \phi_r\},$$

Z and T for which conditions (5.4), (5.5) hold and

$$\zeta \leq \frac{\sum_{j=1}^r \eta_j N^j + 1 - T}{Z - T},$$

with $T < Z$ (where of course we are no longer assuming that $\zeta = F_{\vec{\phi},\theta}(\vec{\eta})$). If $\vec{\eta} \notin C(\vec{\phi})$ then we trivially have the required conclusion that

$$f_{\phi_1}^{\eta_1}, \ldots, f_{\phi_r}^{\eta_r} \triangleright f_\theta^\zeta.$$

So let us assume that $\vec{\eta} \in C(\vec{\phi})$. Then, by the proved rational case for rational points, we will have $\vec{p} \in C(\vec{\phi})$ close to $\vec{\eta}$ and

$$q = \frac{\sum_{j=1}^r p_j N^j + 1 - T}{Z - T},$$

$$f_{\phi_1}^{p_1}, \ldots, f_{\phi_r}^{p_r} \triangleright f_\theta^q.$$

Hence

$$F_{\vec{\phi},\theta}(\vec{p}) \geq \frac{\sum_{j=1}^r p_j N^j + 1 - T}{Z - T}$$

and by continuity

$$F_{\vec{\phi},\theta}(\vec{\eta}) \geq \frac{\sum_{j=1}^r \eta_j N^j + 1 - T}{Z - T} \geq \zeta,$$

giving the required conclusion that

$$f_{\phi_1}^{\eta_1}, \ldots, f_{\phi_r}^{\eta_r} \triangleright f_\theta^\zeta.$$

When ζ and/or all the η_i are zero, for $i \in \{1, ..., r\}$, we have a trivial equivalent version.

If only some of the η_i (not all) are zero then the equivalent version in classical propositional calculus reduces trivially to the one above by only considering those η_i that are non zero. ■

5.4 A complete proof system for $\triangleright$

We now introduce the proof system $\Vdash$ which we will shortly show is sound and complete with respect to $\triangleright$. It consists of some complete and sound set of rules and axioms for classical propositional logic[4], say in a natural deduction format to tie in with what follows, together with the following probability rules and axioms:

[4]Which we assume has rules which allow us to freely add to and remove repeats from the left hand side of sequents.

Fraction Rules:

Let $\{\phi_1, ..., \phi_r\} \subseteq SL$ and, for each $i \in \{1, ..., r\}$, let $\chi_k = \phi_i$ for all

$$k \in \{\sum_{j=1}^{i-1} N^j + 1, ..., \sum_{j=1}^{i} N^j\}$$

(where N^j is the number of copies of ϕ_j).

$$\frac{\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=Z}} \bigwedge_{j \in S} \chi_j \vdash \bot}{f^{\eta_1}_{\chi_1}, ..., f^{\eta_1}_{\chi_{N^1}}, ..., f^{\eta_r}_{\chi_{\sum_{j=1}^{r-1} N^j+1}}, ..., f^{\eta_r}_{\chi_N} \,|\, f^{\zeta}_{\psi}}$$

where

$$\bigvee_{\substack{S \subseteq \{1,...,N\} \\ |S|=T}} \bigwedge_{j \in S} \chi_j \vdash \psi$$

,

$$T(1-\zeta) \leq \sum_{i=1}^{r} \eta_i N^i - \zeta Z + 1 \;\; and \;\; 0 \leq T < Z.$$

Negation Replacement Axiom:

$$\neg f^{\eta}_{\phi} \,|\, f^{1-\eta}_{\neg\phi}$$

Introduction Axiom:

$$|\, f^{0}_{\phi}$$

Elimination Axiom:

$$f^{\eta}_{\phi} \wedge f^{0}_{\phi} \,|\, f^{\eta}_{\phi}$$

Tautology Axiom:

$$|\, f^{\eta}_{\top}$$

We define $\Gamma \Vdash g$ to hold if there is a formal proof of $\Delta \,|\, g$ using these rules and axioms for some $\Delta \subseteq \Gamma$; that is, if there is a finite sequence

$$\Delta_1 \,|\, g_1, ..., \Delta_r \,|\, g_r,$$

with the Δ_i finite, $\Delta_r \,|\, g_r = \Delta \,|\, g$, and each $\Delta_i \,|\, g_i$ is either an axiom or follows by one of the rules from previous sequents.

Theorem 68 *Let Γ be a finite subset of $\mathcal{SFL}$ and $g \in \mathcal{SFL}$. Then*

$$\Gamma \rhd g \iff \Gamma \Vdash g.$$

Proof. That the rules and axioms are sound for these semantics is easy to check.[5]

In the other direction suppose that $\Gamma \rhd g$. Appealing to the *Disjunctive* and *Conjunctive Normal Form Theorems*, in order to prove that $\Gamma \Vdash g$ it will be enough to show that

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} \pm f_{\phi_{ij}}^{\eta_{ij}} \Vdash \bigwedge_{u=1}^{r} \bigvee_{v=1}^{h_u} \pm f_{\theta_{uv}}^{\zeta_{uv}}$$

whenever this holds with $\rhd$ in place of $\Vdash$. From this we have that

$$\bigwedge_{j=1}^{k_i} \pm f_{\phi_{ij}}^{\eta_{ij}} \rhd \bigvee_{v=1}^{h_u} \pm f_{\theta_{uv}}^{\zeta_{uv}}$$

for each $i \in \{1, ..., m\}$ and $u \in \{1, ..., r\}$ and again by appealing to the propositional calculus it will be enough to show this holds with $\Vdash$ in place of $\rhd$.

If every $f_{\theta_{uv}}^{\zeta_{uv}}$ here is negated and every $f_{\phi_{ij}}^{\eta_{ij}}$ is not negated then we will have that

$$\bigwedge_{j=1}^{k_i} f_{\phi_{ij}}^{\eta_{ij}} \wedge \bigwedge_{v=1}^{h_u} f_{\theta_{uv}}^{\zeta_{uv}} \rhd f_{\perp}^{1}$$

and, assuming that all the η_{ij} and ζ_{uv} are greater than zero, the required result with $\Vdash$ in place of $\rhd$ will follow by applying some Fraction Rule (if it were not the case that all the η_{ij} and ζ_{uv} be greater than zero then we should apply some of the axioms above in a trivial way).

On the other hand if some $f_{\theta_{uv}}^{\zeta_{uv}}$ here is not negated (or some $f_{\phi_{ij}}^{\eta_{ij}}$ is) we can shuffle these literals across and leave ourselves to show something of the form

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d} \neg f_{\varphi_i}^{\delta_i} \Vdash f_{\theta}^{\beta}$$

[5]Notice that in the Negation Replacement Axiom, $\neg f_{\phi}^{\eta} \,|\, f_{\neg\phi}^{1-\eta}$, we appear to lose some information since the negation of f_{ϕ}^{η} amounts to $w(\phi) < \eta$, equivalently $w(\neg\phi) > 1 - \eta$ and the rule replaces this by $w(\neg\phi) \geq 1 - \eta$.

where this holds with $\rhd$ in place of $\Vdash$.

If

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d} \neg f_{\varphi_i}^{\delta_i}$$

is not satisfiable then

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d-1} \neg f_{\varphi_i}^{\delta_i} \rhd f_{\varphi_d}^{\delta_d}$$

and we can repeat the above argument (in a case with fewer literals). On the other hand if this conjunction is satisfiable then Lemma 69 below tells us that

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d} f_{\neg\varphi_i}^{1-\delta_i} \rhd f_{\theta}^{\beta}$$

and, assuming that all the γ_i are greater than zero and all the δ_i less than one, applications of some Fraction Rule and Negation Replacement Axioms give us the required proof. ■

Lemma 69 *If*

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d} \neg f_{\varphi_i}^{\delta_i} \rhd f_{\theta}^{\beta}$$

and

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d} \neg f_{\varphi_i}^{\delta_i}$$

is satisfiable then

$$\bigwedge_{i=1}^{q} f_{\psi_i}^{\gamma_i} \wedge \bigwedge_{i=1}^{d} f_{\neg\varphi_i}^{1-\delta_i} \rhd f_{\theta}^{\beta}$$

Proof. Let us proceed by *reductio ad absurdum* assuming that there is some probability function w_1 such that $w_1(\psi_i) \geq \gamma_i$ for $i \in \{1, ..., q\}$ and $w_1(\neg\varphi_i) \geq 1 - \delta_i$ for $i \in \{1, ..., d\}$ but $w_1(\theta) < \beta$.

Let w_2 be such that $w_2(\psi_i) \geq \gamma_i$ for $i \in \{1, ..., q\}$ and $w_2(\neg\varphi_i) > 1 - \delta_i$ for $i \in \{1, ..., d\}$. The latter condition is equivalent to $\neg(w_2(\varphi_i) \geq \delta_i)$ for $i \in \{1, ..., d\}$. Thus $w_2(\theta) \geq \beta$.

Hence we could find a convex combination $w^* = (1-\epsilon)w_1 + \epsilon w_2$ $(\epsilon > 0)$ with $w^*(\psi_i) \geq \gamma_i$ for $i \in \{1, ..., q\}$ and $w^*(\neg\varphi_i) > 1 - \delta_i$ for $i \in \{1, ..., d\}$. But then $w^*(\theta) < \beta$, which contradicts the assumption we started with. ■

It is easy to see that we cannot extend this to infinite Γ (in the forward direction) since for example

$$\{f_p^{1-\frac{1}{n}} \mid n \in \mathbb{N}\} \triangleright f_p^1$$

but since this fails for any *finite* subset of the left hand side we cannot have

$$\{f_p^{1-\frac{1}{n}} \mid n \in \mathbb{N}\} \Vdash f_p^1.$$

Chapter 6

${}^{\eta}\triangleright_{\zeta}$ for infinite languages

In this chapter we consider a countably infinite propositional language L^{∞} along with its corresponding set of sentences SL^{∞} (finite boolean combinations of the primitive propositions in L^{∞}) and study some properties of the consequence relation ${}^{\eta}\triangleright_{\zeta}$ when considering possibly infinite sets of premises (in particular we prove the *non-compactness* property of ${}^{\eta}\triangleright_{\zeta}$). We finish the chapter with a representation theorem for the functions of the form $F_{\Gamma,\theta}$, for $\Gamma \subseteq SL^{\infty}$ (possibly infinite) and $\theta \in SL^{\infty}$.

We start with a somewhat brief discussion that aims at proving the above mentioned representation theorem.

Let $0 \leq \mu \leq 1$ and let $\mathcal{F} : [0, \mu] \longrightarrow [0, 1]$ be a function with the following properties:

1. $\mathcal{F}(0) = 0$.

2. $\mathcal{F}$ is increasing.

3. For $\mu > 0$, $\mathcal{F}$ is continuous and convex on $[0, \mu]$.

4. The greatest element $\delta \in [0, \mu]$ for which $\mathcal{F}(\delta) = 0$ is less than 1 and the line segment joining the points $(\delta, 0)$ and $(1, 1)$ is a lower bound of $\mathcal{F}$.

Let $\mathcal{L}$ be the set of straight lines $ax + b$ for which the following conditions hold:

1. $a, b \in \mathbb{Q}$.

2. $b \leq 0$, $1 - b \leq a$.

3. $ax + b \leq \mathcal{F}(x)$ for all $x \in [0, \mu]$.

We will denote these straight lines in $\mathcal{L}$ by l_{ab} (that is to say, $l_{ab}(x) = ax + b$).

Take $l_{ab} \in \mathcal{L}$. Since a and b are rational numbers we know that the points at which it intersects the lines $y = 0$ and $y = 1$ are rational pairs: $(\frac{-b}{a}, 0)$ and $(\frac{1-b}{a}, 1)$ respectively.

Let $a = \frac{p_a}{q_a}$ and $b = \frac{p_b}{q_b}$ with $p_a > 0 \geq p_b$. In terms of p_a, q_a, p_b and q_b the pairs $(\frac{-b}{a}, 0)$ and $(\frac{1-b}{a}, 1)$ become $(\frac{-q_a p_b}{q_b p_a}, 0)$ and $(\frac{q_b q_a - q_a p_b}{q_b p_a}, 1)$ respectively.

Let L be a finite propositional language such that $2q_b p_a < 2^{|L|}$ and SL its corresponding set of sentences.[1]

Recall from Theorem 24 that we can find a finite set of sentences $\Gamma_{ab} \subseteq SL$ and $\theta_{ab} \in SL$ such that $F_{\Gamma_{ab}, \theta_{ab}}$ behaves as follows:

$$F_{\Gamma_{ab},\theta_{ab}}(\eta) = \begin{cases} 0 & if \;\; 0 \leq \eta \leq \frac{-b}{a} \\ l_{ab}(\eta) & if \;\; \frac{-b}{a} < \eta < \frac{1-b}{a} \\ 1 & if \;\; \frac{1-b}{a} \leq \eta \end{cases}$$

In order to define Γ_{ab} and θ_{ab} we will proceed as in Section 3.4 by first defining a suitable matrix for Γ_{ab} and θ_{ab}. The method will be very much based on that implemented in Theorem 24, only that this time we will define Γ_{ab} and θ_{ab} in a slightly different way from the matrix.

We will set a matrix with $q_b p_a + 1$ rows and $2q_b p_a$ columns. The first $q_b p_a$ coordinates of the row $q_b p_a + 1$ will be $0's$ and the other $q_b p_a$ coordinates will be $1's$. For the other rows we will have exactly $-q_a p_b$ $1's$ within the first $q_b p_a$ columns and exactly $q_b q_a - q_a p_b$ $1's$ for the last $q_b p_a$ columns distributed as follows: For the i^{th} row we will have $-q_a p_b$ consecutive $1's$ starting at the i^{th} coordinate (when we reach the column $q_b p_a$ we go back to the first one and carry on until we complete a sequence of $-q_a p_b$ consecutive $1's$) and $q_b q_a - q_a p_b$ consecutive $1's$ starting at the coordinate $q_b p_a + i$ (when we reach the column $2q_b p_a$ we go back to the column $q_b p_a + 1$ and go on until we complete a sequence of $q_b q_a - q_a p_b$ consecutive $1's$).

[1] We could impose the condition that $2q_b p_a \leq 2^{|L|}$ but, for further proofs, it will make things easier if we assume that $2q_b p_a < 2^{|L|}$. This, as we will see, translates into the fact that we will have atoms in L that do not logically imply any sentence in $\Gamma_{ab} \cup \{\theta_{ab}\}$.

Let us see how it works with an example.

> Let us suppose that $\frac{-q_a p_b}{q_b p_a} = \frac{2}{5}$ and $\frac{q_b p_a - q_a p_b}{q_b p_a} = \frac{3}{5}$. Then the matrix described above would correspond to

$$\left(\begin{array}{ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

For our definition of Γ_{ab} and θ_{ab} we will need $2q_b p_a$ atoms,

$$\alpha_1, ..., \alpha_{q_b p_a}, \alpha_{q_b p_a + 1}, ..., \alpha_{2q_b p_a} \subseteq At^L.$$

Define $L^* = L \cup \{s\}$ and At^{L^*} in the obvious way. We want $\Gamma_{ab} \subseteq SL^*$ and $\theta_{ab} \in SL^*$. First, we set $\theta_{ab} = s$. For the i^{th} column we take the following atom:

1. If $1 \leq i \leq q_b p_a$ we take the atom $\alpha_i \wedge \neg s$.

2. If $q_b p_a + 1 \leq i \leq 2q_b p_a$ we take the atom $\alpha_i \wedge s$.

We then define Γ_{ab} as the set of $q_b p_a$ sentences given by the corresponding disjunctions of atoms in At^{L^*} (that is, take the i^{th} row of the matrix and define $\phi_i \in \Gamma_{ab}$ as the disjunction of the atoms for the columns whose i^{th} coordinate is 1).

As seen in Theorem 24, Γ_{ab} and θ_{ab} thus defined give us $F_{\Gamma_{ab}, \theta_{ab}}$.

Let $k \in \mathbb{N}$. Let us consider a collection of finite propositional languages $L_i = \{p_1^i, ..., p_{k_i}^i\}$, with $k_i \in \mathbb{N}$, for all $i \in \{1, ..., k\}$. We will assume that for all $i, j \in \{1, ..., k\}$, $i \neq j$, $L_i \cap L_j = \emptyset$ (that is to say, such languages are pairwise disjoint). The set of atoms of L_i will be denoted by At^{L_i}.

Let us define, for each i, the language $L_i^* = L_i \cup \{s\}$ and its respective set of atoms $At^{L_i^*}$.

Let us take now a collection of sets of sentences $\Gamma_i = \{\phi_1^i, ..., \phi_{m_i}^i\} \subseteq SL_i^*$ and $\theta = s$, with $m_i \in \mathbb{N}$, for all $i \in \{1, ..., k\}$.

Set $L^* = \cup_{i=1}^k L_i^*$ and $\Gamma = \cup_{i=1}^k \Gamma_i \subseteq SL^*$.

Claim 70 *Let $\eta \in [0,1]$. If w is a probability function on L^* such that $w(\Gamma) \geq \eta$ then we have what follows:*

$$w(s) \geq max\{F_{\Gamma_i,s}(\eta) \,|\, 1 \leq i \leq k\}.$$

Proof. Let w be a probability function on L^* such that $w(\Gamma) \geq \eta$.

For each i we can restrict w to a probability function w_i on ${L_i}^*$. We will then have that $w_i(\alpha^i) = w(\alpha^i)$ for all $\alpha^i \in At^{{L_i}^*}$.

As a result we will have that $w_i(\Gamma_i) \geq \eta$ since $w_i(\alpha^i) = w(\alpha^i)$ for all $\alpha^i \in At^{{L_i}^*}$ and also

$$w(s) = w_i(s) \geq F_{\Gamma_i,s}(\eta).$$

We can then conclude that $w(s) \geq max\{F_{\Gamma_i,s}(\eta) \,|\, 1 \leq i \leq k\}$. ■

For the next claim we need to consider our sets of sentences Γ_i to be of the form $\Gamma_{a_ib_i}$, for $i \in \{1, ..., k\}$, with $a_i = \frac{p_{a_i}}{q_{a_i}}$ and $b_i = \frac{p_{b_i}}{q_{b_i}}$.

Claim 71 *Let $\eta \in [0,1]$. We claim that there exists a probability function w on L^* such that $w(\Gamma) \geq \eta$ and*

$$w(s) = max\{F_{\Gamma_{a_ib_i},s}(\eta) \,|\, 1 \leq i \leq k\}.$$

Proof. For each i let w_i be a probability function on L_i^* such that $w_i(\Gamma_{a_ib_i}) \geq \eta$ and $w_i(s) = F_{\Gamma_{a_ib_i},s}(\eta)$.

Let us assume that

$$w_j(s) = max\{w_i(s) \,|\, 1 \leq i \leq k\},$$

for some $j \in \{1, ..., k\}$.

We can define a probability function w on L^* in a way that $w(\alpha^j) = w_j(\alpha^j)$ for all $\alpha \in At^{L_j^*}$ such that $w(\Gamma) \geq \eta$ and $w(s) = F_{\Gamma_{a_jb_j},s}(\eta)$.

If $w_i(s) > 0$ we know that for $\Gamma_{a_ib_i}$ thus defined the only atoms of the form $\alpha^i \wedge s$, with $\alpha^i \in At^{L_i}$, for which $w_i(\alpha^i \wedge s) > 0$ are

$$\alpha^i_{q_{b_i}p_{a_i}+1} \wedge s, ..., \alpha^i_{2q_{b_i}p_{a_i}} \wedge s.$$

Further, by an argument similar to that employed in the proof of Theorem 24, the probability assigned to each such atom by w_i can be assumed to be the same:

$\frac{w_i(s)}{q_{b_i}p_{a_i}}$ for each atom of the form $\alpha^i \wedge s$. Likewise, when $w_i(s) > 0$, we will have that the only atoms of the form $\alpha^i \wedge \neg s$ for which $w_i(\alpha^i \wedge \neg s) > 0$ are

$$\alpha_1^i \wedge \neg s, ..., \alpha^i_{q_{b_i}p_{a_i}} \wedge \neg s.$$

The probability assigned to these atoms by w_i can be assumed to be the same too: $\frac{w_i(\neg s)}{q_{b_i}p_{a_i}}$ for each atom of the form $\alpha^i \wedge \neg s$.[2]

Let us define the following sets for each $i \in \{1, ..., k\}$:

$$A^i = \{\alpha^i_r \in At^{L_i} \,|\, 1 \le r \le q_{b_i}p_{a_i}\},$$

$$B^i = \{\alpha^i_r \in At^{L_i}, \,|\, q_{b_i}p_{a_i} + 1 \le r \le 2q_{b_i}p_{a_i}\}.$$

Consider the cartesian product $B^1 \times ... \times B^k$. Its cardinality is $\Pi_{i=1}^k q_{b_i}p_{a_i}$. For each k-tuple $(\alpha^1, ..., \alpha^k) \in B^1 \times ... \times B^k$ we will set w in a way that

$$w(s \wedge \bigwedge_{i=1}^k \alpha^i) = \frac{w_j(s)}{\Pi_{i=1}^k q_{b_i}p_{a_i}}.$$

Consider now the cartesian product $A^1 \times ... \times A^k$. For each k-tuple $(\alpha^1, ..., \alpha^k) \in A^1 \times ... \times A^k$ we will set w in a way that

$$w(\neg s \wedge \bigwedge_{i=1}^k \alpha^i) = \frac{w_j(\neg s)}{\Pi_{i=1}^k q_{b_i}p_{a_i}}.$$

Notice that for each $i \in \{1, ..., k\}$ and each $r \in \{q_{b_i}p_{a_i} + 1, ..., 2q_{b_i}p_{a_i}\}$ we will have that

$$w(s \wedge \alpha^i_r) = \frac{w_j(s)}{q_{b_i}p_{a_i}} \ge \frac{w_i(s)}{q_{b_i}p_{a_i}}.$$

Thus, since given $\phi^i_j \in \Gamma_{a_i b_i}$ for $j \in \{1, ..., m_i\}$ there is a larger number of atoms of the form $s \wedge \alpha^i$ (with $\alpha^i \in B^i$) logically implying ϕ^i_j than atoms of the form $\neg s \wedge \alpha^i$ (with $\alpha^i \in A^i$) implying ϕ^i_j, the probability function w will be as desired. ■

These two claims amount to saying that given a finite collection of sets of

[2]Notice that, although not necessarily the case when $w_i(s) = 0$ we can assume, without loss of generality, that w_i has this property since we have atoms of the form $\alpha^i \wedge \neg s$ not logically implying any sentences in $\Gamma_{a_i b_i, s}$ that could be assigned probability greater than 0 by w_i –recall that $2q_{b_i}p_{a_i} < 2^{|L_i|}$.

sentences of the form
$$\{\Gamma_{a_ib_i} \subseteq SL_i^* \,|\, 1 \leq i \leq k\}$$
as described above,
$$F_{\bigcup_i \Gamma_{a_ib_i},s}(\eta) = max\{F_{\Gamma_{a_ib_i},s}(\eta) \,|\, 1 \leq i \leq k\},$$
for $\eta \in [0,1]$.

Let $\bigcup \Gamma_{ab} = \bigcup \{\Gamma_{ab} |\, l_{ab} \in \mathcal{L}\}$.

For any finite subset $\Delta \subset \{\Gamma_{ab} \,|\, l_{ab} \in \mathcal{L}\}$ we have that, for any $\eta \in [0,1]$,
$$F_{\bigcup \Delta,s}(\eta) = max\{F_{\Gamma_{ab},s}(\eta) |\, \Gamma_{ab} \in \Delta\}.$$

The set $\{\Gamma_{ab}|\, l_{ab} \in \mathcal{L}\}$ is countable. Let us define the sequence $\{\Gamma^n\}$ by numbering all its elements.

Let $\eta \in [0,\mu]$. We can define a sequence of probability functions $\{w^n\}$ in a way that
$$w^n(\bigcup_{i=1}^{n} \Gamma^i) \geq \eta$$
and
$$w^n(s) = F_{\bigcup_{i=1}^{n} \Gamma^i,s}(\eta) = max\{F_{\Gamma^i,s}(\eta) |\, 1 \leq i \leq n\}$$
for all $n \in \mathbb{N}$.

Notice that $\{w^n(s)\}$ is actually an increasing sequence bounded above by $\mathcal{F}(\eta)$ with limit $sup\{F_{\Gamma^n,s}(\eta) \,|\, n \in \mathbb{N}\}$.

Thus, $F_{\bigcup \Gamma_{ab},s}(\eta) = sup\{F_{\Gamma_{ab},s}(\eta) \,|\, l_{ab} \in \mathcal{L}\}$.

Let us define recursively the sequence $\{\Delta^n\}$ as follows:

1. $\Delta^1 = \Gamma^1$.
2. $\Delta^n = \Delta^{n-1} \cup \{\Gamma^n\}$.

$\{L^n\}$ will be the corresponding sequence of languages (that is to say, L^n will be the language necessary to define all the sets of sentences $\Gamma \in \Delta^n$ in the way shown earlier, with s among its propositions).

Theorem 72 *Let $\eta \in [0,\mu]$. $F_{\bigcup \Gamma_{ab},s}(\eta) = \mathcal{F}(\eta)$.*

Proof. We can define a sequence of probability functions $\{w^n\}$, with w^n a probability function on L^n, such that $w^n(\bigcup \Delta^n) \geq \eta$ and $w^n(s) \leq \mathcal{F}(\eta)$ for all $n \in \mathbb{N}$, with $\{w^n(s)\}$ increasing and with limit $\mathcal{F}(\eta)$. In the limit we will have that $F_{\bigcup \Gamma_{ab},s}(\eta) = \mathcal{F}(\eta)$. ∎

Proposition 73 *Let $\Gamma \subseteq SL^\infty$ be infinite, $\theta \in SL^\infty$ and assume that $\Gamma\,{}^{\eta}{\triangleright}_{\zeta}\,\theta$, for some $\eta, \zeta \in [0,1]$. It is not generally the case that there exists a finite subset $\Delta \subset \Gamma$ such that $\Delta\,{}^{\eta}{\triangleright}_{\zeta}\,\theta$.*

Proof. Let $\mathcal{F}$ be as above and suppose that there exists $\eta \in [0,\mu] \cap \mathbb{Q}$ such that $\mathcal{F}(\eta)$ is irrational. The previous theorem proves that $F_{\bigcup \Gamma_{ab},s}(\eta) = \mathcal{F}(\eta)$ and, therefore, that $\bigcup \Gamma_{ab}\,{}^{\eta}{\triangleright}_{\mathcal{F}(\eta)}\,s$. However, for any finite subset $\Delta \subset \{\Gamma_{ab} \,|\, l_{ab} \in \mathcal{L}\}$, $\Delta\,{}^{\eta}{\not\triangleright}_{\mathcal{F}(\eta)}\,s$ since $F_{\Delta,s}(\eta) \in \mathbb{Q}$ (see Proposition 20) and $F_{\Delta,s}(\eta) < \mathcal{F}(\eta)$. [3] ∎

6.1 A representation theorem for $F_{\Gamma,\theta}$ within L^∞

Let $\Gamma = \{\phi_n \,|\, n \in \mathbb{N}\} \subseteq SL^\infty$ and $\theta \in SL^\infty$.

Let us define the sequence $\{\Delta^n\}$ recursively as follows:

1. $\Delta^1 = \{\phi_1\}$.

2. $\Delta^n = \Delta^{n-1} \cup \{\phi_n\}$.

$\{L^n\}$ will be the corresponding sequence of languages (that is to say, L^n will be a language large enough to define all sentences in Δ^n).

Let $\eta \in [0, mc(\Gamma)]$.[4]

Notice that we can define a sequence $\{w^n\}$, with w^n a probability function on L^n such that $w^n(\Delta^n) \geq \eta$ and $w^n(\theta) = F_{\Delta^n,\theta}(\eta)$. The sequence $\{w^n(\theta)\}$ is increasing and certainly bounded above and we will have that

$$F_{\Gamma,\theta}(\eta) = \lim_{n\to\infty} F_{\Delta^n,\theta}(\eta).$$

In short we will write $F_{\Gamma,\theta} = \lim_{n\to\infty} F_{\Delta^n,\theta}$.

[3] Notice that we could have done with a much simpler construction of Γ to prove this proposition. Here we use the construction of Γ originally intended for the representation theorem of graphs of functions of the form $F_{\Gamma,\theta}$, with $\Gamma \subseteq SL^\infty$ and $\theta \in SL^\infty$.

[4] In [26] Knight gives a nice characterization of η-consistency for infinite sets of sentences: Γ is η-consistent if and only if every finite subset of Γ is η-consistent.

Next we state and prove some properties of $F_{\Gamma,\theta}$.

Proposition 74 *$F_{\Gamma,\theta}(0) \in \{0,1\}$, $F_{\Gamma,\theta}(1) \in \{0,1\}$ and on $(mc(\Gamma),1]$ $F_{\Gamma,\theta}$ has constant value* 1.

Proof. Let us first prove that $F_{\Gamma,\theta}(0) \in \{0,1\}$. Let us proceed by *reductio ad absurdum* and assume that $F_{\Gamma,\theta}(0) = \zeta$, with $0 < \zeta < 1$. Then we could find $n \in \mathbb{N}$ such that $0 < F_{\Delta^n,\theta}(0) < \zeta$, which contradicts the fact that $F_{\Delta^n,\theta}(0) \in \{0,1\}$ (by Proposition 21).

That $F_{\Gamma,\theta}(1) = 1$ is clear if $mc(\Gamma) < 1$. If $mc(\Gamma) = 1$ we can proceed in the same way as above to prove that $F_{\Gamma,\theta}(1) \in \{0,1\}$.

That $F_{\Gamma,\theta}$ has constant value 1 on $(mc(\Gamma),1]$ follows from the fact that

$$mc(\Gamma) = \inf\{mc(\Delta^n) \mid n \in \mathbb{N}\}.$$[5]

■

Proposition 75 *$F_{\Gamma,\theta}$ is increasing.*

Proof. That $F_{\Gamma,\theta}$ is increasing on $(mc(\Gamma),1]$ is clear. Let us proceed by *reductio ad absurdum* to prove that $F_{\Gamma,\theta}$ is also increasing on $[0, mc(\Gamma)]$.

Assume that $0 \le \eta^- < \eta^+ \le mc(\Gamma)$ and that $F_{\Gamma,\theta}(\eta^+) < F_{\Gamma,\theta}(\eta^-)$. Thus we can find $n \in \mathbb{N}$ for which $F_{\Gamma,\theta}(\eta^+) < F_{\Delta^n,\theta}(\eta^-) \le F_{\Gamma,\theta}(\eta^-)$. But $F_{\Delta^n,\theta}$ is increasing (see Proposition 16) and therefore $F_{\Delta^n,\theta}(\eta^-) \le F_{\Delta^n,\theta}(\eta^+)$, yielding $F_{\Gamma,\theta}(\eta^+) < F_{\Delta^n,\theta}(\eta^+)$. Contradiction. ■

Proposition 76 *$F_{\Gamma,\theta}$ is convex on $[0, mc(\Gamma)]$.*

Proof. Let us argue again by *reductio ad absurdum.*

Assume that $0 \le \eta^- < \eta^+ \le mc(\Gamma)$ and that there exists $\lambda \in [0,1]$ for which

$$F_{\Gamma,\theta}(\lambda\eta^- + (1-\lambda)\eta^+) > \lambda F_{\Gamma,\theta}(\eta^-) + (1-\lambda)F_{\Gamma,\theta}(\eta^+).$$

We could find $n \in \mathbb{N}$ for which $F_{\Delta^n,\theta}(\eta^-) \le F_{\Gamma,\theta}(\eta^-)$, $F_{\Delta^n,\theta}(\eta^+) \le F_{\Gamma,\theta}(\eta^+)$ and

$$F_{\Delta^n,\theta}(\lambda\eta^- + (1-\lambda)\eta^+) > \lambda F_{\Gamma,\theta}(\eta^-) + (1-\lambda)F_{\Gamma,\theta}(\eta^+),$$

[5]Although quite straightforward see [26] for a proof if desired.

yielding

$$F_{\Delta^n,\theta}(\lambda\eta^- + (1-\lambda)\eta^+) > \lambda F_{\Delta^n,\theta}(\eta^-) + (1-\lambda)F_{\Delta^n,\theta}(\eta^+).$$

But $F_{\Delta^n,\theta}$ needs to be convex (see Proposition 18). Contradiction. ■

Proposition 77 *$F_{\Gamma,\theta}$ is continuous on* $[0, mc(\Gamma)]$.

Proof. $F_{\Gamma,\theta}$ is increasing and convex on $[0, mc(\Gamma)]$. Thus, to prove continuity on $[0, mc(\Gamma)]$, it suffices to show that $F_{\Gamma,\theta}$ is continuous from the left at $mc(\Gamma)$.[6]

Let us proceed by *reductio ad absurdum* and assume that $F_{\Gamma,\theta}$ is not continuous from the left at $mc(\Gamma)$. Thus, for all $\eta \in [0, mc(\Gamma))$,

$$F_{\Gamma,\theta}(mc(\Gamma)) - F_{\Gamma,\theta}(\eta) > \epsilon$$

for some $\epsilon > 0$.

Let $n \in \mathbb{N}$ be such that

$$F_{\Gamma,\theta}(mc(\Gamma)) - F_{\Delta^n,\theta}(mc(\Gamma)) = \epsilon^- < \epsilon.$$

By continuity of $F_{\Delta^n,\theta}$ (see Proposition 19) we can find $\eta \in [0, mc(\Gamma))$ such that

$$F_{\Delta^n,\theta}(mc(\Gamma)) - F_{\Delta^n,\theta}(\eta) < \epsilon - \epsilon^-,$$

yielding $F_{\Gamma,\theta}(\eta) < F_{\Delta^n,\theta}(\eta)$. Contradiction. ■

Proposition 78 *Assume that $F_{\Gamma,\theta}(0) = 0$ and that $\delta < 1$ is the greatest element in the interval $[0, mc(\Gamma)]$ for which $F_{\Gamma,\theta}(\delta) = 0$. $F_{\Gamma,\theta}$ is bounded below by the line segment joining the points $(\delta, 0)$ and $(1, 1)$.*

Proof. Let us proceed again by *reductio ad absurdum*.

Assume that there exists $\mu > \delta$ such that the pair $(\mu, F_{\Gamma,\theta}(\mu))$ lies below the straight segment joining $(\delta, 0)$ and $(1, 1)$.

Consider the straight line through $(\mu, F_{\Gamma,\theta}(\mu))$ and $(1, 1)$, say $ux + v$, and suppose that $u\lambda + v = 0$.

Let us define the sequence $\{\delta^n\}$, for

$$\delta^n = max\{\eta \in [0, 1] \mid F_{\Delta^n,\theta}(\eta) = 0\}.$$

[6] See [44].

Notice that $\{\delta^n\}$ converges to δ. Thus there has to exist n for which $\delta^n < \lambda$. Therefore $F_{\Delta^n,\theta}$ is bounded below by the straight segment joining $(\delta^n, 0)$ and $(1, 1)$ (see Proposition 22), yielding $F_{\Delta^n,\theta}(\mu) > F_{\Gamma,\theta}(\mu)$. Contradiction. ■

Theorem 79 *A function $\mathcal{F} : [0, 1] \longrightarrow [0, 1]$ is of the form $F_{\Gamma,\theta}$ for some $\Gamma \subseteq SL^\infty$ and $\theta \in SL^\infty$ if and only if it has the following properties, for $\mu \in [0, 1]$:*

1. *$\mathcal{F}(0) \in \{0, 1\}$, $\mathcal{F}(1) \in \{0, 1\}$ and, on the interval $(\mu, 1]$, it has constant value 1.*
2. *$\mathcal{F}$ is increasing.*
3. *For $\mu > 0$, $\mathcal{F}$ is continuous and convex on $[0, \mu]$.*
4. *If $\mathcal{F}(0) = 0$ and the greatest element in the interval $[0, \mu]$ at which $\mathcal{F}$ has value 0 is $\delta < 1$ then it is bounded below by the line segment joining the points $(\delta, 0)$ and $(1, 1)$.*

Proof. The right implication follows mostly from the discussion in the previous section culminating in Theorem 72. The only cases not covered in the initial discussion were $\mathcal{F}(x) = 0$ for all $x \in [0, 1]$ and $\mathcal{F}(x) = 1$ for all $x \in [0, 1]$. For the former we can set θ to be a contradiction and Γ any consistent set of sentences and for the latter we can set θ to be a tautology and Γ any set of sentences.

The right implication follows from the above propositions (Propositions 74, 75, 76, 77 and 78) and the left implication follows from the discussion in the previous section culminating in Theorem 72. ■

Chapter 7

Fuzzy logics

In this chapter we develop some of the ideas presented in previous sections within the frame of fuzzy logics, particularly within the frame of Gödel and Łukasiewicz logics. The weight functions to be considered in this section will be *truth valuations* and thus we will be dealing with *degrees of truth* instead of degrees of belief.

We start by introducing the logic BL (short for *Basic Logic*).

7.1 Basic Logic, BL

Let $L = \{p_1, ..., p_l\}$ be a finite propositional language.
We define SL recursively as follows:

1. $\bar{0} \in SL$.

2. Let $p \in L$. Then $p \in SL$.

3. Let θ, $\phi \in SL$. Then $\neg\phi$, $\phi \to \theta$, $\phi \& \theta$, $\phi \wedge \theta$, $\phi \vee \theta \in SL$.

$\theta \wedge \phi$ is short for $\theta \& (\theta \to \phi)$.
$\theta \vee \phi$ abbreviates $((\theta \to \phi) \to \phi) \wedge ((\phi \to \theta) \to \theta)$.
$\neg\phi$ abbreviates $\phi \to \bar{0}$.[1].

Thus we only have two primitive connectives and a constant from which all the other connectives can be defined ($\&$, $\to$, $\bar{0}$). For convenience though we will deal with all of them in most results.

[1]See [21] for a fuller account of this language.

Let us first define the basic fuzzy logic, BL, as in [21].

Definition 80 *BL is given by the following axiom schemas:*

1. $(\phi \to \psi) \to ((\psi \to \xi) \to (\phi \to \xi))$
2. $(\phi \wedge \psi) \to \phi$
3. $(\phi \wedge \psi) \to (\psi \wedge \phi)$
4. $(\phi \wedge (\phi \to \psi)) \to (\psi \wedge (\psi \to \phi))$
5. $(\phi \to (\psi \to \xi)) \to ((\phi \wedge \psi) \to \xi)$
6. $((\phi \wedge \psi) \to \xi) \to (\phi \to (\psi \to \xi))$
7. $((\phi \to \psi) \to \xi) \to (((\phi \to \psi) \to \xi) \to \xi)$
8. $\bar{0} \to \phi$

The deduction rule is Modus Ponens.

7.2 Gödel logic, G

Let us now define the logic G (Gödel logic).

Definition 81 *G is an extension of the axiom system BL by the axiom schema*

$$\phi \to (\phi \wedge \phi).$$

The deduction rule is, as in BL, Modus Ponens.

The notion of *proof* in G is defined in the obvious way.

G proves the equivalence $(\phi \& \psi) \equiv (\phi \wedge \psi)$.[2] Thus we can dispense with one of these connectives.

Definition 82 *Let $w : SL \longrightarrow [0,1]$. We say that w is a G-valuation on L if, for $\theta, \phi \in SL$, we have what follows:*

1. $w(\phi \& \theta) = min\{w(\phi), w(\theta)\}$

[2] See [21] for a proof of this result.

2. $w(\phi \rightarrow \theta) = \begin{cases} 1 & if \;\; w(\phi) \leq w(\theta) \\ w(\theta) & otherwise \end{cases}$

3. $w(\bar{0}) = 0$

From these three clauses we can deduce the behaviour of G-valuations with respect to the other connectives:

1. $w(\phi \wedge \theta) = w(\phi \& \theta) = min\{w(\phi), w(\theta)\}$
2. $w(\phi \vee \theta) = max\{w(\phi), w(\theta)\}$
3. $w(\neg\phi) = \begin{cases} 1 & if \;\; w(\phi) = 0 \\ 0 & otherwise \end{cases}$

We will call those sentences in SL that are given value 1 by all G-valuations G-tautologies whereas G-contradictions will be those sentences that are given value 0 for all G-valuations.

We may wonder if we could derive some sort of consistency measure from G-valuations in the same way as Knight did from probability functions in [25] and [26] (see Chapter 2).

Throughout let $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$ and $\theta \in SL$.

Definition 83 *Let $\eta \in [0, 1]$. We say that Γ is G_η-consistent if and only if there exists a G-valuation w such that $w(\Gamma) \geq \eta$.*

Definition 84 *Let $\eta \in [0, 1]$. We say that Γ is maximally G_η-consistent if and only if Γ is G_η-consistent and there is no $\lambda > \eta$ such that Γ is G_λ-consistent.*

Notice that saying that Γ is G_η-consistent is equivalent to claiming that there exists a G-valuation w such that $w(\bigwedge \Gamma) \geq \eta$ ($\bigwedge \Gamma$ is short for $\phi_1 \wedge ... \wedge \phi_k$).

Here and throughout we will denote the set of propositional variables that occur in at least one sentence of Γ by L_Γ and the propositional variables that occur in a sentence θ by L_θ.

Let us define recursively the following infinite collection of sets of sentences in SL:

- $C_1 = L \cup \{\bar{0}\}$.

- $C_n = C_{n-1} \cup \{\phi \to \psi, \phi \& \psi |\, \phi, \psi \in C_{n-1}\}$, $n \in \mathbb{N}$.

We will denote this collection by $\mathcal{C}$ (that is, $\mathcal{C} = \{C_n \,|\, n \in \mathbb{N}\}$)

Proposition 85 *Let w be a G-valuation such that $w(\theta) > 0$. Let us define the G-valuation w^* in the following way:*

For $p \in L$,

$$w^*(p) = \begin{cases} 1 & if\;\; w(p) > 0 \\ 0 & otherwise \end{cases}$$

We claim that w^* is such that $w^*(\theta) = 1$.

Proof. Let us proceed by induction on $\mathcal{C}$.

Let us first assume that $\theta \in C_1$ is such that there exists a G-valuation w for which $w(\theta) > 0$ and define w^* as above. Notice that θ has to be a propositional variable since $w(\bar{0}) = 0$. Thus clearly $w^*(\theta) = 1$.

Assume now that the result is true for all sentences in C_{n-1}. That is to say, if $\phi \in C_{n-1}$ is such that there exists a G-valuation w for which $w(\phi) > 0$ then w^* defined from w as above will be such that $w^*(\phi) = 1$.

Let $\theta \in C_n - C_{n-1}$ for some $n \in \mathbb{N}$. Let w be such that $w(\theta) > 0$ and define w^* as above from w.

Let us suppose first that $\theta = \phi \to \psi$ for some $\phi, \psi \in C_{n-1}$. Since by assumption $w(\theta) > 0$ we will have that either $w(\phi) = w(\psi) = 0$ or $w(\psi) > 0$. If $w(\phi) = w(\psi) = 0$ then $w^*(\phi) = w^*(\psi) = 0$ and $w^*(\theta) = 1$. If $w(\psi) > 0$ then, by inductive hypothesis, we will have that $w^*(\psi) = 1$ and, therefore, that $w^*(\theta) = 1$.

Suppose now that $\theta = \phi \& \psi$ for some $\phi, \psi \in C_{n-1}$. We know that $w(\phi) > 0$ and that $w(\psi) > 0$. By inductive hypothesis $w^*(\phi) = 1$ and $w^*(\psi) = 1$. Thus $w^*(\theta) = 1$. ■

Corollary 86 *Let Γ be G_η-consistent, for $\eta \in (0, 1]$. Then Γ is maximally G_1-consistent.*

Proposition 87 *Let $\eta \in (0, 1]$. Γ is G_η-consistent if and only if Γ is classically consistent.*

Proof. Any classical valuation is also a G-valuation and thus the left implication holds. To prove the right implication assume that Γ is G_η-consistent, $\eta \in (0, 1]$.

By Proposition 85 we can define a new G-valuation w^* (which is also a classical valuation) from any w being such that $w(\Gamma) \geq \eta$. Thus it follows that Γ is classically consistent. ■

Corollary 88 *Γ is maximally G_0-consistent if and only if Γ is classically inconsistent.*

Thus we see that G-valuations are not *good* candidates for measuring *degrees* of consistency. They behave classically and turn out to be useless when it comes to determining any presumable difference among distinct inconsistent sets.

Proposition 89 *Let w be a G-valuation such that $w(\theta) = \eta$ for some $\eta \in (0, 1)$ Then there exists a propositional variable $p \in L_\theta$ such that $w(p) = \eta$.*

Proof. The result can be proved recursively in a pretty trivial way. Let w be as stated above.

If θ is a propositional variable, say p, then $w(p) = \eta$.

If $\theta = \phi \& \psi$ for some $\phi, \psi \in SL$ then $w(\phi) = \eta$ or $w(\psi) = \eta$.

If $\theta = \phi \rightarrow \psi$ for some $\phi, \psi \in SL$ then $w(\psi) = \eta$.

Notice that θ can not be $\bar{0}$ since $w(\bar{0}) = 0$ for all G-valuations w. ■

Corollary 90 *Let w be a G-valuation and $\mathcal{A} = \{w(p) \,|\, p \in L_\theta\} \cup \{0, 1\}$. We have that $w(\theta) \in \mathcal{A}$.*

Proposition 91 *Let w be a G-valuation such that $0 < w(\theta) < 1$ and $\lambda \in (0, 1]$. We claim that there exists a G-valuation w^* such that $w^*(\theta) = \lambda$.*

Proof. Let w be a G-valuation such that $w(\theta) = \eta$ for some $\eta \in (0, 1)$.

That there exists a G-valuation w^* such that $w^*(\theta) = 1$ was proved in Proposition 85. Let us take $\lambda \in (0, 1)$ and prove that we can define a G-valuation w^* such that $w^*(\theta) = \lambda$.

Let $h : [0, 1] \longrightarrow [0, 1]$. The map h has the following properties:

1. $h(0) = 0$, $h(1) = 1$.
2. h is strictly increasing.
3. $h(\eta) = \lambda$.

Let w^* be such that $w^*(p) = h(w(p))$ for all $p \in L$.

We claim that h will be such that $w^*(\phi) = h(w(\phi))$ for all $\phi \in SL$ and thus $w^*(\theta) = \lambda$. To see this we can proceed by induction on $\mathcal{C}$.

Let $\phi \in SL$.

Assume first that $\phi \in C_1$. If $\phi = p$ for some $p \in L$ then, by definition, $w^*(p) = h(w(p))$. On the other hand, if $\phi = \bar{0}$ then clearly $w^*(\bar{0}) = h(w(\bar{0})) = 0$.

Assume now that $\phi \in C_n - C_{n-1}$ for some $n \in \mathbb{N}$ and that $w^*(\psi) = h(w(\psi))$ for all $\psi \in C_{n-1}$.

Let $\phi = \psi \to \xi$ for some $\psi, \xi \in C_{n-1}$. If $w^*(\phi) = 1$ then $w^*(\psi) \leq w^*(\xi)$. Thus, $h(w(\psi)) \leq h(w(\xi))$ and, since h is strictly increasing, $w(\psi) \leq w(\xi)$. Hence, $w(\phi) = h(w(\phi)) = 1$. If $w^*(\phi) < 1$ then $w^*(\psi) > w^*(\xi) = w^*(\phi)$. Thus, $h(w(\psi)) > h(w(\xi))$ and, since h is strictly increasing, $w(\psi) > w(\xi) = w(\phi)$. But $w^*(\xi) = h(w(\xi))$. Therefore, $w^*(\phi) = h(w(\phi))$.

Let $\phi = \psi \& \xi$ for some $\psi, \xi \in C_{n-1}$. Assume that $w^*(\phi) = w^*(\psi)$ -that is, $w^*(\psi) \leq w^*(\xi)$. Thus, $h(w(\psi)) \leq h(w(\xi))$. Since h is strictly increasing we can conclude that $w(\psi) \leq w(\xi)$ and therefore $w(\phi) = w(\psi)$. But $w^*(\psi) = h(w(\psi))$. Thus, $w^*(\phi) = h(w(\phi))$. ■

We define now an inference relation based on G-valuations similar to ${}^{\eta}\triangleright_{\zeta}$.

Let $\eta, \zeta \in [0, 1]$.

Definition 92 *We say that* Γ (η, ζ)*-implies* θ *(denoted* $\Gamma^{\eta} \trianglerighteq_{\zeta} \theta$*) if and only if, for all* G*-valuations* w*, if* $w(\Gamma) \geq \eta$ *then* $w(\theta) \geq \zeta$*.*

Next we introduce the equivalent to $F_{\Gamma,\theta}$ for ${}^{\eta} \trianglerighteq_{\zeta}$, which we denote by $\mathcal{G}_{\Gamma,\theta}$.

Definition 93 *The function* $\mathcal{G}_{\Gamma,\theta} : [0, 1] \longrightarrow [0, 1]$ *is defined as follows:*

$$\mathcal{G}_{\Gamma,\theta}(\eta) = sup\{\zeta \mid \Gamma^{\eta} \trianglerighteq_{\zeta} \theta\}.$$

Let us consider first the following example. Take $\Gamma = \{p\}$ and $\theta = p \vee \neg p$. Notice that $\mathcal{G}_{\Gamma,\theta}(0) = 0$ but there is no G-valuation w such that $w(\theta) = 0$. To see this take $\epsilon > 0$. We can always find w, a G-valuation, such that $0 < w(\theta) < \epsilon$ (set for example $w(p) = \frac{\epsilon}{2}$). This tells us that the supremum, the value given by the function $\mathcal{G}_{\Gamma,\theta}$ at some point in the interval $[0, 1]$, is not always attained by a G-valuation.

Proposition 94 *$\mathcal{G}_{\Gamma,\theta}$ is increasing.*

Proof. It follows trivially from the definition of ${}^{\eta} \trianglerighteq_{\zeta}$. ∎

Proposition 95 $\mathcal{G}_{\Gamma,\theta}(1) \in \{0, 1\}$.

Proof. Let us proceed by *reductio ad absurdum* and assume that $0 < \mathcal{G}_{\Gamma,\theta}(1) = \mu < 1$. Thus there has to exist a G-valuation w such that $w(\bigwedge \Gamma) = 1$ and $1 > w(\theta) \geq \mu$. We can define a new G-valuation w^* the way we did in Proposition 91 such that $w^*(\bigwedge \Gamma) = 1$ and $w^*(\theta) = \lambda < \mu$. Notice that we can define this valuation taking any value of λ in the interval $(0, \mu)$. Hence $\mathcal{G}_{\Gamma,\theta}(1) = 0$, which contradicts the assumption we started with. ∎

Proposition 96 $\mathcal{G}_{\Gamma,\theta}(0) \in \{0, 1\}$.

Proof. Let us proceed again by *reductio ad absurdum* by assuming that $0 < \mathcal{G}_{\Gamma,\theta}(0) = \mu < 1$. This means that there exists a G-valuation w such that $1 > w(\theta) \geq \mu$. We can define a new G-valuation w^* as in Proposition 91 such that $w^*(\theta) = \lambda < \mu$ for any $\lambda \in (0, \mu)$. Hence we have that $\mathcal{G}_{\Gamma,\theta}(0) = 0$, which contradicts the assumption above. ∎

Proposition 97 *Let $\eta \in [0, 1]$. Then $\mathcal{G}_{\Gamma,\theta}(\eta) \in \{0, \eta, 1\}$.*

Proof. By Proposition 96 we know that $\mathcal{G}_{\Gamma,\theta}(0) \in \{0, 1\}$ and by Proposition 95 that $\mathcal{G}_{\Gamma,\theta}(1) \in \{0, 1\}$.

Let $\eta \in (0, 1)$.

Let us proceed by *reductio ad absurdum* and assume first that $\eta < \mathcal{G}_{\Gamma,\theta}(\eta) = \zeta < 1$. Since $\mathcal{G}_{\Gamma,\theta}$ is increasing, $\mathcal{G}_{\Gamma,\theta}(1) = 1$ and $\mathcal{G}_{\Gamma,\theta}(\eta) < 1$ there has to exist a G-valuation w such that $w(\bigwedge \Gamma) = \eta + u$ and $w(\theta) = \zeta + v$, with $u \in [0, 1 - \eta)$ and $v \in [0, 1 - \zeta)$. If $\zeta + v < \eta + u$ then we can define a new G-valuation w^* as seen in Proposition 91 such that $w^*(\bigwedge \Gamma) = \eta$ and $w^*(\theta) < \eta$. If $\eta + u < \zeta + v$ we can define a new G-valuation w^* the way we did in Proposition 91 such that $w^*(\bigwedge \Gamma) = \eta$ and $\eta < w^*(\theta) = \lambda$ for any $\lambda \in (\eta, \zeta]$. If $\eta + u = \zeta + v$ we can define w^* in a way that $w^*(\bigwedge \Gamma) = w^*(\theta) = \eta$. Thus, we can conclude that $\mathcal{G}_{\Gamma,\theta}(\eta) \leq \eta$.

Let us suppose now that $0 < \mathcal{G}_{\Gamma,\theta}(\eta) = \zeta < \eta$. By what was said above, there exists a G-valuation w such that $w(\bigwedge \Gamma) = \eta + u$ and $w(\theta) = \zeta + v$, with $u \in [0, 1 - \eta)$ and $v \in [0, \eta - \zeta)$. We can define a G-valuation w^* as in Proposition 91 such that $w^*(\bigwedge \Gamma) = \eta$ and $w^*(\theta) = \lambda$ for each $\lambda \in (0, \zeta]$. Thus, $\mathcal{G}_{\Gamma,\theta}(\eta) = 0$. This contradicts the assumption above.

Therefore we can conclude that $\mathcal{G}_{\Gamma,\theta}(\eta) \in \{0, \eta, 1\}$. ∎

Proposition 98 *$\mathcal{G}_{\Gamma,\theta}$ is continuous on* $(0,1]$.

Proof. Let us first assume that $\mathcal{G}_{\Gamma,\theta}$ is not continuous from the right at $\eta \in (0,1)$. This means that, for some $\epsilon > 0$, $\mathcal{G}_{\Gamma,\theta}(x) - \mathcal{G}_{\Gamma,\theta}(\eta) > \epsilon$ for all $x \in (\eta, 1]$. In view of the previous proposition there are only a few cases in which this could happen:

1. $\mathcal{G}_{\Gamma,\theta}(\eta) = \eta$ and $\mathcal{G}_{\Gamma,\theta}(x) = 1$ for all $x \in (\eta, 1]$.

   In this case there has to exist a G-valuation w such that $w(\bigwedge \Gamma) = \eta$ and $\eta \leq w(\theta) < 1$. We can then define a G-valuation w^* as in Proposition 91 such that $\eta < w^*(\bigwedge \Gamma) < 1$ and $\eta < w^*(\theta) < 1$. That contradicts the assumption above.

2. $\mathcal{G}_{\Gamma,\theta}(\eta) = 0$ and $\mathcal{G}_{\Gamma,\theta}(x) = x$ or $\mathcal{G}_{\Gamma,\theta}(x) = 1$ for all $x \in (\eta, 1]$.

   Notice that $\bigwedge \Gamma$ cannot be a G-contradiction. Let w be such that $w(\bigwedge \Gamma) \geq \eta$ and $w(\theta) < \eta$. Thus we could define a new G-valuation w^* from w as in Proposition 91 such that $w^*(\theta) < \eta < w^*(\bigwedge \Gamma)$. This would contradict what we assumed at the beginning.

Let us assume now that $\mathcal{G}_{\Gamma,\theta}$ is not continuous from the left at $\eta \in (0,1]$. That is, for some $\epsilon > 0$, $\mathcal{G}_{\Gamma,\theta}(\eta) - \mathcal{G}_{\Gamma,\theta}(x) > \epsilon$ for all $x \in [0, \eta)$.

For $\eta \in (0,1)$ we can proceed as we did above:

1. $\mathcal{G}_{\Gamma,\theta}(x) = 0$ for all $x \in [0, \eta)$ and $\mathcal{G}_{\Gamma,\theta}(\eta) = \eta$ or $\mathcal{G}_{\Gamma,\theta}(\eta) = 1$.

   There has to exist a G-valuation w such that $0 \leq w(\theta) < w(\bigwedge \Gamma) < \eta$. We can define a new G-valuation w^* such that $w^*(\theta) < \eta < w^*(\bigwedge \Gamma)$, which contradicts the assumption above.

2. $\mathcal{G}_{\Gamma,\theta}(x) = x$ for all $x \in [0, \eta)$ and $\mathcal{G}_{\Gamma,\theta}(\eta) = 1$.

   In this case there has to exist a G-valuation w such that $0 < w(\bigwedge \Gamma) < \eta$ and $w(\bigwedge \Gamma) \leq w(\theta) < 1$. We could define a new G-valuation w^* as we did above which would contradict what we assumed.

For $\eta = 1$:

1. $\mathcal{G}_{\Gamma,\theta}(x) = 0$ for all $x \in [0, 1)$ and $\mathcal{G}_{\Gamma,\theta}(1) = 1$.

   Let us first suppose that $0 < w(\bigwedge \Gamma) < 1$ for some G-valuation w. In this case, since by assumption $\mathcal{G}_{\Gamma,\theta}(1) = 1$, there has to exist w' for which

$0 < w'(\bigwedge \Gamma) < 1$ and $w'(\theta) = 0$. We can define a new G-valuation w^* as in Proposition 85 such that $w^*(\bigwedge \Gamma) = 1$ and $w^*(\theta) = 0$, which contradicts the assumption we started with.

Let us assume now that $w(\bigwedge \Gamma) \in \{0, 1\}$ for all G-valuations w. Notice that if $\bigwedge \Gamma$ were a G-contradiction then $\mathcal{G}_{\Gamma,\theta}(x) = 1$ for all $x \in (0, 1]$. If $\bigwedge \Gamma$ were a G-tautology then $\mathcal{G}_{\Gamma,\theta}(x) = 1$ or $\mathcal{G}_{\Gamma,\theta}(x) = 0$ for all $x \in [0, 1]$. Let us assume now that $\bigwedge \Gamma$ is neither a G-tautology nor a G-contradiction. In this case, assuming that $\mathcal{G}_{\Gamma,\theta}(1) = 1$, for all $x \in (0, 1]$ we would have that $\mathcal{G}_{\Gamma,\theta}(x) = 1$ since, by assumption, $w(\bigwedge \Gamma) \in \{0, 1\}$.

We have proved continuity of $\mathcal{G}_{\Gamma,\theta}$ on $(0, 1]$. ■

The next proposition gives us a representation for the functions of the form $\mathcal{G}_{\Gamma,\theta}$.

Proposition 99 $\mathcal{G}_{\Gamma,\theta}$ *is of one of the following forms:*

1. $\mathcal{G}_{\Gamma,\theta}(\eta) = 0$ *for all* $\eta \in [0, 1]$
2. $\mathcal{G}_{\Gamma,\theta}(\eta) = \eta$ *for all* $\eta \in [0, 1]$
3. $\mathcal{G}_{\Gamma,\theta}(\eta) = 1$ *for all* $\eta \in [0, 1]$
4. $\mathcal{G}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & if \ \ \eta = 0 \\ 1 & otherwise \end{cases}$

Proof. It follows directly from the previous propositions (94, 95, 96, 97 and 98). ■

That the forms above are possible graphs for some Γ and θ can easily be showed.

Let $p \in L$.

1. Let $\Gamma = \{p\}$ and $\theta = \neg(p \to p)$. In this case $\mathcal{G}_{\Gamma,\theta}(\eta) = 0$ for all $\eta \in [0, 1]$.
2. Let $\Gamma = \{p\}$ and $\theta = p$. In this case $\mathcal{G}_{\Gamma,\theta}(\eta) = \eta$ for all $\eta \in [0, 1]$.
3. Let $\Gamma = \{p\}$ and $\theta = p \to p$. Here $\mathcal{G}_{\Gamma,\theta}(\eta) = 1$ for all $\eta \in [0, 1]$.
4. Let $\Gamma = \{\neg(p \to p)\}$ and $\theta = p$. For this example $\mathcal{G}_{\Gamma,\theta}(0) = 0$ and $\mathcal{G}_{\Gamma,\theta}(\eta) = 1$ for all $\eta \in (0, 1]$.

7.3 Łukasiewicz logic, Ł

Let us define Ł (short for Łukasiewicz logic).

Definition 100 *Ł is obtained by extending BL by the axiom schema*

$$\neg\neg\phi \rightarrow \phi.$$

The deduction rule is Modus Ponens.

In Ł the only primitive connective is $\rightarrow$. The other connectives can be defined from $\rightarrow$ and $\bar{0}$. In particular, $\neg\theta$ stands for $\theta \rightarrow \bar{0}$ and $\theta\&\phi$ for $\neg(\theta \rightarrow \neg\phi)$.

We can introduce a new connective, $\underline{\vee}$, defined from the previous ones. $\theta\underline{\vee}\phi$ will be short for $\neg\theta \rightarrow \phi$.

Definition 101 *Let $w : SL \longrightarrow [0,1]$. We say that w is an Ł-valuation if, for ϕ, $\theta \in SL$, we have what follows:*

1. $w(\phi \rightarrow \theta) = min\{1, 1 - w(\phi) + w(\theta)\}$
2. $w(\bar{0}) = 0$

From these two clauses we can define the behaviour of Ł-valuations for the other connectives:

1. $w(\phi \wedge \theta) = min\{w(\phi), w(\theta)\}$
2. $w(\phi \vee \theta) = max\{w(\phi), w(\theta)\}$
3. $w(\neg\phi) = 1 - w(\phi)$
4. $w(\phi\&\theta) = max\{0, w(\phi) + w(\theta) - 1\}$
5. $w(\phi\underline{\vee}\theta) = min\{1, w(\phi) + w(\theta)\}$

Next we state a central theorem in Łukasiewicz logic. In order to do so we need to specify some previous notation.

Let $\theta \in SL$. Assume that $L_\theta = \{p_1, ..., p_n\} \subseteq L$. We will denote this by $\theta(p_1, ..., p_n)$.

Let w be an Ł-valuation.

We have that $w(\theta) = f(w(p_1), ..., w(p_n))$ for some $f : [0,1]^n \rightarrow [0,1]$. We will denote this f by f_θ –we will write sometimes $f_\theta(x_1, ..., x_n)$.

Theorem 102 ***McNaughton's Theorem***

In order that a function $f : [0,1]^n \to [0,1]$ be of the form f_θ for some $\theta \in SL$ it is necessary and sufficient that f satisfy the following two conditions:

1. *f is continuous on $[0,1]^n$.*

2. *There are a finite number of distinct polynomials with integer coefficients λ_i, $1 \leq i \leq \mu$, $\lambda_i = b_i + m_{1_i}x_1 + ... + m_{n_i}x_n$, such that for every $(x_1, ..., x_n)$, $0 \leq x_i \leq 1$ for all $i \in \{1, ..., n\}$, there is λ_j for some $j \in \{1, ..., \mu\}$ such that $f(x_1, ..., x_n) = \lambda_j(x_1, ..., x_n)$.*

For an alternative presentation and a proof of this theorem see [30].

Throughout let $\Gamma \subseteq SL$.

Definition 103 *Let $\eta \in [0,1]$. We say that Γ is $Ł_\eta$-consistent if and only if there exists an Ł-valuation w such that $w(\Gamma) \geq \eta$.*

Definition 104 *Let $\eta \in [0,1]$. We say that Γ is maximally $Ł_\eta$-consistent if and only if Γ is $Ł_\eta$-consistent and there is no $\lambda > \eta$ such that Γ is $Ł_\lambda$-consistent.*

Let us consider now the examples we started with in Chapter 1 and check how this notion of Ł-consistency behaves.

We formulated Kyburg's lottery paradox through the set of sentences

$$\Gamma_n = \{\neg t_1, ..., \neg t_n, t_1 \vee ... \vee t_n\}.$$

Recall from Chapter 2 that Γ_n was maximally $\frac{n}{n+1}$-consistent (and $\frac{n}{n+1}$-coherent). It turns out that Γ_n is maximally $Ł_{\frac{n}{n+1}}$-consistent too (in terms of Ł the set Γ_n would be given by $\{\neg t_1, ..., \neg t_n, t_1 \underline{\vee} ... \underline{\vee} t_n\}$. That is to say, we would take the disjunction of the paradox to be given by the connective $\underline{\vee}$). To see that this is indeed the case consider an Ł-valuation that assigns $\frac{1}{n+1}$ to each propositional variable involved in Γ_n. It is easy to see that Γ_n is $Ł_{\frac{n}{n+1}}$-consistent (in fact maximally $Ł_{\frac{n}{n+1}}$-consistent).

The Sorites paradox was formulated by means of the set of sentences

$$\Gamma_n = \{p_n, p_n \to p_{n-1}, ..., p_2 \to p_1, \neg p_1\}.$$

Γ_n is maximally Ł$_{\frac{n}{n+1}}$-consistent. To see this consider an Ł-valuation w such that $w(p_i) = \frac{i}{n+1}$ for all $i \in \{1, ..., n\}$. In can easily be seen that w gives Γ_n its maximal Ł-consistency.

Thus the notion of Ł-consistency seems to go well with the idea that the bigger the set of sentences the *less* inconsistent –at least in some examples (as we justified in Chapter 2), very much like the notions of η-consistency and η-coherence do. However such an approach to measuring inconsistency has a serious drawback, as the following example shows.

Let us consider the following set of sentences:

$$\Gamma = \{p \to q, \neg p \to q, p \to \neg q, \neg p \to \neg q\}$$

for $p, q \in L$. Γ is Ł$_1$-consistent (take an Ł-valuation w such that $w(p) = w(q) = \frac{1}{2}$) but classically inconsistent.

Furthermore, since not all tautologies (or contradictions) are Ł-tautologies (or Ł-contradictions respectively) –for example, the sentence $p \to (p\&p)$ is not an Ł-tautology– we have that a set of sentences Γ which contains an explicit contradiction does not have to be necessarily maximally Ł$_0$-consistent.

The converse holds though.

Proposition 105 *If Γ is consistent then Γ is maximally Ł$_1$-consistent.*[3]

Proof. Since classical valuations are also Ł-valuations and the connectives in Ł behave classically under such valuations it is clear that if Γ is classically consistent then Γ is maximally Ł$_1$-consistent. ■

Let $\Delta = \{\phi_1, ..., \phi_n\} \subseteq SL$ be finite.

We will use the following abbreviations:

$\bigwedge \Delta$ for $\phi_1 \wedge ... \wedge \phi_n$.

$\bigvee \Delta$ for $\phi_1 \vee ... \vee \phi_n$.

$\underline{\bigvee}\Delta$ for $\phi_1 \underline{\vee} ... \underline{\vee} \phi_n$.

$\&\Delta$ for $\phi_1 \& ... \& \phi_n$.

[3]From a classical point of view the connectives $\wedge$ and $\&$ are equivalent and so are $\vee$ and $\underline{\vee}$.

In some cases we will indicate with superscripts the number of sentences involved for any of the connectives above. For example $\bigvee^n \theta$ stands for $\theta \vee ... \vee \theta$, where θ occurs n times.

It is customary to refer to $\&^n\theta$ (that is, $\theta\&...\&\theta$, where θ occurs n times) by θ^n.

Notice that the Ł-consistency of a set of sentences Γ is the same as the Ł-consistency of the sentence $\bigwedge \Gamma$. We will talk indistinctively about the consistency of sentences and sets of sentences.

Proposition 106 *For all $n \in \mathbb{N}$ we can construct a sentence $\phi \in SL$ (which we will denote by $\phi_{\frac{1}{n}}$) that is maximally $Ł_{\frac{1}{n}}$-consistent.*

Proof. Let us define $\phi_{\frac{1}{n}}$ as follows:

$$\phi_{\frac{1}{n}} = \neg p \wedge p^{n-1}$$

Let us check that $\phi_{\frac{1}{n}}$ is maximally $Ł_{\frac{1}{n}}$-consistent.

Let w be an Ł-valuation such that $w(p) = \frac{n-1}{n}$. This way $w(\neg p) = \frac{1}{n}$ and $w(p^{n-1}) = (n-1)(\frac{n-1}{n}) - (n-2) = \frac{1}{n}$. Thus, $w(\phi_{\frac{1}{n}}) = \frac{1}{n}$.

Clearly, any Ł-valuation w for which $w(p) < \frac{n-1}{n}$ or $w(p) > \frac{n-1}{n}$ is such that $w(\phi_{\frac{1}{n}}) < \frac{1}{n}$. ∎

Proposition 107 *Let $r \in \mathbb{Q} \cap [0,1]$. We can construct a sentence $\phi \in SL$ (which we will denote by ϕ_r) that is maximally $Ł_r$-consistent.*

Proof. Let $r = \frac{u}{v}$ and $p \in L$. Let us define ϕ_r as follows:

$$\phi_r = \underline{\bigvee}^u \phi_{\frac{1}{v}}$$

$\phi_{\frac{1}{v}} = \neg p \wedge p^{v-1}$. By Proposition 106 $\phi_{\frac{1}{v}}$ is maximally $Ł_{\frac{1}{v}}$-consistent. Thus, $\underline{\bigvee}^u \phi_{\frac{1}{v}}$ is maximally $Ł_{\frac{u}{v}}$-consistent. ∎

Although obvious, it is worth mentioning that there exists an Ł-valuation w for which $w(\phi_r) = 0$. Thus, by continuity of f_{ϕ_r} we will have an Ł-valuation w such that $w(\phi_r) = \lambda$ for each $\lambda \in [0, r]$.

7.3.1 ${}^{\eta}\blacktriangleright_{\zeta}$ and the function $\mathcal{L}_{\Gamma,\theta}$

Let $\Gamma = \{\phi_1, ..., \phi_k\} \subseteq SL$, $\theta \in SL$ and $\eta, \zeta \in [0, 1]$.

As with Gödel logic we can define an inference relation based on degrees of truth.

Definition 108 *We say that Γ (η, ζ)-implies θ (denoted $\Gamma^{\eta} \blacktriangleright_{\zeta} \theta$) if and only if, for all Ł-valuations w, if $w(\Gamma) \geq \eta$ then $w(\theta) \geq \zeta$.*[4]

Definition 109 *The function $\mathcal{L}_{\Gamma,\theta} : [0, 1] \longrightarrow [0, 1]$ is defined as follows, for all $\eta \in [0, 1]$:*

$$\mathcal{L}_{\Gamma,\theta}(\eta) = sup\{\zeta \mid \Gamma^{\eta} \blacktriangleright_{\zeta} \theta\}.$$

Next we prove that this supremum is actually attained by a certain Ł-valuation.

Proposition 110 *Let Γ be Ł$_{\eta}$-consistent. There exists an Ł-valuation w such that $w(\bigwedge \Gamma) \geq \eta$ and $w(\theta) = \mathcal{L}_{\Gamma,\theta}(\eta) = \zeta$.*

Proof. We can define a decreasing sequence $\{\zeta_n\}$ whose limit is ζ such that for all $n \in \mathbb{N}$ there exists an Ł-valuation w_n with $w_n(\theta) = \zeta_n$ and $w_n(\bigwedge \Gamma) \geq \eta$. We can characterize every w_n by the values it assigns to the propositional variables in L. We will thus identify w_n with the vector $\vec{w}_n = (w_n(p_1), ..., w_n(p_k))$.

We need to prove now that there exists an Ł-valuation w such that $w(\theta) = \zeta$ and $w(\bigwedge \Gamma) \geq \eta$.

We can take a convergent subsequence $\{\vec{w}^1_{n_k}\}$ in the first coordinates of $\{\vec{w}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0, 1]$ by compactness. Next we can pick a convergent subsequence $\{\vec{w}^2_{n_k}\}$ in the second coordinates of $\{\vec{w}^1_{n_k}\}$. As before, such subsequence needs to exist by compactness. We can proceed in the same way for the other coordinates.

The final subsequence, $\{\vec{w}^{2^l}_{i_k}\}$, will have as limit an Ł-valuation $\vec{w}$ for which $w(\theta) = \zeta$ and $w(\bigwedge \Gamma) \geq \eta$. ∎

Proposition 111 *$\mathcal{L}_{\Gamma,\theta}$ is increasing.*

Proof. It follows directly from the definition of ${}^{\eta}\blacktriangleright_{\zeta}$. ∎

For the next proposition assume that Γ is maximally Ł$_{\lambda}$-consistent, $\lambda > 0$.

[4] In [15] we find a consequence relation similar in nature to this one. It could be presented, in our own terminology and notation, as follows: $\Gamma \models_{\eta} \theta$ if and only if, for all Ł-valuations w and all $\eta \in [0, 1]$, if $w(\Gamma) \geq \eta$ then $w(\theta) \geq \eta$.

Proposition 112 *$\mathcal{L}_{\Gamma,\theta}$ is left continuous on $[0,\lambda]$.*

Proof. Let us proceed by *reductio ad absurdum* by assuming that there exists $\eta \in (0,\lambda]$ and $\epsilon > 0$ such that

$$\mathcal{L}_{\Gamma,\theta}(\eta) - \mathcal{L}_{\Gamma,\theta}(x) > \epsilon$$

for all $x \in [0,\eta)$.

Let $\zeta = sup\,\{\mathcal{L}_{\Gamma,\theta}(x) |\, x < \eta\}$.

We can define an increasing sequence $\{\eta_n\}$ with limit η and a sequence $\{\zeta_n\}$ with limit ζ such that for all $n \in \mathbb{N}$ there exists an Ł-valuation w_n with $w_n(\bigwedge \Gamma) = \eta_n$ and $w_n(\theta) = \zeta_n$. We will identify w_n with the vector $\vec{w}_n = (w_n(p_1), ..., w_n(p_k))$.

We can take a convergent subsequence $\{\vec{w}^1_{n_k}\}$ in the first coordinates of $\{\vec{w}_n\}$. We know such a convergent subsequence needs to exist and converge in the interval $[0,1]$ by compactness. We can proceed in the same way for the other coordinates.

The final subsequence, $\{\vec{w}^{2^l}_{i_k}\}$, will have as limit an Ł-valuation $\vec{w}$ for which $w(\Gamma) = \eta$ and $w(\theta) = \zeta$ since $\mathcal{L}_{\Gamma,\theta}$ is increasing. Therefore $\mathcal{L}_{\Gamma,\theta}$ needs to be continuous from the left at η. ∎

Proposition 113 *$\mathcal{L}_{\Gamma,\theta}$ is of the following form:*

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} a_1\eta + b_1 & if \;\; \eta \leq \lambda_1 \\ ... & \\ a_k\eta + b_k & if \;\; \lambda_{k-1} < \eta \leq \lambda_k \end{cases}$$

with $a_i, b_i, \lambda_i \in \mathbb{Q}$ and $k \in \mathbb{N}$, $1 \leq i \leq k$.

Proof. Let $\mathcal{R} = \langle \mathbb{R}, +, -, <, =, 0, 1\rangle$.[5]

The set of pairs

$$\{(x,y) \in \mathbb{R}^2 |\; y = \mathcal{L}_{\Gamma,\theta}(x)\}$$

is $\mathcal{R}$-definable (notice that, since $\mathcal{R}$ is an elementary extension of the structure $\mathcal{Q} = \langle \mathbb{Q}, +, -, <, =, 0, 1\rangle$, it is $\mathcal{Q}$-definable too).

The theory of $\mathcal{R}$ has quantifier elimination (see for example [29]). Therefore that the set of pairs

$$\{(x,y) \in \mathbb{R}^2 |\; y = \mathcal{L}_{\Gamma,\theta}(x)\}$$

[5]Here by '$-$' we mean the map given by $x \longrightarrow -x$.

is given by a finite boolean combination (which reduces to a finite union of intersections by the complement and distributive laws for sets) of sets of the form

$$\{(x,y) \in \mathbb{R}^2 | \; my < nx + k\}$$

and

$$\{(x,y) \in \mathbb{R}^2 | \; my = nx + k\}$$

for $n, m, k \in \mathbb{Z}$.

Notice that each intersection of sets of such form is convex so, since $\mathcal{L}_{\Gamma,\theta}$ is a function, such intersection has to be a line segment (with coefficients and bounds in $\mathbb{Q}$).

That $\mathcal{L}_{\Gamma,\theta}$ is left continuous follows from Proposition 112. ■

In the next section we give a representation theorem for the functions $\mathcal{L}_{\Gamma,\theta}$. We prove that a function $\mathcal{F} : [0,1] \longrightarrow [0,1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \subseteq SL$ and $\theta \in SL$ if and only if $\mathcal{F}$ satisfies the properties stated in Propositions 111 and 113.

7.3.2 Graphs of $\mathcal{L}_{\Gamma,\theta}$

Basic graphs

We define five basic types of graphs.

Proposition 114 *(Type 1)*

Let r and s be two rational numbers in the interval $[0,1]$. We can find Γ and θ for which $\mathcal{L}_{\Gamma,\theta}$ is as follows:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} s & if \;\; \eta \leq r \\ 1 & otherwise \end{cases}$$

Proof. Let $0 < r = \frac{u_1}{v_1}$ and $0 < s = 1 - \frac{u_2}{v_2} < 1$.

Let $\Gamma = \{\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}\}$, with $\phi_{\frac{1}{v_1}} = \neg p \wedge p^{v_1 - 1}$ and $p \in L$. As seen above, Γ is maximally Ł$_r$-consistent.

On the other hand take $\phi_{\frac{1}{v_2}} = \neg q \wedge q^{v_2 - 1}$, for $q \in L$, $q \neq p$. The sentence $\underline{\bigvee}^{u_2} \phi_{\frac{1}{v_2}}$ is maximally Ł$_{\frac{u_2}{v_2}}$-consistent. Thus there is no Ł-valuation w such that $w(\neg(\underline{\bigvee}^{u_2} \phi_{\frac{1}{v_2}})) < 1 - \frac{u_2}{v_2} = s$.

Thus we can set $\theta = \neg(\underline{\bigvee}^{u_2} \phi_{\frac{1}{v_2}})$.

Clearly, for Γ and θ thus defined, $\mathcal{L}_{\Gamma,\theta}$ is as stated above.

For $r = 0$ we can take $\bigwedge \Gamma$ to be an Ł-contradiction. If $s = 0$ we can take θ to be an Ł-contradiction and, if $s = 1$, an Ł-tautology. ■

It is worth remarking the importance of a subclass of this type of graphs; namely, the graph given when $s = 0$.

Notice that in the above example Γ is not Ł$_1$-consistent. Later on, in order to prove the representation theorem for the functions $\mathcal{L}_{\Gamma,\theta}$, we will need to appeal to graphs of this form for Ł$_1$-consistent sets of premises. From McNaughton's Theorem we can claim that there exist sentences $\bigwedge \Gamma$ and θ involving only one propositional variable –say $p \in L$– with $\bigwedge \Gamma$ Ł$_1$-consistent such that $\mathcal{L}_{\Gamma,\theta}(\eta) = 0$ for $\eta \leq r$ and $\mathcal{L}_{\Gamma,\theta}(\eta) = 1$ for $\eta > r$, for any $r \in [0,1] \cap \mathbb{Q}$. To see this consider $f_{\bigwedge \Gamma}(x)$ and $f_\theta(x)$ to be of the following form:

$$f_{\bigwedge \Gamma}(x) = \begin{cases} a_1 x & if \;\; x \leq \frac{1+b_2}{a_1+a_2} \\ 1 - (a_2 x - b_2) & if \;\; \frac{1+b_2}{a_1+a_2} < x \leq \frac{1+b_2}{a_2} \\ a_3 x - b_3 & if \;\; \frac{1+b_2}{a_2} < x \leq c \\ 1 & otherwise \end{cases}$$

Here a_1, a_2, a_3, b_2, b_3 are positive integers and c is a rational number. Other conditions on these values are that $a_1(\frac{1+b_2}{a_1+a_2}) = 1 - (a_2(\frac{1+b_2}{a_1+a_2}) - b_2) = r$, $1 + b_2 < a_2$,[6] $1 - (a_2(\frac{1+b_2}{a_2}) - b_2) = a_3(\frac{1+b_2}{a_2}) - b_3 = 0$ and $a_3 + b_3 \geq 1$.

$$f_\theta(x) = \begin{cases} 0 & if \;\; x \leq d_1 \\ a_4 x - b_4 & if \;\; d_1 < x \leq d_2 \\ 1 & otherwise \end{cases}$$

Here a_4, b_4 are positive integers and d_1, d_2 are rational numbers. Another conditions on these values are $a_4 d_1 - b_4 = 0$, $a_4 d_2 - b_4 = 1$ and $\frac{1+b_2}{a_1+a_2} \leq d_1 < d_2 \leq \frac{1+b_2}{a_2}$.

For $\bigwedge \Gamma$ and θ of this form the function $\mathcal{L}_{\Gamma,\theta}$ will be as stated. To see this notice that $f_{\bigwedge \Gamma}(\frac{1+b_2}{a_1+a_2}) = r$, $f_\theta(\frac{1+b_2}{a_1+a_2}) = 0$ and, for all $x \in [0,1]$ for which $f_{\bigwedge \Gamma}(x) > r$ we have that $f_\theta(x) = 1$.

Proposition 115 *(Type 2)*

[6]Notice that for any $r \in [0,1] \cap \mathbb{Q}$ we can find positive integers a_1, a_2 and b_2 for which the conditions stated so far hold.

Let $r < s$ be two rational numbers in the interval $[0, 1]$. We can find Γ and θ for which $\mathcal{L}_{\Gamma,\theta}$ is as follows:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & if\ \ \eta \leq r \\ \frac{\eta - r}{s - r} & if\ \ r < \eta < s \\ 1 & otherwise \end{cases}$$

Proof. Let $0 < r = \frac{u_1}{v_1} < s = \frac{u_2}{v_2}$.

Take $s - r = \frac{u_2 v_1 - u_1 v_2}{v_1 v_2}$ and define ψ_1 and θ as follows:

$$\psi_1 = \underline{\bigvee}^{u_2 v_1 - u_1 v_2} \phi_{\frac{1}{v_1 v_2}},$$

$$\theta = \underline{\bigvee}^{v_1 v_2} \phi_{\frac{1}{v_1 v_2}}.$$

Here $\phi_{\frac{1}{v_1 v_2}} = \neg p \wedge p^{v_1 v_2 - 1}$, for $p \in L$.

Define ψ_2 as follows:

$$\psi_2 = \underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}.$$

We take $\phi_{\frac{1}{v_1}}$ to be $\neg q \wedge q^{v_1 - 1}$, for $q \in L$, $q \neq p$.

Set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$.

$\mathcal{L}_{\Gamma,\theta}$ is as stated above. To see this notice that, since ψ_2 is maximally Ł$_r$-consistent, $\mathcal{L}_{\Gamma,\theta}(x) = 0$ for all $x \in [0, r]$ and that any Ł-valuation w for which $w(\psi_1) = s - r$ (its maximal consistency) is such that $w(\theta) = 1$.

If $r = 0$ then we can dispense with ψ_2 and take $\Gamma = \{\psi_1\}$. ■

As with *Type 1* McNaughton's Theorem guarantees the existence of $\bigwedge \Gamma$ Ł$_1$-consistent and θ such that $\mathcal{L}_{\Gamma,\theta}$ is as above. To see this consider $\phi(p)$ and $\theta(p)$ (with $p \in L$) for which $f_\phi(x)$ and $f_\theta(x)$ are of the following form:

$$f_\phi(x) = \begin{cases} bx & if\ \ x \leq \frac{1}{b} \\ 1 & otherwise \end{cases}$$

$$f_\theta(x) = \begin{cases} ax & if\ \ x \leq \frac{1}{a} \\ 1 & otherwise \end{cases}$$

Here $a, b \in \mathbb{N}$ and $\frac{a}{b} = \frac{1}{s-r}$. Notice that $\mathcal{L}_{\{\phi\},\theta}(\eta) = \frac{a\eta}{b}$ for all $\eta \leq \frac{b}{a}$.

We can then set $\Gamma = \{\phi \underline{\vee} \psi_2\}$, where ψ_2 is as in the previous example:

$$\psi_2 = \underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}},$$

with $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1 - 1}$, for $q \in L$, $q \neq p$.

$\mathcal{L}_{\Gamma,\theta}$ will be as stated, with Γ Ł$_1$-consistent.

Proposition 116 *(Type 3)*

Let r and s be two rational numbers in the interval $[0,1]$. We can define Γ and θ for which $\mathcal{L}_{\Gamma,\theta}$ is as follows:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} 0 & if \;\; \eta \leq r \\ \frac{s(\eta - r)}{1-r} & otherwise \end{cases}$$

Proof. Let $r = \frac{u_1}{v_1}$ and $s = \frac{u_2}{v_2}$. We have to distinguish two possible cases here:

<u>Case 1</u>. $\frac{s}{1-r} \leq 1$.

Consider $\frac{s}{1-r} = \frac{u_2 v_1}{v_2(v_1 - u_1)}$.
We first define ψ_1 and θ as follows:

$$\psi_1 = \underline{\bigvee}^{v_2(v_1 - u_1)} \phi_{\frac{1}{v_2(v_1 - u_1)}},$$

$$\theta = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{v_2(v_1 - u_1)}}.$$

Here $\phi_{\frac{1}{v_2(v_1 - u_1)}} = \neg p \wedge p^{v_2(v_1 - u_1) - 1}$, for $p \in L$.

Let us now define ψ_2 for $r > 0$ as follows:

$$\psi_2 = \underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}.$$

In this case $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1 - 1}$, for $q \in L$, $q \neq p$.

Set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$.

Clearly $\mathcal{L}_{\Gamma,\theta}$ is as mentioned above.

Notice that if $r = 0$ then we can dispense with ψ_2 and set $\Gamma = \{\psi_1\}$ to get $\mathcal{L}_{\Gamma,\theta}$ as mentioned.

<u>Case 2</u>. $\frac{s}{1-r} > 1$.

Consider $\frac{1-r}{s} = \frac{v_2(v_1-u_1)}{u_2v_1}$.

We now define ψ_1 and θ in the following way:

$$\psi_1 = \underline{\bigvee}^{v_2(v_1-u_1)} \phi_{\frac{1}{u_2v_1}},$$

$$\theta = \underline{\bigvee}^{u_2v_1} \phi_{\frac{1}{u_2v_1}}$$

with $\phi_{\frac{1}{u_2v_1}} = \neg p \wedge p^{u_2v_1-1}$, for $p \in L$.

If $r > 0$ define ψ_2 as above and set $\Gamma = \{\psi_1 \underline{\vee} \psi_2\}$. $\mathcal{L}_{\Gamma,\theta}$ will be as stated.

As before, if $r = 0$ then we set $\Gamma = \{\psi_1\}$. ■

Proposition 117 *(Type 4)*

Let $r < s$ be two rational numbers in the interval $[0,1]$. We can define Γ and θ for which $\mathcal{L}_{\Gamma,\theta}(\eta) = (s-r)\eta + r$.

Proof. Let $r = \frac{u_1}{v_1} < s = \frac{u_2}{v_2}$. Take $s - r = \frac{u_2v_1-u_1v_2}{v_1v_2}$ and define ψ and θ_1 as follows:

$$\psi = \underline{\bigvee}^{v_1v_2} \phi_{\frac{1}{v_1v_2}},$$

$$\theta_1 = \underline{\bigvee}^{u_2v_1-u_1v_2} \phi_{\frac{1}{v_1v_2}},$$

where $\phi_{\frac{1}{v_1v_2}} = \neg p \wedge p^{v_1v_2-1}$, for $p \in L$.

Let us define θ_2 as follows:

$$\theta_2 = \neg(\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}).$$

Here $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1-1}$, for $q \in L$, $q \neq p$.

Set $\theta = \theta_1 \underline{\vee} \theta_2$ and $\Gamma = \{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as stated above.

If $r = 0$ then we set $\theta = \theta_1$. ■

Proposition 118 *(Type 5)*

Let r and s be two rational numbers in the interval $[0,1]$. We can define Γ and θ for which $\mathcal{L}_{\Gamma,\theta}$ is as follows:

$$\mathcal{L}_{\Gamma,\theta}(\eta) = \begin{cases} \eta(\frac{1-r}{s}) + r & if \ \ \eta \leq s \\ 1 & otherwise \end{cases}$$

Proof. Let $0 < r = \frac{u_1}{v_1}$ and $s = \frac{u_2}{v_2}$. We have to distinguish two possible cases:

<u>Case 1</u>. $\frac{1-r}{s} > 1$.

Consider $\frac{s}{1-r} = \frac{u_2 v_1}{v_2(v_1-u_1)}$ and define ψ and θ_1 as follows:

$$\psi = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{v_2(v_1-u_1)}},$$

$$\theta_1 = \underline{\bigvee}^{v_2(v_1-u_1)} \phi_{\frac{1}{v_2(v_1-u_1)}},$$

with $\phi_{\frac{1}{v_2(v_1-u_1)}} = \neg p \wedge p^{v_2(v_1-u_1)-1}$, for $p \in L$.

On the other hand define θ_2 as follows:

$$\theta_2 = \neg(\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}}),$$

with $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1-1}$, for $q \in L$, $q \neq p$.

Set $\theta = \theta_1 \underline{\vee} \theta_2$ and $\Gamma = \{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as stated above.

If $r = 0$ then we can set $\theta = \theta_1$.

As with *Type 1* and *Type 2*, McNaughton's Theorem guarantees the existence of sentences $\bigwedge \Gamma$ and θ in one variable (say $p \in L$), with $\bigwedge \Gamma$ Ł$_1$-consistent, such that $\mathcal{L}_{\Gamma,\theta}$ is as above. To see this consider ϕ and ψ for which $f_\phi(x)$ and $f_\psi(x)$ are as follows:

$$f_\phi(x) = \begin{cases} bx & if \ \ x \leq \frac{1}{b} \\ 1 & otherwise \end{cases}$$

$$f_\psi(x) = \begin{cases} ax & if \ \ x \leq \frac{1}{a} \\ 1 & otherwise \end{cases}$$

Here $a, b \in \mathbb{N}$ and $\frac{a}{b} = \frac{1-r}{s}$.

Set $\Gamma = \{\phi\}$ and $\theta = \{\psi \underline{\vee} \theta_2\}$, where

$$\theta_2 = \neg(\underline{\bigvee}^{u_1} \phi_{\frac{1}{v_1}})$$

and $\phi_{\frac{1}{v_1}} = \neg q \wedge q^{v_1 - 1}$, for $q \in L$, $q \neq p$.

Clearly $\mathcal{L}_{\Gamma,\theta}$ will be as stated, with Γ Ł$_1$-consistent.

Again, if $r = 0$ then we can set $\theta = \psi$.

Case 2. $\frac{1-r}{s} \leq 1$.

Consider $\frac{1-r}{s} = \frac{v_2(v_1-u_1)}{u_2 v_1}$ and define ψ and θ_1 as follows:

$$\psi = \underline{\bigvee}^{u_2 v_1} \phi_{\frac{1}{u_2 v_1}},$$

$$\theta_1 = \underline{\bigvee}^{v_2(v_1-u_1)} \phi_{\frac{1}{u_2 v_1}},$$

where $\phi_{\frac{1}{u_2 v_1}} = \neg p \wedge p^{u_2 v_1 - 1}$, for $p \in L$.

Define θ_2 as in *Case 1* and set $\theta = \theta_1 \underline{\vee} \theta_2$ and $\Gamma = \{\psi\}$. The function $\mathcal{L}_{\Gamma,\theta}$ will be as desired.

For $r = 0$ we dispense again with θ_2. ■

Compound graphs

Let L_1, L_2 be two disjoint languages and SL_1, SL_2 their respective sets of sentences. Take $\Gamma_1 \subseteq SL_1$, $\Gamma_2 \subseteq SL_2$ and $\theta_1 \in SL_1$, $\theta_2 \in SL_2$. Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ is maximally Ł$_\lambda$-consistent.

Proposition 119 *$max\{\mathcal{L}_{\Gamma_1,\theta_1}(\eta), \mathcal{L}_{\Gamma_2,\theta_2}(\eta)\} = \mathcal{L}_{\Gamma_1\cup\Gamma_2,\theta_1\vee\theta_2}(\eta)$ for all $\eta \in [0,1]$.*

Proof. It follows trivially given the way the connective $\vee$ is defined. ■

Proposition 120 *$min\{\mathcal{L}_{\Gamma_1,\theta_1}(\eta), \mathcal{L}_{\Gamma_2,\theta_2}(\eta)\} = \mathcal{L}_{\Gamma_1\cup\Gamma_2,\theta_1\wedge\theta_2}(\eta)$ for all $\eta \in [0,\lambda]$.*

Proof. It follows trivially given the definition of the connective $\wedge$. ■

We can extend these propositions to any finite collection of sets of sentences $\Gamma_1 \subseteq SL_1, ..., \Gamma_k \subseteq SL_k$ and $\theta_1 \in SL_1, ..., \theta_k \in SL_k$, for some $k \in \mathbb{N}$, with $L_1, ..., L_k$ a collection of pairwise disjoint languages.

Theorem 121 *The function $\mathcal{F} : [0,1] \longrightarrow [0,1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \subseteq SL$ and $\theta \in SL$ if and only if $\mathcal{F}$ is an increasing function of the following form:*

$$\mathcal{F}(x) = \begin{cases} a_1 x + b_1 & if \;\; x \leq \lambda_1 \\ ... \\ a_k x + b_k & if \;\; \lambda_{k-1} < x \leq \lambda_k \end{cases}$$

with $a_i, b_i, \lambda_i \in \mathbb{Q}$ and $k \in \mathbb{N}$, $i \in \{1, ..., k\}$.

Proof. If $\mathcal{F} : [0,1] \longrightarrow [0,1]$ is of the form $\mathcal{L}_{\Gamma,\theta}$ for some $\Gamma \subseteq SL$ and $\theta \in SL$ then we know, by Propositions 111 and 113, that $\mathcal{F}$ will be an increasing function of the form stated above.

Let us prove now the left implication.

Let $\mathcal{F} : [0,1] \rightarrow [0,1]$ be as stated.

We will denote the line segment given by $a_i x + b_i$ and $\lambda_{i-1} < x \leq \lambda_i$ by l_i, for $i \in \{2, ..., k\}$ (l_1 will be the line segment given by $a_1 x + b_1$ and $x \leq \lambda_1$).

Let us define Γ and θ for which $\mathcal{L}_{\Gamma,\theta}(\eta) = \mathcal{F}(\eta)$ for all $\eta \in [0,1]$.

First, let l_i be a line segment of $\mathcal{F}$, $i \in \{1, ..., k\}$ (without loss of generality we can assume that $i \neq 1$). We can define $\Gamma_i \subseteq SL$ Ł$_1$-consistent and $\theta_i \in SL$ for which $\mathcal{L}_{\Gamma_i,\theta_i}$ is as follows:

$$\mathcal{L}_{\Gamma_i,\theta_i}(x) = \begin{cases} a_i \lambda_{i-1} + b_i & if \;\; x \leq \lambda_{i-1} \\ a_i x + b_i & if \;\; \lambda_{i-1} < x \leq \lambda_i \\ 1 & otherwise \end{cases}$$

To see this set

$$\mathcal{L}_{\Gamma_i,\theta_i}(\eta) = max\{\mathcal{L}_{\Delta_1,\psi_1}(\eta), max\{\mathcal{L}_{\Delta_2,\psi_2}(\eta), \mathcal{L}_{\Delta_3,\psi_3}(\eta)\}\}$$

for all $\eta \in [0,1]$, with $\Delta_j \subseteq SL_j$ Ł$_1$-consistent and $\psi_j \in SL_j$ for all $j \in \{1,2,3\}$, where L_1, L_2, L_3 are pairwise disjoint languages.

$\mathcal{L}_{\Delta_1,\psi_1}$ and $\mathcal{L}_{\Delta_2,\psi_2}$ are of *Type 1*:

$$\mathcal{L}_{\Delta_1,\psi_1}(x) = \begin{cases} 0 & if \;\; x \leq \lambda_i \\ 1 & otherwise \end{cases}$$

$$\mathcal{L}_{\Delta_2,\psi_2}(x) = \ a_i\lambda_{i-1} + b_i \quad for\ all\ x \in [0,1]$$

The nature of the straight line $a_i x + b_i$ will determine the type of graph of $\mathcal{L}_{\Delta_3,\psi_3}$. We will choose Δ_3 and ψ_3 such that the graph of $\mathcal{L}_{\Delta_3,\psi_3}$ contains the straight segment $a_i x + b_i$, for $\lambda_{i-1} < x \leq \lambda_i$. That $\mathcal{L}_{\Delta_3,\psi_3}$ will be of one of the types described in the previous subsection is clear.

It can easily be seen that

$$\mathcal{F}(\eta) = \mathcal{L}_{\bigcup \Gamma_i, \bigwedge \theta_i}(\eta) = min\{\mathcal{L}_{\Gamma_i,\theta_i}(\eta) \,|\, i \in \{1, ..., k\}\}$$

for all $\eta \in [0,1]$, with $\Gamma_1 \subseteq SL_1, ..., \Gamma_k \subseteq SL_k$, $\theta_1 \in SL_1, ..., \theta_k \in SL_k$ and $L_1, ..., L_k$ a pairwise disjoint collection of languages. ■

Chapter 8

Conclusion

In Chapter 2 we have presented the notion of η-coherence as a measure of consistency, which we argue improves on the inconsistency measure given by Schotch and Jennings in [45], and have proved its mathematical equivalence to η-consistency, with the advantage over the latter that it rests entirely within propositional calculus without involving probabilities.

In Chapter 3 we have studied the consequence relation ${}^{\eta}\!\triangleright_{\zeta}$ as a model to reasoning by a *rational* agent holding possibly inconsistent beliefs. We have found an equivalent to ${}^{\eta}\!\triangleright_{\zeta}$ (for any values $\eta, \zeta \in [0,1]$) within classical propositional logic and have given a representation theorem for the functions of the form $F_{\Gamma,\theta}$.

In Chapter 4 we have compared ${}^{\eta}\!\triangleright_{\zeta}$ with other consequence relations in the literature with respect to distinct criteria. One of these criteria was consistency. We have proved that for certain values η, ζ the consequence relation ${}^{\eta}\!\triangleright_{\zeta}$ yields consistent sets of consequences and that on those grounds it behaves pretty well with respect to other well known consequence relations. We have seen also that for particular values η, ζ our inference relation ${}^{\eta}\!\triangleright_{\zeta}$ satisfies some desirable structural rules.

In Chapter 5 we have defined new inference relations very much inspired by ${}^{\eta}\!\triangleright_{\zeta}$ on the basis that distinct sentences in our belief base were allowed distinct probability thresholds. We have first defined a consequence relation to deal with belief bases given by sentences of the form $w(\phi) \geq \eta$ (for $\phi \in SL$ and $\eta \in [0,1]$, with the intended meaning 'the probability of ϕ is at least η') and have given a propositional equivalent to it, analogous to that for ${}^{\eta}\!\triangleright_{\zeta}$ in Chapter 3. Next we extended the language and defined a new consequence relation for belief bases given by boolean combinations of sentences of the form $w(\phi) \geq \eta$ and gave a proof system for it.

In Chapter 6 we have considered countably infinite propositional languages and have studied some properties of ${}^{\eta}\triangleright_{\zeta}$ when considering possibly infinite sets of premises. In particular we have given a representation theorem for the functions of the form $F_{\Gamma,\theta}$ with Γ possibly infinite.

In Chapter 7 we have defined consequence relations similar in nature to ${}^{\eta}\triangleright_{\zeta}$ in terms of degrees of truth (truth values) rather than degrees of belief (probabilities). We have done so for Gödel and Łukasiewicz logics and have given representation theorems for the functions $\mathcal{G}_{\Gamma,\theta}$ and $\mathcal{L}_{\Gamma,\theta}$ (analogous to $F_{\Gamma,\theta}$ in our probabilistic approach).

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