A note on automorphisms of the Lindenbaum Algebra of SL and BL.

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An Example

The sole purpose of this note is to give and example of an automorphism of the Lindenbaum Algebra of SL which cannot be extended to a BL automorphism. Unexplained terms in this note can be found in [1].

Let $L = \{P\}$ with P unary. Define a map τ on constituents by:

$$\tau\big(\exists x\, P(x) \wedge \exists x\, \neg P(x) \wedge \bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)\big) = \exists x\, P(x) \wedge \exists x\, \neg P(x) \wedge P(a_1)^{1-\epsilon_1} \wedge \bigwedge_{i=2}^{n} P^{\epsilon_i}(a_i),$$

$$\tau\big(\forall x\, P(x)\big) = \forall x\, P(x), \quad \tau\big(\forall x\, \neg P(x)\big) = \forall x\, \neg P(x).$$

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Extend τ to disjunctions of constituents, Υ_i by

$$\tau(\bigvee_{j=1}^{m} \Upsilon_j) = \bigvee_{j=1}^{m} \tau(\Upsilon_j)$$

and hence to the Lindenbaum Algebra by the obvious:[1]

$$\tau\left[\bigvee_{j=1}^{m} \Upsilon_j\right] = \left[\tau(\bigvee_{j=1}^{m} \Upsilon_j)\right].$$

Claim τ is an isomorphism of the Lindenbaum Algebra.

Claim τ does not extend to an automorphism of BL.

The argument here is that suppose it did, so there would be some bijection $\gamma : \mathcal{T}L \to \mathcal{T}L$ such that for $\theta \in SL$,

$$\gamma\{M \in \mathcal{T}L \mid M \vDash \theta\} = \{M \in \mathcal{T}L \mid M \vDash \tau(\theta)\}.$$

In particular then for the $K \in \mathcal{T}L$ such that $K \models \neg P(a_1)$, $K \models P(a_i)$ for all $i > 1$, we would have

$$\gamma(K) \in \bigcap_{j=1}^{\infty} \left[\exists x \, \neg P(x) \wedge \bigwedge_{i=1}^{j} P(a_i)\right] = \emptyset,$$

yielding the required contradiction, since

$$K \vDash \exists x \, \neg P(x) \ \text{ and for all } j, \ K \vDash \neg P(a_1) \wedge \bigwedge_{i=2}^{j} P(a_i),$$

whilst for all $j > 1$,

[1] Following the established convention, in the context of Lindenbaum-Tarski algebras square brackets around a sentence denote the class of sentences logically equivalent to the sentence enclosed by the brackets. However, following the convention introduced in [1], in the context of BL, square brackets around a sentence denote the set of structures from $\mathcal{T}L$ in which the sentence is satisfied.

$$
\begin{aligned}
\tau\big[\exists x\, \neg P(x) \wedge \neg P(a_1) \wedge \bigwedge_{i=2}^{j} P(a_i)\big] & \\
&= \tau\big[\exists x\, P(x) \wedge \exists x\, \neg P(x) \wedge \neg P(a_1) \wedge \bigwedge_{i=2}^{j} P(a_i)\big] \\
&= \big[\exists x\, P(x) \wedge \exists x\, \neg P(x) \wedge \bigwedge_{i=1}^{j} P(a_i)\big] \\
&= \big[\exists x\, \neg P(x) \wedge \bigwedge_{i=1}^{j} P(a_i)\big].
\end{aligned}
$$

References

[1] Paris, J.B. & Vencovská, A., *Pure Inductive Logic*, in the Association of Symbolic Logic Perspectives in Mathematical Logic Series, Cambridge University Press, April 2015.