A proof that FVP_n does not imply FVP_{n+1}

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The purpose of this note is simply to sketch a proof of the fact asserted in [2] that FVP_n does not imply FVP_{n+1}. We assume that the reader is already fully familiar with [2].

This result for $n = 1$ is given in the main paper [2]. We shall give the construction in the case $n = 2$. The corresponding construction for $n = 3, 4, \ldots,$ is analogous but with co-linear replaced by co-planar etc.. An example for $n = 0$ follows from the example given for $n = 2$ using the method described below (or by an easy direct construction).

Let L be the language with a single ternary relation symbol R. Let M^+ be the structure with universe the Cartesian plane and R interpreted as holding between three points just if they are distinct and co-linear. Let M be an elementary substructure of M^+ with countable universe $\{e_i \,|\, i \in \mathbb{N}^+\}$. Notice that we can define equality in M^+ and M.

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For any points e_i, e_j, e_k, e_r with $i = j$ just if $k = r$, there is a translation plus a rotation of the plane plus a scaling which sends e_i to e_k and e_j to e_r. Since this mapping is co-linearity preserving it gives an isomorphism of M^+ and hence for any $\theta(a_1, a_2) \in SL^{(2)}$,

$$M^+(\text{or } M) \models \theta(e_i, e_j) \iff M^+(\text{or } M) \models \theta(e_k, e_r). \tag{1}$$

Also for any e_i, e_j, e_k, e_r which form a proper quadrilateral with no parallel sides we can define in M the point which is the intersection of the lines formed by e_i, e_j and by e_k, e_r and so on for the other disjoint pairs from e_i, e_j, e_k, e_r. In turn we can define the intersection points of the lines determined by the original e_i, e_j, e_k, e_r and the first intersection points to form some new 'second' intersection points, and so on. In general then we can find for each $m \in \mathbb{N}$ a sentence $\theta(a_1, a_2, \ldots, a_5) \in SL^{(5)}$ such that

$$M \models \theta(e_i, e_j, e_k, e_r, e_s) \iff e_s \text{ is an } m\text{'th but not an } n\text{'th intersection point for any } n < m. \tag{2}$$

Now define a probability function V_M on L by

$$V_M(\theta(a_1, a_2, \ldots, a_n)) = \begin{cases} 1 & \text{if } M \models \theta(e_1, e_2, \ldots, e_n), \\ 0 & \text{otherwise,} \end{cases}$$

and in turn a further function w on SL by

$$w(\theta(a_1, a_2, \ldots, a_n)) = \sum_{\tau} V_M(\theta(a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(n)})) \cdot \prod_{i=1}^{n} 2^{-\tau(i)}$$

where τ runs over all maps from $\{1, 2, \ldots, n\}$ into $\mathbb{N}^+$. By a theorem of Gaifman, see [1] or [3, Chapter 26], w is a probability function on L satisfying Ex. From (1) $w(\theta(a_1, a_2))$ must be a sum of none, one or both of

$$\sum_{\tau(1) \neq \tau(2)} 2^{-\tau(1)-\tau(2)}, \qquad \sum_{\tau(1) = \tau(2)} 2^{-2\tau(1)},$$

so w satisfies FVP_2. However from (2) w must give non-zero probability to a_5 being an m'th (but not n'th for $n < m$) intersection point from a_1, a_2, a_3, a_4 so w does not satisfy FVP_5.

From this it follows that there is some $2 \leq n \leq 4$ such that w satisfies FVP_n but not FVP_{n+1}. If $n = 2$ then we already have what we want. So suppose $n = 3$ (the other case will be similar) and for notational convenience that e_i as above in (2) is e_1. Now extend the language L to L' by adding relation symbols $R_1, \ldots, R_6$ and form a structure M' for L' with universe $\{e_j \mid j > 1\}$ by interpreting the relation symbols as:

$$\begin{aligned}
M' \models R(e_r, e_s, e_t) &\iff M \models R(e_r, e_s, e_t),\\
M' \models R_1(e_r) &\iff M \models R(e_r, e_1, e_1),\\
M' \models R_2(e_r) &\iff M \models R(e_1, e_r, e_1),\\
M' \models R_3(e_r) &\iff M \models R(e_1, e_1, e_r),\\
M' \models R_4(e_r, e_s) &\iff M \models R(e_r, e_s, e_1),\\
M' \models R_5(e_r, e_s) &\iff M \models R(e_r, e_1, e_s),\\
M' \models R_6(e_r, e_s) &\iff M \models R(e_1, e_r, e_s).
\end{aligned}$$

By induction on the length of θ we can now show that for $\theta(a_1, a_2, \ldots, a_{n+1}) \in SL$ there is a sentence $\theta'(a_1, a_2, \ldots, a_n) \in SL'$ such that for any $i_1, i_2, i_3, \ldots, i_n > 1$,

$$M \models \theta(e_1, e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \iff M' \models \theta'(e_{i_1}, e_{i_2}, \ldots, e_{i_n}). \qquad (3)$$

Conversely, using the fact that equality is definable in M, for any sentence $\phi'(a_1, a_2, \ldots, a_n) \in SL'$ there is a sentence $\phi(a_1, a_2, \ldots, a_{n+1}) \in SL$ such that for any $i_1, i_2, i_3, \ldots, i_n > 1$,

$$M' \models \phi'(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) \iff M \models \phi(e_1, e_{i_1}, e_{i_2}, \ldots, e_{i_n}). \qquad (4)$$

Now, mimicking the above construction, define the probability function $V_{M'}$ on L' by

$$V_{M'}(\psi(a_1, a_2, \ldots, a_n)) = \begin{cases} 1 & \text{if } M' \models \psi(e_2, e_3, \ldots, e_{n+1}),\\ 0 & \text{otherwise,} \end{cases}$$

and in turn a further function w' on SL' by

$$w'(\psi(a_1, a_2, \ldots, a_n)) = \sum_{\tau} V_{M'}(\psi(a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(n)})) \cdot \prod_{i=1}^{n} 2^{-\tau(i)}.$$

Again w' satisfies Ex. Also the set of values of the $w'(\phi'(a_1, a_2))$ is finite by (4) and the assumption that w satisfies FVP_3 so w' satisfies FVP_2. However since w fails FVP_4 the set of values of the $w(\theta(a_1, a_2, a_3, a_4))$ is infinite so by (3) the set of values of the $w'(\theta'(a_1, a_2, a_3))$ is infinite and w' fails FVP_3.

References

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