A Useful Lemma in Unary Predicate Logic

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The sole purpose of this short note is to make readily accessible a representation result from Unary Predicate Logic which find frequent use in Pure Inductive Logic.

Let L be a language for first order logic with (just) the unary relation symbols $R_1, R_2, \ldots, R_m$ and no constants, function symbols or equality. Let FL denote the set of formulae of L. For $\phi \in FL$ write ϕ^1 for ϕ and ϕ^0 for $\neg\phi$. We call a formula of L an *atom* (for $R_1, R_2, \ldots, R_m$) if it has the form

$$R_1^{\epsilon_1}(x_1) \wedge R_2^{\epsilon_2}(x_1) \wedge \ldots \wedge R_m^{\epsilon_m}(x_1)$$

for some $\epsilon_1, \epsilon_2, \ldots, \epsilon_m \in \{0, 1\}$. Since there are 2^m choices for the finite sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_m \in \{0, 1\}$ there are 2^m such atoms, which we shall denote by $\alpha_1(x_1), \alpha_2(x_1), \ldots, \alpha_{2^m}(x_1)$.

Now suppose that we are given an interpretation for L, that is a structure for L together with an assignment of elements of the universe of the structure to the free variables $x_1, x_2, x_3, \ldots$.[1] Let $\theta(x_1, \ldots, x_n) \in FL$ and $1 \leq i \leq m$. Then exactly one of $R_1(x_i), \neg R_1(x_i)$ is true in this interpretation. In

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[1]We shall use $w_1, w_2, w_3, \ldots$ for bound variables so no confusion can arise.

other words there is a unique $\epsilon_1 \in \{0,1\}$ such that $R_1^{\epsilon_1}(x_i)$ is true in this interpretation. Similarly there is a unique $\epsilon_2 \in \{0,1\}$ such that $R_2^{\epsilon_2}(x_i)$ is true in this interpretation. Continuing in this way we see that there is a unique atom α_{h_i} such that $\alpha_{h_i}(x_i)$ is true in this interpretation.

Similarly for each $1 \leq k \leq 2^m$ exactly one of $\exists w_1\, \alpha_k(w_1)$ and $\neg\exists w_1\, \alpha_k(w_1)$ is true in this interpretation. In other words there is a unique $\delta_k \in \{0,1\}$ such that $(\exists w_1\, \alpha_k(w_1))^{\delta_k}$ is true in this interpretation. Putting these observations together then there are unique finite sequences $j_1, j_2, \ldots, j_n \in \{1, 2, \ldots, 2^m\}$ and $\delta_1, \delta_2, \ldots, \delta_{2^m} \in \{0,1\}$ such that the formula

$$\bigwedge_{i=1}^{n} \alpha_{j_i}(x_i) \wedge \bigwedge_{j=1}^{2^m} (\exists w_1\, \alpha_j(w_1))^{\delta_j} \tag{1}$$

is true in this interpretation. Call a formula of this form for some $j_1, j_2, \ldots, j_n \in \{1, 2, \ldots, 2^m\}$ and $\delta_1, \delta_2, \ldots, \delta_{2^m} \in \{0,1\}$ a *diagram* for $x_1, \ldots, x_n$.

Lemma 1.* *Let $\theta(x_1, \ldots, x_n) \in FL$ and suppose that the relation symbols occurring in θ are $R_1, R_2, \ldots, R_m$ and these are all unary. Then $\theta(x_1, \ldots, x_n)$ is logically equivalent to a disjunction of diagrams for $x_1, \ldots, x_n$.*

Before we give the proof it is worth noticing that not all diagrams are satisfiable since a diagram might for example have conjuncts $\alpha_1(x_1)$ and $\neg\exists w_1\, \alpha_1(w_1)$. Clearly we could, without loss, drop these unsatisfiable diagrams from the representation given in this theorem.

Proof. The proof is by induction on the length, $|\theta(x_1, \ldots, x_n)|$, of $\theta(x_1, \ldots, x_n) \in FL$ (where each x_i, w_i, R_i has length 1 etc.).

In the case that $\theta = R_r(x_i)$, with, say, $1 \leq i \leq n$, we have from the above discussion that in any interpretation

$$\begin{aligned} \theta \text{ is true} \quad &\Longleftrightarrow \quad R_r(x_i) \text{ is true} \\ &\Longleftrightarrow \quad \text{some atom } \textstyle\bigwedge_{k=1}^{m} R_k^{\epsilon_k}(x_i) \text{ with } \epsilon_r = 1 \text{ is true} \\ &\Longleftrightarrow \quad \text{some diagram (1) where } \alpha_{j_i} = \textstyle\bigwedge_{k=1}^{m} R_k^{\epsilon_k}(x_j) \\ &\qquad\quad \text{with } \epsilon_r = 1 \text{ is true .} \end{aligned}$$

In other words $R_r(x_i)$ is logically equivalent to the disjunction of all such diagrams.

Now suppose that $\theta(x_1, \ldots, x_n) = (\phi(x_1, \ldots, x_n) \wedge \psi(x_1, \ldots, x_n))$. By inductive hypothesis ϕ is logically equivalent to a disjunction of diagrams for $\vec{x} = x_1, x_2, \ldots, x_n$ so given an interpretation ϕ is true in that interpretation just if the unique diagram which is true in that interpretation is one of these disjuncts. Similarly for ψ.

Hence θ is true in an interpretation just if the diagram true in that interpretation is a disjunct for both ϕ and ψ. Or to put it another way θ is logically equivalent to the disjunction of diagrams which appear in the corresponding forms for both ϕ and ψ. The cases for the other connectives are exactly analogous.

The tricky cases concern the quantifiers. So now suppose that $\theta(x_1, \ldots, x_n) = \exists w_j\, \phi(x_1, \ldots, x_n, w_j)$. By inductive hypothesis then there are diagrams for $x_1, \ldots, x_n, x_{n+1}$, say $\xi_g(x_1, \ldots, x_n, x_{n+1})$ for $g = 1, \ldots, u$, such that

$$\phi(x_1, \ldots, x_{n+1}) \equiv \bigvee_{g=1}^{u} \xi_g(x_1, \ldots, x_n, x_{n+1}).$$

Then using well known logical equivalents

$$\begin{aligned} \exists w_j\, \phi(x_1, \ldots, x_n, w_j) &\equiv \exists w_2 \left(\bigvee_{g=1}^{u} \xi_g(x_1, \ldots, x_n, w_2) \right) \\ &\equiv \left(\bigvee_{g=1}^{u} \exists w_2\, \xi_g(x_1, \ldots, x_n, w_2) \right). \end{aligned} \tag{2}$$

Since each $\xi_g(x_1, \ldots, x_n, w_2)$ is a conjunction of expressions only one of which actually mentions w_2, and that one has the form $\alpha_v(w_2)$ for some atom $\alpha_v(x_1)$, standard logical equivalents give that this $\exists w_2\, \xi_g(x_1, \ldots, x_n, w_2)$ is logically equivalent to a formula of the form

$$\exists w_2\, \alpha_v(w_2) \wedge \zeta(x_1, \ldots, x_n) \tag{3}$$

where ζ is a diagram for $x_1, \ldots, x_n$. If $(\exists w_1\, \alpha_v(w_2))^1$ already appears in ζ then (3) is logically equivalent to ζ. On the other hand if $(\exists w_1\, \alpha_v(w_1))^1$ does not already appear in ζ then $(\exists w_1\, \alpha_v(w_1))^0$, i.e. $\neg\exists w_1\, \alpha_v(w_1)$, must appear in ζ and in that case (3) is not satisfiable.

Using standard logical equivalents it now follows that $\exists w_j\, \phi(x_1, \ldots, x_n, w_j)$ is logically equivalent to the disjunction of the (distinct) diagrams for which the (3) yielded a satisfiable ζ, giving the required result.

Finally in the case $\theta(x_1, \ldots, x_n) = \forall w_j \, \phi(x_1, \ldots, x_n, w_j)$ we have that $\theta(x_1, \ldots, x_n) = \neg\exists w_j \, \neg\phi(x_1, \ldots, x_n, w_j)$. To treat this formula it is simplest to use three of the cases already covered, namely going from $\phi(x_1, \ldots, x_n, x_{n+1})$ (where we can use the Inductive Hypothesis) to
$\neg\phi(x_1, \ldots, x_n, x_{n+1})$ (for which we then have the Inductive Hypothesis), thence to $\exists w_j \, \neg\phi(x_1, \ldots, x_n, w_j)$, and finally to $\neg\exists w_j \, \neg\phi(x_1, \ldots, x_n, w_j)$.

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