A note on *-conditioning

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Introduction

The purpose of this note is to prove a technical result concerning a variation
on Bayesian conditioning (*-conditioning) which needs be referenced in some
other papers. We shall assume familiarity with the context and notation of
Pure Inductive Logic as given for example in [1], [2].

The Main Result

Let \( w \) be a probability function on \( SL \) satisfying \( \text{Ex} \). For \( \Gamma(a_1, a_2, \ldots, a_n) \) a
state description with \( w(\Gamma) > 0 \) we define \( w_{\ast \Gamma} : SL \rightarrow [0, 1] \) by

\[
w_{\ast \Gamma}(\theta(a_1, a_2, \ldots, a_m)) = w(\theta(a_{n+1}, a_{n+2}, \ldots, a_{n+m}) \mid \Gamma(a_1, a_2, \ldots, a_n)).
\]

Theorem 1. \( w_{\ast \Gamma} \) is a probability function on \( SL \).

Proof. To be a probability function we need to show that \( w_{\ast \Gamma} \) satisfies the
defining conditions:

(P1) If \( \vdash \theta \) then \( w_{\ast \Gamma}(\theta) = 1 \)

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(P2) If $\vdash \neg(\theta \land \phi)$ then $w_{s\Gamma}(\theta \lor \phi) = w_{s\Gamma}(\theta) + w_{s\Gamma}(\phi)$

(P3) $w_{s\Gamma}(\exists x \psi(x)) = \lim_{r \to \infty} w_{s\Gamma}(\bigvee_{i=1}^{r} \psi(a_{i}))$

It is clear that $w_{s\Gamma}$ satisfies (P1-2) so we only have to show it satisfies (P3). In terms of $w$ then this requires

$$\lim_{r \to \infty} w \left( \bigvee_{i=1}^{r} \psi(a_{n+1}, a_{n+2}, \ldots, a_{n+m}, a_{n+1}) \land \Gamma(a_{1}, \ldots, a_{n}) \right)$$

$$= w(\exists x \psi(a_{n+1}, a_{n+2}, \ldots, a_{n+m}, x) \land \Gamma(a_{1}, \ldots, a_{n}))$$

$$= \lim_{r \to \infty} w \left( \bigvee_{i=1}^{r} \psi(a_{n+1}, a_{n+2}, \ldots, a_{n+m}, a_{i}) \land \Gamma(a_{1}, \ldots, a_{n}) \right).$$

If this was to fail there would have to be some $\beta > 0$ such that for all $r \geq n + 1$,

$$w \left( \bigvee_{i=1}^{r} \psi(a_{n+1}, a_{n+2}, \ldots, a_{n+m}, a_{i}) \land \Gamma(a_{1}, \ldots, a_{n}) \right)$$

$$- w \left( \bigvee_{i=n+1}^{r} \psi(a_{n+1}, a_{n+2}, \ldots, a_{n+m}, a_{i}) \land \Gamma(a_{1}, \ldots, a_{n}) \right) \geq \beta.$$

Equivalently

$$w \left( \bigvee_{i=1}^{n} \psi(a_{n+1}, \ldots, a_{n+m}, a_{i}) \land \neg \left( \bigvee_{i=n+1}^{r} \psi(a_{n+1}, \ldots, a_{n+m}, a_{i}) \land \Gamma(a_{1}, \ldots, a_{n}) \right) \right) \geq \beta. \quad (1)$$

So certainly we must also have

$$w \left( \bigvee_{i=1}^{n} \psi(a_{n+1}, \ldots, a_{n+m}, a_{i}) \land \neg \left( \bigvee_{i=n+1}^{r} \psi(a_{n+1}, \ldots, a_{n+m}, a_{i}) \right) \right) \geq \beta. \quad (2)$$

Purely to simplify matters suppose that $m = 1$ here, so (2) becomes

$$w \left( \bigvee_{i=1}^{n} \psi(a_{n+1}, a_{i}) \land \neg \left( \bigvee_{i=n+1}^{r} \psi(a_{n+1}, a_{i}) \right) \right) \geq \beta. \quad (3)$$
Now take $r = 1 + (k + 1)n$. Then by Ex for $w$ with $0 \leq s < k$, interchanging the block $a_1, a_2, \ldots, a_n$ with $a_{n+1+ns+1}, a_{n+1+ns+2}, \ldots, a_{n+1+n(s+1)}$ in (3) gives that

$$w \left( \bigvee_{i=n+1+ns+1}^{n+1+n(s+1)} \psi(a_{n+1}, a_i) \land \neg \left( \bigvee_{i=1}^{n} \psi(a_{n+1}, a_i) \lor \bigvee_{i=n+1}^{n+1+ns} \psi(a_{n+1}, a_i) \lor \bigvee_{i=n+2+n(s+1)}^{r} \psi(a_{n+1}, a_i) \right) \right)$$

is also at least $\beta$. But the sentences in (4) as we vary $s$ are disjoint and there are $k$ of them so making $k > \beta^{-1}$ gives a contradiction, as required.

$\square$

References
