A Note on Gaifman's Condition

The purpose of this note is to give a proof of the theorem below related to Gaifman's Condition (P3) in the definition of a probability function in the context of [1].

Theorem 1 *Let $z : SL \to [0,1]$ satisfy (P1-2) and let L^+ be L augmented by additional constants b_i, $i \in \mathbb{N}^+$. Then there is a probability function w on SL^+ satisfying (P1-3) which agrees with z on SL.*

Proof We work in a non-standard universe, more precisely a non-standard ω_1-saturated elementary extension, U^* say, of a sufficiently large portion U of the set theoretic universe containing z, L, $\mathbb{N}$ and anything else that we might need as we go along.

Let ν be a nonstandard natural number in U^*. Working in U^* and measuring the length $|\theta|$ of sentences of $\theta \in SL$ in the sense of U^* in the usual way (with constants getting length 1) let $\mathcal{H}$ be the set of subsets H of $\{\pm\theta \in SL \,|\, |\theta| \leq \nu\}$ such that H is consistent and for each $\theta \in SL$ with $|\theta| \leq \nu$, either $\theta \in H$ or $\neg\theta \in H$. For each $H \in \mathcal{H}$ pick[1], in U^*, a model M_H of H.

Still in U^* list the constants $a_1, a_2, a_3, \ldots$ of L and the new constants $b_1, b_2, b_3, \ldots$ of $L^+ - L$ as

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots$$

and using this ordering of the constants enumerate the sentences of SL^+ which start with $\exists x$ in order of their Gödel numbers, say $\phi_1, \phi_2, \phi_3 \ldots$. Now assign values $\pi(a_i), \pi(b_i) \in M_H$ to these constants as follows: For a_i, $\pi(a_i)$ is just the value given to the constant a_i in M_H. For b_i let $\phi_i = \exists x\, \psi_i(x, \vec{a}, \vec{b})$ and if

$$M_H \models \exists x\, \psi_i(x, \pi(\vec{a}), \pi(\vec{b}))$$

pick $\pi(b_i) \in M_H$ such that

$$M_H \models \psi_i(\pi(b_i), \pi(\vec{a}), \pi(\vec{b})).$$

[1] With a little extra care any use of the Axiom of Choice in this proof can be avoided.

(Notice that ϕ_i can only involve constants to which π has already assigned a value.) If

$$M_H \models \neg\exists x\, \psi_i(x, \pi(\vec{a}), \pi(\vec{b}))$$

just assign $\pi(b_i)$ arbitrarily.

We now have

$$M_H \models \exists x\, \psi_i(x, \pi(\vec{a}), \pi(\vec{b})) \iff M_H \models \psi_i(\pi(b_i), \pi(\vec{a}), \pi(\vec{b})). \quad (1)$$

Returning now to the standard universe U define $w : SL^+ \to [0,1]$ by

$$w(\theta) = {}^o\sum\{z(\bigwedge H) \mid H \in \mathcal{H} \text{ and } M_H \models \theta\}$$

where as usual ^{o} denotes the standard part operation. Clearly w satisfies (P1-2) and furthermore by (1) and the uniformity in the choice of b_i,

$$w(\exists x\, \psi_i(x, \vec{a}, \vec{b})) = w(\psi_i(b_i, \vec{a}, \vec{b})), \quad (2)$$

so Gaifman's condition (P3) holds. Furthermore for standard $\theta \in SL$ one of θ, $\neg\theta$ will be in each the $H \in \mathcal{H}$ so

$$w(\theta) = {}^o\sum\{z(\bigwedge H) \mid H \in \mathcal{H} \text{ and } M_H \models \theta\} = {}^oz(\theta) = z(\theta).$$

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Reference

[1] Paris, J.B. & Vencovská, A., *Pure Inductive Logic,* to appear in the ASL Perspectives in Logic Series, Cambridge University Press, 2013.