A note on $*$-conditioning

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Introduction

The purpose of this note is to prove a technical result concerning a variation on Bayesian conditioning ($*$-conditioning) which needs be referenced in some other papers. We shall assume familiarity with the context and notation of Pure Inductive Logic as given for example in [1], [2].

The Main Result

Let w be a probability function on SL satisfying Ex. For $\Gamma(a_1, a_2, \ldots, a_n)$ a state description with $w(\Gamma) > 0$ we define $w_{*\Gamma} : SL \to [0, 1]$ by

$$w_{*\Gamma}(\theta(a_1, a_2, \ldots, a_m)) = w(\theta(a_{n+1}, a_{n+2}, \ldots, a_{n+m}) \,|\, \Gamma(a_1, a_2, \ldots, a_n)).$$

Theorem 1. *$w_{*\Gamma}$ is a probability function on SL.*

Proof. To be a probability function we need to show that $w_{*\Gamma}$ satisfies the defining conditions:

(P1) If $\vDash \theta$ then $w_{*\Gamma}(\theta) = 1$

(P2) If $\models \neg(\theta \wedge \phi)$ then $w_{*\Gamma}(\theta \vee \phi) = w_{*\Gamma}(\theta) + w_{*\Gamma}(\phi)$

(P3) $w_{*\Gamma}(\exists x\, \psi(x)) = \lim_{r\to\infty} w_{*\Gamma}(\bigvee_{i=1}^{r} \psi(a_i))$

It is clear that $w_{*\Gamma}$ satisfies (P1-2) so we only have to show it satisfies (P3). In terms of w then this requires

$$\begin{aligned}
&\lim_{r\to\infty} w\left(\bigvee_{i=1}^{r} \psi(a_{n+1}, a_{n+2}, \dots, a_{n+m}, a_{n+i}) \wedge \Gamma(a_1, \dots, a_n)\right) \\
&= w(\exists x\, \psi(a_{n+1}, a_{n+2}, \dots, a_{n+m}, x) \wedge \Gamma(a_1, \dots, a_n)) \\
&= \lim_{r\to\infty} w\left(\bigvee_{i=1}^{r} \psi(a_{n+1}, a_{n+2}, \dots, a_{n+m}, a_i) \wedge \Gamma(a_1, \dots, a_n)\right).
\end{aligned}$$

If this was to fail there would have to be some $\beta > 0$ such that for all $r \geq n+1$,

$$\begin{aligned}
&w\left(\bigvee_{i=1}^{r} \psi(a_{n+1}, a_{n+2}, \dots, a_{n+m}, a_i) \wedge \Gamma(a_1, \dots, a_n)\right) \\
&\qquad - w\left(\bigvee_{i=n+1}^{r} \psi(a_{n+1}, a_{n+2}, \dots, a_{n+m}, a_i) \wedge \Gamma(a_1, \dots, a_n)\right) \geq \beta.
\end{aligned}$$

Equivalently

$$w\left(\bigvee_{i=1}^{n} \psi(a_{n+1}, \dots, a_{n+m}, a_i) \wedge \neg\left(\bigvee_{i=n+1}^{r} \psi(a_{n+1}, \dots, a_{n+m}, a_i)\right) \wedge \Gamma(a_1, \dots, a_n)\right) \geq \beta. \tag{1}$$

So certainly we must also have

$$w\left(\bigvee_{i=1}^{n} \psi(a_{n+1}, \dots, a_{n+m}, a_i) \wedge \neg\left(\bigvee_{i=n+1}^{r} \psi(a_{n+1}, \dots, a_{n+m}, a_i)\right)\right) \geq \beta. \tag{2}$$

Purely to simplify matters suppose that $m = 1$ here, so (2) becomes

$$w\left(\bigvee_{i=1}^{n} \psi(a_{n+1}, a_i) \wedge \neg\left(\bigvee_{i=n+1}^{r} \psi(a_{n+1}, a_i)\right)\right) \geq \beta. \tag{3}$$

Now take $r = 1 + (k+1)n$. Then by Ex for w with $0 \leq s < k$, interchanging the block $a_1, a_2, \ldots, a_n$ with $a_{n+1+ns+1}, a_{n+1+ns+2}, \ldots, a_{n+1+n(s+1)}$ in (3) gives that

$$w\left(\bigvee_{i=n+1+ns+1}^{n+1+n(s+1)} \psi(a_{n+1}, a_i) \;\wedge\; \neg\left(\bigvee_{i=1}^{n} \psi(a_{n+1}, a_i) \vee \bigvee_{i=n+1}^{n+1+ns} \psi(a_{n+1}, a_i) \vee \bigvee_{i=n+2+n(s+1)}^{r} \psi(a_{n+1}, a_i)\right)\right) \tag{4}$$

is also at least β. But the sentences in (4) as we vary s are disjoint and there are k of them so making $k > \beta^{-1}$ gives a contradiction, as required.

□

References

[1] Paris, J.B., Pure Inductive Logic, in *The Continuum Companion to Philosophical Logic*, eds. L.Horsten & R.Pettigrew, Continuum International Publishing Group, London, 2011, pp428–449.

[2] Paris, J.B. & Vencovská, A., *Pure Inductive Logic*, to appear in the Association of Symbolic Logic Series *Perspectives in Mathematical Logic*, Cambridge University Press.