Postscript to 'Symmetry's End?'

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Construction of λ for general unary languages

The construction of the automorphism λ in [1] given for a single unary predicate can be summed up by the diagram[1]:

$00 \longrightarrow 0$	$1 \longrightarrow 11$
$01 \longrightarrow 10$	

In this case we have 2 atoms which correspond to 0 and 1 in this scheme. In the case of 3 'atoms', 0,1,2, we can build on this to produce an analogous scheme[2]

$00 \longrightarrow 0$	$1 \longrightarrow 11$	$20 \longrightarrow 202$
$01 \longrightarrow 10$		$210 \longrightarrow 201$
$02 \longrightarrow 200$		$211 \longrightarrow 12$
		$212 \longrightarrow 21$
		$22 \longrightarrow 22$

and from right to left

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[1]Here we use $0, 1$ where in the paper we used $1, 2$.

[2]Of course we are only really interested in the case of 2^n atoms for some n but the inductively defined construction we shall give will need to go through all numbers of 'atoms' between.

$0 \longleftarrow 00$	$10 \longleftarrow 01$	$200 \longleftarrow 02$
	$11 \longleftarrow 1$	$201 \longleftarrow 210$
	$12 \longleftarrow 211$	$202 \longleftarrow 20$
		$21 \longleftarrow 212$
		$22 \longleftarrow 22$

In the case for 4 atoms we first treat 2 and 3 separately just the way 2 was treated in the 3 atom case the *partial* pairings of initial segments from left to right being:

$00 \longrightarrow 0$	$1 \longrightarrow 11$	$20 \longrightarrow 202$	$30 \longrightarrow 303$
$01 \longrightarrow 10$		$210 \longrightarrow 201$	$310 \longrightarrow 301$
$02 \longrightarrow 200$		$211 \longrightarrow 12$	$311 \longrightarrow 13$
$03 \longrightarrow 300$		$212 \longrightarrow 21$	$313 \longrightarrow 31$
		$22 \longrightarrow 22$	$33 \longrightarrow 33$

This is only partial, we need to satisfactorily pair the remaining $\{213, 23, 32, 312\}$ with $\{203, 23, 32, 302\}$. The following works:

$213 \longrightarrow 2311$	$2300 \longrightarrow 203$	$2301 \longrightarrow 2310$
$2302 \longrightarrow 302$	$2303 \longrightarrow 230$	$231 \longrightarrow 2312$
$232 \longrightarrow 232$	$233 \longrightarrow 233$	$312 \longrightarrow 2313$
$32 \longrightarrow 32$		

There is a pattern emerging here. Suppose that we have produced such a pairing $\mathcal{P}_{\{2,3,\ldots,q\}}$ for $q+1$ atoms, $\{0, 1, \ldots, q\}$ say, which acts as above on $00, 01, 1$. Let $\mathcal{Q}$ be the partial pairing

$$\bigcup_{\substack{S \subset \{2,3,\ldots,q+1\} \\ |S|=q-1}} \mathcal{P}_S.$$

It will be clear from what follows that doing this will not destroy the requirement that the pairing be one to one. Assume as part of the inductive hypothesis that the (minimal) unpaired strings in $\mathcal{Q}$ are of the form $a_2 a_3 \ldots a_{q+1}$ and $a_2 a_3 \ldots a_q \, 0 \, a_{q+1}$ on one side and the $a_2 a_3 \ldots a_{q+1}$ and $a_2 a_3 \ldots a_q \, 1 \, a_{q+1}$ on the other side where $a_2, a_3, \ldots, a_{q+1}$ are $2, 3, 4, \ldots, q+1$ in some cyclic order. [So there are q possibilities here.]

Without loss of generality suppose that the strings with 0 appear on the left.

To extend $\mathcal{Q}$ to a full pairing replace the left hand side copy of $2\,3\,4 \ldots (q+1)$ by

$$\begin{array}{ll}
2\,3\,4\ldots(q+1)\,0 & 2\,3\,4\ldots(q+1)\,1\,0\\
2\,3\,4\ldots(q+1)\,2 & 2\,3\,4\ldots(q+1)\,1\,1\\
2\,3\,4\ldots(q+1)\,3 & 2\,3\,4\ldots(q+1)\,1\,2\\
.\ .\ .\ .\ .\ .\ .\ .\ . & .\ .\ .\ .\ .\ .\ .\ .\ .\\
2\,3\,4\ldots(q+1)\,(q+1) & 2\,3\,4\ldots(q+1)\,1\,q\\
 & 2\,3\,4\ldots(q+1)\,1\,(q+1)
\end{array}$$

and similarly on the right hand side to obtain

$$\begin{array}{ll}
2\,3\,4\ldots(q+1)\,1 & 2\,3\,4\ldots(q+1)\,0\,0\\
2\,3\,4\ldots(q+1)\,2 & 2\,3\,4\ldots(q+1)\,0\,1\\
2\,3\,4\ldots(q+1)\,3 & 2\,3\,4\ldots(q+1)\,0\,2\\
.\ .\ .\ .\ .\ .\ .\ .\ . & .\ .\ .\ .\ .\ .\ .\ .\ .\\
2\,3\,4\ldots(q+1)\,(q+1) & 2\,3\,4\ldots(q+1)\,0\,q\\
 & 2\,3\,4\ldots(q+1)\,0\,(q+1)
\end{array}$$

Then the number of left hand side strings containing all of $0, 1, 2, 3, \ldots, q+1$ is just 1 and can be paired with the corresponding right hand side string.

The number of left hand side strings containing just $0, 2, 3, \ldots, q+1$ is $q+1$ whilst the number of left hand side strings containing just $1, 2, 3, \ldots, q+1$ is also $q+1$, so these can be successfully paired with the corresponding strings on the other side.

All the remaining unpaired strings contain each of $2, 3, 4, \ldots, q+1$ and being of equal number on each side can again be paired off.

Finally notice that if we had carried out this whole construction in a 'language' with $q+2$ atoms these last two splits above would have left us with single (unpaired) strings $2\,3\,4\ \ldots, (q+1)\,(q+2)$ and $2\,3\,4\ \ldots, (q+1)\,1\,(q+2)$ on the left and $2\,3\,4\ \ldots, (q+1)\,(q+2)$ and $2\,3\,4\ \ldots, (q+1)\,0\,(q+2)$ on the right thus confirming the inductive hypothesis.

References

[1] Paris, J.B. & Vencovská, A., Symmetry's End? Submitted to *Erkenntnis*.