A multivariate natural probability distribution based on the propositional calculus

by

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Abstract

In this paper we rework the results in our earlier paper [1] using an alternate initial set of connectives.
We show that this yields a more satisfactory limiting prior probability distribution than our earlier paper
and that this distribution extends naturally to more variables. We then consider these multivariate prior
probability distributions in the light of established desiderata.

Introduction and Notation

In this paper we shall continue the investigations into 'natural prior probability distributions' initiated in
[1], [2]. Briefly in these papers we considered the problem of estimating the probability that a 'naturally
encountered' probability function $P$ on the set $SL_1$ of sentences of the proposition language $L_1$ with a
single propositional variable $q_1$ would have $P(q_1)$, equivalently the expected (truth) value of $q_1$, lying in
a real interval $(a, b)$. [Recall that for a language $L_n = \{q_1, ..., q_n\}$ a probability function $P$ on $SL_n$
is determined by its values on the atoms $\alpha_1, ..., \alpha_{2^n}$, that is sentences of the form $\bigwedge_{i=1}^n \pm q_i$ (see [3]).
In particular then in the case $n = 1$ $P$ is determined simply by its value on $q_1$ since the atoms in this case
are just $q_1, \neg q_1$ and $P(\neg q_1) = 1 - P(q_1).$] In another guise then this is the well known problem of picking
a prior in the situation of total ignorance.

Whilst this concept of a 'distribution of naturally occurring probabilities' may be difficult to accept
(we give a detailed discussion on this point in [1]) we can certainly see finite approximations to this
notion if, for example, in a medical context we were to take a histogram of the frequencies of patients
being diagnosed at a particular clinic as suffering from various conditions.

In [1] we developed the idea that naturally occurring probability distributions on $SL_1$ arise as very
complex combinations of basic random processes and modelled these complex combinations as large
sentences of the propositional calculus built up from random, independent, propositional variables $p_i$
using the connectives and ($\land$) and or ($\lor$) (where, as usual, a (truth) value of one corresponds to true and
a value of zero to false). Initially the $p_i$ were taken to be independently distributed with expected (truth)
value $1/2$, equivalently probability $1/2$ of being true, and we showed that as the sizes of the language and
the sentences tended (independently) to infinity so the corresponding probability of a random sentence
having expected value in an interval $(a, b)$ tended to a fixed limit $D(a, b)$. More precisely let $A^n_m$
be the set of sentences formed from the propositional variables $\pm p_1, ..., \pm p_m$ using at most $n$
occurrents of the connectives $\land, \lor$ and for $\theta \in A^n_m$ let $h(\theta)$ be the expected (truth) value of $\theta$ when the propositional

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variables \( p_i \) are assumed to be independently distributed with expected values \( \frac{1}{2} \). Alternatively, in this case, \( h(\theta) \) is the probability that a randomly chosen valuation \( V \) on \( \{p_1, ..., p_n\} \) will satisfy \( \theta \) i.e. \( V(\theta) = 1 \). [So, for example if \( \theta = (p_1 \land p_3) \lor (p_2 \lor p_1) \) then \( h(\theta) = \frac{7}{8} \).] Then we show in [1] that for \( 0 \leq a \leq b \leq 1 \)

\[
D(a, b) = \lim_{n,m \to \infty} \frac{\{\theta \in A^n_m | h(\theta) \in (a, b)\}}{|A^n_m|}
\]

exists (and similarly for closed and half open intervals) and that \( D \) extends to a countably additive measure on the Borel subsets of \([0, 1]\). In order to prove this result we introduced another family of measures \( D_n \) on \([0, 1]\) defined by

\[
D_n(a, b) = \frac{|\{\theta \in A_n | h(\theta) \in (a, b)\}|}{|A_n|},
\]

where \( A_n \) is the set of sentences in \( A_n^{n+1} \) in which exactly the propositional variables \( p_1, ..., p_{n+1} \) appear, in that order, when reading from left to right and show that for \( 0 \leq a \leq b \leq 1 \), \( \lim_{n \to \infty} D_n(a, b) \) exists and equals \( D(a, b) \) (and similarly for closed intervals etc.). The advantage of going via these discrete measures \( D_n \) is that they are simply defined by the recursion

\[
D_0 = U_\frac{1}{2},
\]

\[
D_{n+1} = \sum_{i=0}^{n} \mu_i^{n+1} D_i \ast D_{n-i},
\]

(1)

where, for \( a \in [0, 1] \), \( U_a \) is the (normalised) measure which heaps all the measure on the point \( a \) and places none anywhere else, the operation \( \ast \) is defined on the \( U_a \) by

\[
U_a \ast U_b = \frac{1}{2} (U_{ab} + U_{a+b-ab}),
\]

and is extended to measures \( F = \sum_{i=1}^{n} c_i U_{a_i}, G = \sum_{j=1}^{m} d_j U_{b_j} \) by

\[
F \ast G = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j U_{a_i} \ast U_{b_j},
\]

and \( \mu_i^{n+1} = b_i b_{n-i}/b_{n+1} \), where \( b_n \) is the Catalan Number \( \left( \begin{array}{c} 2n \\ n \end{array} \right) / (n + 1) \). [See [1].] As far as we are able to calculate a histogram of \( D \) resemble a catenary with a hump at \( \frac{1}{2} \), smaller humps at \( \frac{1}{3}, \frac{5}{8} \), and so on. The apparent smoothness here is however misleading since as we show in [1] \( D \) has infinite derivative at infinitely many points and so does not possess a corresponding density function.

Whilst \( D(a, b) \) provides a possible estimate of the probability that a natural 0,1 valued random variable has an expected value in \( (a, b) \) it suffers from the criticism that the original decision to take the \( p_i \) to have expected value \( \frac{1}{2} \) was rather arbitrary. In view of the subsequent developments it would seem more sensible to have started with \( D \) as the distribution for the expected values of the \( p_i \) (rather than the measure which loads everything on \( 1/2 \)). As we show in [1] if we now repeat the process starting with \( D (= D^1, \text{say}) \), where \( D^0 = U_\frac{1}{2} \), we again obtain a limit measure, \( D^1 \), and we can carry on in this fashion to obtain \( D^0, D^1, D^2, ... \) with limit \( D^\infty \). \( D^\infty \) is now above this criticism since, essentially, this same limit \( D^\infty \) will be obtained in this way no matter what initial (symmetric) distribution we impose on the expected values of the \( p_i \). Unfortunately however, \( D^\infty \) is (to within an infinitesimal) simply the discrete distribution which divides all the measure equally between the points 0 and 1. In other words all randomness has disappeared!

This is clearly no longer a satisfactory model for the distribution of 'natural' random variables. However once the above criticism of \( D \) is accepted then it seems hard to propose any other choice from \( D^0, D^1, D^2, ... \) except \( D^\infty \).

A second question which we did not consider in [1] (although it was briefly remarked on in [2]) was how to extend these results from natural univariate probability distributions on \( SL_1 \) to multivariate probability distributions on \( SL_n \). Since (as already mentioned above) a probability function on \( SL_n \) is determined by its values on the atoms...
\[ \{ \bigwedge_{i=1}^{n} q_i^\epsilon | \epsilon_1, ..., \epsilon_n \in \{0, 1\} \} \]

where, as usual, \( q^1 = q, q^0 = \neg q \), this is equivalent to asking for the 'probability' that a randomly encountered probability function \( P \) on \( SL_n \) satisfies

\[ \langle P(\alpha_1), ..., P(\alpha_{2^n}) \rangle \in B \]

where \( B \) is an open (say) subset of

\[ \mathbb{D}_n = \{ <x_1, ..., x_{2^n}> | 0 \leq x_i, \sum x_i = 1 \} \]

This is the question which original motivated the work in [1] and [2] because of its relevance to the theory and evaluation of probabilistic expert systems. In situations where expert systems are customarily envisaged as operating the features, or variables, are frequently highly dependent and attempting to capture, simulate and explain such dependencies is a key requirement of the modeling in this case.

Treated as the problem of picking a prior the most frequent approach (as we have previously remarked in [2] and [4]) is simply to take the uniform distribution on \( \mathbb{D}_n \). Unfortunately this choice suffers from the serious criticism that it does not marginalise. That is, if we assume that all probability functions on \( SL_n \) are equally likely to be encountered and then subsequently restrict our attention to sentences from the sublanguage \( SL_{n-1} \) then we will not retain the uniform distribution, these distributions on \( SL_{n-1} \) will no longer be equally likely. Instead they will be distributed as a certain Dirichlet distribution. From this it follows that if we assume the uniform distribution for \( SL_n \) then we shall have to settle for (different) Dirichlet distributions for any subsequent marginalisations (and similarly if we wish to enlarge the language). Hence if we want marginalisation and the uniform distribution in some \( SL_n \) we are forced to assume a family of Dirichlet distributions. This is clearly unsatisfactory in the context of naturally encountered probability distributions, firstly because there is usually no clear, fixed, number of propositional variables under consideration (for example in a medical context new conditions could appear, or even disappear, at any time) so it would seem perverse for this parameter to play such a crucial role, and secondly because it would seem to be hard to justify the uniform distribution for any one particular language whilst not so doing for other languages.

A second criticism of assuming a Dirichlet distribution (as already remarked in [2]), at least in the sort of contexts where expert systems would normally be seen as being applicable, is that for probability functions \( P \) on \( SL_n \), representative of such situations we would expect the \( P(q_i) \) to be rather variable and the distinct \( q_i, q_j \) to be, at least somewhat, dependent i.e. \( P(q_i \land q_j) - P(q_i)P(q_j) \) should be relatively large. However for the uniform distribution on \( P \) we have the expected values

\[ E((P(q_i) - \frac{1}{2})^2) = \frac{1}{4(2^n + 1)} \]

\[ E(P(q_i \land q_j) - P(q_i)P(q_j))^2 = \frac{2^n}{16(2^n + 3)(2^n + 1)} \]

Essentially then for a random probability function \( P \) chosen according to this uniform distribution we can expect the \( P(q_i) \) to be very close to 1/2 and \( q_i, q_j \) to be practically independent. This is certainly not the sort of fertile ground on which expert systems germinate and thrive. As such then the uniform distribution (and more generally the Dirichlet distributions) seems inappropriate as a model, or source, of the frequency of naturally encountered probability functions, at least as far as the objective testing of expert systems is concerned.

Our plan in this paper is first to rework, and extend, the results of [1], but using instead of the connectives \( \land, \lor \) of that paper the connectives \( \land, \lor, \leftrightarrow, \not \) (where \( p \not q \) is equivalent to \( p \leftrightarrow \neg q \), the dual of \( \leftrightarrow \)). We will show that this has a profound effect on the final distribution \( D^\infty \) that we obtain. [See also the Note at the end of this section.] We shall then go on to develop an idea first proposed in [2] for extending these results to \( SL_n \), and show that these resulting distributions have many desirable properties in this context.

The idea of using the connectives \( \land, \lor, \leftrightarrow, \not \) in place of \( \land, \lor \) was one which was mentioned briefly in the conclusion of [1]. Certainly these two additional connectives would seem to be as justifiable in this context as our original choice since in practice we quite frequently meet situations (e.g. mating,
bonding of magnets) which only produce a positive outcome if the two parties are of the same/different sort. Surprisingly the addition of these two connectives (which for the purposes of this paper actually act as if they were two copies of the same connective) has a significant effect on the nature of the resulting distributions. Essentially the reason for this is that if \( q_i \) has probability \( 1/2 \) (of being true) then \( q_i \leftrightarrow q_j, q_i \Downarrow q_j \) also have probability \( 1/2 \), independent of the probability of \( q_j \). Consequently, as we now form the \( D_n \) for this expanded set of connectives so measure is constantly being 'reduced' to \( 1/2 \) and as a result \( D \) (and also \( D^2, D^3, \ldots \)) comes out to be discrete.

The next section of this paper will be devoted to reworking the results of sections 3 and 5 of [1] up to theorem 5.6 of that paper in the context of our enlarged set of connectives. Since this largely involves essentially reworking old results the reader may wish to simply accept this section on a first reading and only refer back to it as necessary later. Naturally we assume throughout a familiarity with [1].

Note [Added later.] In subsequent work for his MSc thesis, [6], Paul Watton extended the results of this paper to a general weighted combination of \( \land \lor \) and \( \leftrightarrow / \updownarrow / \downarrow \) connectives. A brief survey of his conclusions, which are much in line with what one would have expected on the basis of the results in the main body of this paper, are summarised in Appendix B.

Reworking with the Additional Connectives

To avoid a proliferation of notation we shall use the same symbols \( \Lambda^m_n, \Lambda^m_n, D_n \) etc as in [1] but now applying to the new enlarged set of connectives. [One possible point of confusion that we should remove however concerns the propositional languages we are using. In [1] the sentences in \( \Lambda^m_n \) etc used the propositional variables \( p_i \) and we shall continue to use \( p_i \) for this purpose (albeit starting from \( p_1 \) rather than \( p_0 \)), where necessary denoting the language with just the propositional variables \( p_1, \ldots, p_m \) by \( L \{ p_1, \ldots, p_m \} \) and the sentences from this language by \( SL \{ p_1, \ldots, p_m \} \). At the same time, as already indicated, we shall be describing natural prior probability distributions on the probability functions on the sentences of a propositional languages \( L_n \) with the \( n \) propositional variables \( q_1, \ldots, q_n \).]

Thus \( \Lambda^m_n \) is now the set of sentences formed from \( \pm p_i, i = 1, \ldots, m \) using at most (exactly) \( n \) choices of connectives from \( \land, \lor, \leftrightarrow, \downarrow \). Hence \( \Lambda^m_n \) is now \( m^{n+1} 2^n n \frac{1}{n+1} \), rather than \( m^{n+1} 2^n n \frac{1}{n+1} \) as in [1]. The measure \( D \) (assuming for the moment it exists) is again defined by

\[
D(a,b) = \lim_{n,m \to \infty} \frac{|\{ \theta \in \Lambda^m_n | h(\theta) \in (a,b) \}|}{|\Lambda^m_n|}
\]

and the \( D_n \) by

\[
D_n(a,b) = \frac{|\{ \theta \in \Lambda_n | h(\theta) \in (a,b) \}|}{|\Lambda_n|}.
\]

Again the \( D_n \) may usefully also be defined by the same recursion (1) except that now the + operation also incorporates these two new connectives. I.e. Precisely we define the operations \( \land, \lor, \leftrightarrow, \downarrow \) and \( + \) first on the \( U_a \) by

\[
\begin{align*}
U_a \land U_b &= U_{ab} \\
U_a \lor U_b &= U_{a+b} & U_a \uparrow U_b &= U_{a+b-\downarrow} \\
U_a \uparrow U_b &= \frac{1}{4}(U_a \land U_b + U_a \lor U_b + U_a \leftrightarrow U_b + U_a \Downarrow U_b) \\
\end{align*}
\]

noindent and then on \( F, G \in F, F = \sum_{i=1}^{n} c_i U_{a_i}, \ G = \sum_{j=1}^{m} d_j U_{b_j} \) by

\[
F \oplus G = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i d_j U_{a_i} \oplus U_{b_j}
\]

where \( \oplus \) is any of \( \land, \lor, \leftrightarrow, \downarrow, \uparrow \). I.e. In particular then

\[
F \ast G = \frac{1}{4}(F \land G + F \lor G + F \leftrightarrow G + F \Updownarrow G).
\]

\[
F \ast G = \frac{1}{4}(F \land G + F \lor G + F \leftrightarrow G + F \Downarrow G).
\]
Our plan at this point is to run through the key lemmas of [1] regarding the $D_n$ and indicate how they are altered by the addition of these additional connectives. Essentially the new feature that the connectives $\leftrightarrow, \not\leftrightarrow$ introduce is that because

$$U_{\frac{1}{2}} \leftrightarrow U_{\frac{1}{2}} = U_{\frac{1}{2}} \not\leftrightarrow U_{\frac{1}{2}} = U_{\frac{1}{2}}$$

in the formation of the $D_n$ the point 1/2 is constantly attracting new measure to itself. An immediate result of this is that Lemma 3.2 of [1] now becomes

$$D_n(a) > 0 \iff a = \frac{i}{2^{n+i}}, \text{ for some } 0 < i < 2^{n+1}. $$

For the new connectives we have, as an extension of Lemma 3.3 of [1], that for $F, G \in F$,

$$F \leftrightarrow G(a) = F \not\leftrightarrow G(1-a) = F \leftrightarrow G(1-a),$$

and Lemma 3.4(iii) of that paper changes to

$$E_{F \leftrightarrow G}(x^k) = \frac{1}{4}(1 + (-1)^k)E_F(x^k)E_G(x^k) + \sum_{r=0}^{k-1} \frac{1}{4}(-1)^r E_F(x^r)E_G(x^{k-r}) \binom{k}{r}$$

$$+ \frac{1}{2} \sum_{r=0}^{k-1} E_F(x^r(1-x)^{k-r})E_G(x^r(1-x)^{k-r}) \binom{k}{r}. \quad (4)$$

Notice that the coefficient of $E_F(x^k)$ on the right hand side is

$$\frac{1}{4}(1 + (-1)^k)E_G(x^k) + \frac{1}{2}E_G((x - (1-x))^k). \quad (5)$$

Since this expression is bounded above by $\frac{1}{2}(E_G(x^k) + 1)$, which is strictly less than 1 for $0 \leq k$, Lemma 3.7(ii) of [1] still holds since in place of the expression in Lemma 3.7(i) we now have

$$|E_{D_{n+1}}(x^k) - E_{D_{n+1}}(x^k)| \leq \sum_{i=0}^{d} 2\mu_i \left( \frac{1}{2} E_{D_i}(x^k) + \frac{1}{2} \right) \cdot |E_{D_{n-1}}(x^k) - E_{D_{n-1}}(x^k)| + \epsilon. \quad (6)$$

Whilst only minor modifications are required to amend the proof of Lemma 3.7 of [1] the following lemma, Lemma 3.8, is more substantially altered by the introduction of the connectives $\leftrightarrow$ and $\not\leftrightarrow$. The difficulty arises because, as already mentioned, in the construction of the $D_n's$ the point 1/2 (and hence all its 'descendants') gathers measure to itself, a phenomenon which was not seen for $\land, \lor$. Thus each $D_{n+1}(1/2)$ will have a contribution from $D_0$ to $\mu_{n+1}^{-1}D_0(1/2)$ (which tends in the limit to 1/4 as $n \to \infty$) and further contributions from $D_1(1/2), D_2(1/2), \ldots$. As a result the limit measure $D$ (assuming it exists) will be not continuous in the sense of Theorem 3.10 of [1].

Instead of Lemma 3.8 we need to consider rather more specialised intervals than just $(a - \epsilon, a + \epsilon) \cap [0, 1]$, in particular the nature of the point $a$ is now relevant.

We call $a \in [0, 1]$ a bad point if $D_n(a) > 0$ for some $n$. Otherwise $a$ is good. [So by an earlier remark the bad points are the $i/2^n$ for $0 < i < 2^n, n > 0$.]

Now consider $D_j \ast D_{i-n-1}[a, b]$, where $0 \leq a \leq b \leq 1$.

For each $\epsilon$ such that $D_j(\epsilon) > 0$ this receives 'contributions'
and similarly for \((a, b), (a, b)\) etc.

Much as in the proof of Lemma 3.8 of [1] we can now show that for any \(a \in [0, 1]\) and \(\delta > 0\) there exists \(\epsilon > 0\) and \(n_0\) such that for all \(n \geq n_0\)

\[
D_n((a, a + \epsilon) \cap [0, 1]) < \delta,
\]

\[
D_n((a - \epsilon, a) \cap [0, 1]) < \delta.
\]

For example, in the case of first of these two inequalities, we look at the contribution made to, say,

\[
D_j \ast D_{i_{n-j-1}}((a, a + 1/(n+1)) \cap [0, 1])
\]

by \(D_j(\epsilon) > 0, j = 0, \ldots, d\). We may assume that if \(D_j(\epsilon) > 0\) with \(0 \leq j \leq d\) then \(\epsilon \notin (a, a + 1/(n+1)) \cap [0, 1]\), in which case all the contributing intervals listed above are small and for any particular such \(\epsilon\) (as in the proof of Theorem 3.8) at most one of the contributing intervals for \(\Lambda\) is non-empty. In this case the proof runs as before.

For a good a similar result also holds for \((a - \epsilon, a + \epsilon)\). In the case of a bad, instead of Lemma 3.8, we show that for all \(\delta > 0\) there exists \(n_0\) such that for all \(n, m > n_0\),

\[
|(D_n - D_m)(a)| < \delta.
\]

Again we suppose that this fails for some \(a\) and that \(i_n, i'_n\) are unbounded sequences such that

\[
|(D_{i_n} - D_{i'_n})(a)| \geq \delta,
\]

but that there exists no \(b\) and no such sequences \(j_n, j'_n\) for which

\[
|(D_{j_n} - D_{j'_n})(b)| \geq \delta(1 + \nu)
\]

for some suitably small \(\nu > 0\). In other words there is an \(N_b\) such that for \(j, j' \leq N_b\),

\[
|(D_j - D_{j'})(b)| < \delta(1 + \nu).
\]

Again we find that for \(D_j(\epsilon) > 0\), \(0 \leq j \leq d\), the only way both intervals for \(\Lambda\) and \(\vee\) can contribute to

\[
|(D_j \ast D_{i_{n-j-1}} - D_j \ast D_{i'_n-j-1})(a)|
\]

(7)

is if \(a = e\), in which case the contributing 'intervals' are \(\{0\}, \{1\}\) which are not bad, and so do not contribute after all! As for the other contributions to (7), arising from \(\leftrightarrow\) and \(\Downarrow\), if \(a \neq 1/2\) these will again come out to be single points and a proof similar to that of Theorem 3.8 of [1] can again be pushed through.

Turning finally to the case \(a = 1/2\), the contributions from \(\leftrightarrow\) and \(\Downarrow\) to (7) from \(D_j(1/2), 0 \leq j \leq d\), are

\[
D_j(1/2) \cdot |D_{i_n-j-1}(0, 1) - D_{i'_n-j-1}(0, 1)| = 0,
\]

whilst from \(D_j(b), 0 \leq j \leq d, b \neq 1/2\) they are

\[
D_j(b) \cdot |D_{i_n}(1/2) - D_{i'_n}(1/2)|
\]

so the above method of proof again goes through.

By using these results as in Theorem 3.9 of [1] we can now show that for \(a, b\) good the limit as \(n \to \infty\) of \(D_n(a, b)\) exists. For, say, a bad \(b\) good we can consider the limiting behaviour of \(D_n(a, a + \epsilon), D_n(a + \epsilon, b)\) to again show that the limit must exist (and similarly for closed intervals etc., etc.).

A key difference, of course, in this case is that we may have \(D(a) > 0\), although it is now easy to see that this holds exactly for the bad \(a\). Also for any \(a \in [0, 1]\),

\[
\lim_{\epsilon \to 0} D(a, a + \epsilon) = \lim_{\epsilon \to 0} D(a - \epsilon, a) = 0
\]

so \(D\) is still countably additive.

Turning to the connection between \(D\) and \(S\) in this case, we can, as in Lemma 2.6 of [1], show that for suitably large \(j, q, q\) for all \(m, n\) eventually, almost all \(\theta \in A_\alpha^n\) have the form

\[
*1(\phi_1, *2(\phi_2, *3(\phi_3, \ldots, *3j(\phi_{3j}, \eta) \ldots)))
\]
with the \( \ast, \in \{ \land, \lor, \leftrightarrow, \supset \} \), \( \phi_i \in A^m_n \) with no propositional variable in common. From this it follows, for large \( j \), that almost all \( \theta \in A^m_n \) have the form

\[
\ast_1(\phi_1, \ast_2(\phi_2, \ast_3(\phi_3, ..., \ast_r(\phi_r, ...))).
\]

where the \( \phi_i \in A^m_n \), \( \ast \in \{ \land, \lor \} \) and exactly \( j \) of \( \ast_1, ..., \ast_r \) are \( \land \) or \( \lor \). We now let \( Z^m_n \) be this set of \( \theta \) and, as in the proof of Lemma 2.9 in [1], use this form of \( \theta \) to define \( \overline{\theta} \) (keeping the remaining \( \ast_i \), which must be of the forms \( \leftrightarrow, \supset \), fixed). In this case the proof runs as before to show that

\[
\lim_{n, m \to \infty} |E_{D_n}(x^k) - E_{D_n^m}(x^k)| = 0.
\]

Unfortunately, however, since \( D \) is not continuous at bad points this only suffices (as in the proof of Theorem 3.11 of [1]) to conclude that for good \( a, b \)

\[
\lim_{n, m \to \infty} D^m_n(a, b) = D(a, b),
\]

(and similarly for \( [a, b], (a, b], [a, b) \)) where \( D^m_n \) is the measure on \([0, 1]\) defined by

\[
D^m_n(a, b) = \frac{[\theta \in A^m_n \mid h(\theta) \in (a, b)]}{|A^m_n|}.
\]

What (possibly) may happen for bad \( a \) is that some of the measure \( D(a) \) moves only infinitessimally close to \( a \) in this limit. Nevertheless for practical purposes we may, we believe, justifiably identify \( D \) with this limit.

[Aside. We point out the following relevant result from measure theory at this point (see for example [5]). Suppose that the countably additive measures \( M_n \) on (the Borel subsets of) \([0, 1]\) are such that for each \( k \), \( \lim_{n \to \infty} E_{M_n}(x^k) \) exists. Then there is a unique countably additive measure \( M \) on \([0, 1]\) such that for all \( k \)

\[
\lim_{n \to \infty} E_{M_n}(x^k) = E_M(x^k),
\]

called the weak limit of the \( M_n \) and satisfying that

\[
M[a, b] = \lim_{\epsilon \to 0} \lim\sup_{n} M_n(a - \epsilon, b + \epsilon) = \lim\inf_{n} M_n(a - \epsilon, b + \epsilon).
\]

This result then says that the weak limit of the \( D^m_n \) exists and equals \( D \).

The extension of the * operator from \( F \), the symmetric measures of the form \( \sum_{i=1}^{n} c_i U_{a_i} \), to \( H \), the symmetric countably additive measures on \([0, 1]\) now proceeds just as in [1] and we can again prove the analogues of Theorem 5.1 through to Corollary 5.5 of that paper. As for Theorem 5.6 we again (apparently) run into the problem that the limit \( T \) may only be infinitessimally close to \( D \). Again however we can conclude that \( D \) is the weak limit of the \( T_n \). Finally, by using the identities \( W \ast W = 2W - D_0 \), \( D = W \ast D \), we can again obtain a recursive formula for the \( E_{D_n}(x^k) \) and hence obtain approximations to \( D \), albeit of a rather more complicated form than previously.

We now turn to the analogue of Theorem 5.7 of [1] (adapting the nomenclature already introduced earlier when discussing the analogue of Lemma 3.8). First notice that counting up the contribution to \( T_{n+1}(0) \) gives

\[
T_{n+1}(0) = \frac{1}{4} \sum_{i=0}^{n} \frac{\mu_{i+1}^n}{(T_i(0) + T_{n-i}(0) + 4T_i(0)T_{n-i}(0))} = \sum_{i=0}^{n} \frac{\mu_{i+1}^n}{2} (T_i(0) + 2T_i(0)T_{n-i}(0)).
\]

Now let \( d \) be such that \( \sum_{i=0}^{d} 2\mu_i > 1 - \epsilon \) and \( n_0 \) so large that for \( n \geq n_0 \), \( |\mu_{i+1}^n - \mu_i| < \frac{\epsilon}{d} \) for all \( i \leq d \). Then for \( n, m \geq n_0 \),

\[
|T_{n+1}(0) - T_m(0)| - \sum_{i=0}^{d} 2\mu_i T_i(0)(T_{n-i}(0) - T_{m-i}(0)) < 4\epsilon.
\]

Since \( \sum_{i=0}^{d} 2\mu_i T_i(0) \leq \frac{1}{2} \), by considering \( \lim\sup_{n, m} |(T_n - T_m)(0)| \) we see that \( \lim_{n \to \infty} T_n(0) \) exists.

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Having established this notice that for $0 \leq a < 1$ out of each pair of contributions to $T_{n_0}^0(a, a + \epsilon)$,

$$2\mu_j^* T_j^{0(m)}(c) T_{i_n-j-1}^0(a, a + \epsilon), \quad 2\mu_j^* T_j^{0(m)}(c) T_{i_n-j-1}^0 \left( \frac{a - \epsilon}{1 - \epsilon} \right) - \frac{a + \epsilon - \epsilon}{1 - \epsilon},$$

arising from $\wedge, \vee$ and $T_j^{0(m)}(c) > 0$ (notice that having already fixed $d$ as in the proof of Theorem 5.7 in [1] we can fix $m$ large) at least one of these will be null (even if $e = 0, 1$) and proceeding as in [1] gives that for all $\delta > 0$ there is $\epsilon > 0$ and $n_0$ such that for all $n \geq n_0$, $T_n^0(a, a + \epsilon) < \delta$. Similarly, of course, for $(a - \epsilon, a)$, and also for $(a - \epsilon, a + \epsilon)$ provided $0 < a < 1$ and $a$ is good (i.e. in this context no $T_r^{0(m)}(a)$ is non-zero).

Turning now to the limit of the $T_n^0(a)$ for bad $a$ (in this sense), $a \neq 1/2$, we proceed as earlier to consider $|(T_n^{0(r)} - T_m^{0(r)})(a)|$, again supposing that there was an unbounded sequence $i_n, i'_n, i''_n$ (with $i''_n$ sufficiently large compared with $i_n, i'_n$) such that

$$|(T_n^{0(i''_n)} - T_m^{0(i''_n)})(a)| \geq \delta \quad \text{etc}.$$

For $T_j^{0(m)}(c) > 0$ the only way the intervals for both $\wedge, \vee$ can contribute to

$$|(T_j^{0(i'_n)} \ast T_{i_n-j-1}^{0(i'_n)} - T_j^{0(i''_n)} \ast T_{i_n-j-1}^{0(i''_n)})(a)|$$

is if $a = c$, in which case the contributing intervals are $\{0\}, \{1\}$. But for $i_n, i'_n$ large $T_{i_n-j-1}^{0(i'_n)}(0)$, $T_{i_n-j-1}^{0(i'_n)}(0)$ and $T_{i_n-j-1}^{0(i''_n)}(1)$, $T_{i_n-j-1}^{0(i''_n)}(1)$ are, as we showed above, very close, so the contribution from these to (8) can be made as small as we wish and the usual argument goes through. Finally for $a = 1/2$ the earlier analysis as for the $D_0$ can be applied to show that even here $\lim_{n,m \to \infty} T_{i_n}^{0(m)}(1/2)$ exists.

By showing (as for the $D_0$) that $\lim_{n,m \to \infty} E_{T_n^{0}}(x^k)$ exists for all $k$ we can go on to show that $T_n^0$ is symmetry, countably additive and satisfies

$$T_{n+1}^0 = \sum_{i=0}^{\infty} 2\mu_i^* T_i^0 \ast T_0^0.$$

Of course this effectively suffices to show that all the $T_n^0$ are well defined countably additive symmetric measures.

Notice that by analogy with the results in [1] we can now show that the $T_n^0$ satisfy

$$T_{n+1}^0 = W^n \ast T_0^{n+1}, \quad W^n \ast W^n = 2W^n - T_0^n.$$

The Limit Distribution J

To briefly recall that the results of the previous section showed that if we start with a measure $T \in \mathcal{H}$

and define

$$T_{n+1} = \sum_{i=0}^{n} \mu_i^{n+1} T_i \ast T_{n-i},$$

where the operation $\ast$ incorporates the connectives $\wedge, \vee, \leftrightarrow, \downarrow$, then the limit $T^1$ of the $T_n$ exists and is in

$\mathcal{H}$ (i.e. is countably additive and symmetric). Furthermore an inessential generalisation of the extension of Lemma 3.12 given in the previous section shows that $T^1$ is the weak limit of the measures which on an interval $I$ give value

$$\frac{\{ \theta \in A_{\xi}^n \mid h(\theta) \in I \}}{|A_{\xi}^n|},$$

where $h(\theta)$ is the expected (truth) value of $\theta$ (equivalently the probability that $\theta$ is true) when the expected values of the $p_i$ are independently and identically distributed according to $T$. 

8
As in [1] we will be particularly interested in the case where

\[ D_0 = D^*_0 = U_\frac{1}{4}, \]

\[ D^k_{n+1} = \sum_{i=0}^n \mu_i D^k_1 + D^k_{n-i}, \]

\[ D^{k+1} = D^{k+1}_0 = \lim_{n \to \infty} D^k_n. \]  

(11)

In this case (9) becomes,

\[ D^{n+1} = \sum_{i=0}^\infty 2\mu_i D^n_i * D^{n+1} \]

(12)

Recalling our criticisms of the corresponding measures in [1], where the weak limit of the \( D_n \) (or in fact from any starting point \( T_0^0 \in \mathcal{H} \)) turned out to be \( \frac{1}{2}(U_0 + U_1) \) we will now show that the (weak) limit of the \( D_n \) exists in our present situation and is non-trivial. We will then go on to show that this weak limit is the same for any non-trivial starting measure \( T_0^0 \in \mathcal{H} \).

In the case considered in [1] it was easy to show, by considering second moments, that the ultimate weak limit must be the trivial measure \( \frac{1}{2}(U_0 + U_1) \). In our present situation however the corresponding limit will not simply be determined by a single moment and we shall have to work rather harder to show that it exists. Before embarking on this task however we shall prove a result about the form of the intermediate \( D_n \) which is of independent interest.

For \( T \in \mathcal{H} \) let,

\[ m(T) = \sum_{T(a) > 0} T(a), \]

so \( m(T) \) is the discrete measure in \( T \). Notice that it follows from the results in the previous section that the only points \( a \in [0, 1] \) at which we can have \( D^k_n(a) > 0 \) are of the form \( i/2^{n+1} \) for some \( 0 < i < 2^{n+1} \).

**Theorem 1** (i) \( D^k_n(1/2) > 0 \) for all \( k, n \).

(ii) \( \lim_{n \to \infty} D^k_n(1/2) = 0 \).

(iii) \( m(D^k_n) = 1 \) for all \( k, n \) (i.e. the \( D^k_n \) are are completely discrete).

(iv) \( \lim_{n \to \infty} D^k_n(a) = 0 \) for all \( a \in [0, 1] \).

**Proof.** (i) Clearly if \( D^k_0(1/2) > 0 \) then \( D^k_n(1/2) > 0 \) for all \( n \) so it suffices to prove by induction on \( k \) that \( D^k_n(1/2) > 0 \). This is immediate for \( k = 0 \) so assume \( D^k_0(1/2) > 0 \). Then by direct calculation using (12),

\[ D^{k+1}(1/2) = \sum_{i=0}^\infty \mu_i (D^k_i(1/2) + D^{k+1}_i(1/2) - D^k_i(1/2)D^{k+1}_i(1/2)) \]

since \( 1/2 \) can never be of the form \( ab \) or \( a + b - ab \) for \( a, b \in [0, 1] \) of the form \( j/2m + 1 \) for some \( 0 < j < 2^m+1 \) (recall that if \( D^k_i(c) > 0 \) then \( c = j/2^{k+1} \) for some such \( k, j \)) and can only be of the form \( a(b+1) - ab \) or \( a(1-b) + (1-a) \) for such \( a, b \) in the case either \( a = 1/2 \) or \( b = 1/2 \).

Simplifying gives,

\[ D^{k+1}(1/2) \cdot \left( \frac{1}{2} + \sum_{i=0}^\infty \mu_i D^k_i(1/2) \right) = \sum_{i=0}^\infty \mu_i D^k_i(1/2) \]

(13)

so \( D^{k+1}(1/2) > 0 \) as required.

(ii) We show by induction on \( k \) that \( D^k(1/2) \leq 1/(k+1) \) for all \( k \). This is true for \( k = 0 \) so assume it holds for \( k \). Then \( D^k_n(1/2) \leq 1/(k+1) \) for all \( n \) since if \( D^k_0(1/2) \leq 1/(k+1) \) for \( i \leq n \) then by direct calculation from (11),

\[ D^k_{n+1}(1/2) = \sum_{i=0}^n \frac{1}{2} \mu_i D^k_i(1/2) + D^k_{n-i}(1/2) - D^k_i(1/2)D^k_{n-i}(1/2) \]

\[ \leq \sum_{i=0}^n \frac{1}{2} \cdot \mu_i D^k_i(1/2) = \frac{1}{k+1} \]

Hence, using (13),
\[ D^{k+1}(1/2) = \left( \frac{1}{2} + \sum_{i=0}^{\infty} \mu_i D_i^k(1/2) \right)^{-1} \cdot \sum_{i=0}^{\infty} \mu_i D_i^k(1/2) \]

\[ \leq \left( \frac{1}{2} + \sum_{i=0}^{\infty} \frac{\mu_i}{k+1} \right)^{-1} \cdot \frac{1}{2(k+1)} = \frac{1}{2} \cdot \frac{1}{2(k+1)} = \frac{1}{k+2}, \]

as required.

(iii) To show that \( m(D^k_n) = 1 \) for all \( n, k \) it suffices to show that \( m(D^k) = 1 \) for all \( k \) since clearly if \( m(D^k) = 1 \) then \( m(D^k_n) = 1 \) for all \( n \). We prove this by induction on \( k \). Clearly it holds for \( k = 0 \). Now assume \( m(D^k) = 1 \). Then by direct calculation using (12),

\[ m(D^{k+1}) = \sum_{i=0}^{\infty} \mu_i [m(D^k_i)m(D^{k+1}) + (m(D^k) - D^k_i(1/2))(m(D^{k+1}) - D^{k+1}(1/2))] \]

\[ + (D^k_i(1/2) + D^{k+1}(1/2) - D^k_i(1/2)D^{k+1}(1/2)) \]

where the first terms in this sum come from \( \wedge, \vee \) and the later terms from \( \leftrightarrow, \dagger \). Putting \( m(D^k_i) = 1 \) and simplifying gives

\[ m(D^{k+1}) = \sum_{i=0}^{\infty} 2\mu_i [m(D^k_i) + \frac{1}{2} D^k_i(1/2)(1 - m(D^{k+1}))] \]

and hence, by (1), \( m(D^{k+1}) = 1 \), as required.

(iv) Clearly it is sufficient to prove this for \( a \in [0, 1] \) of the form \( j/2^{s+1} \), \( j \) odd. By (ii) it is enough to show that there is a number \( p(a) > 0 \) such that for all \( k \), \( D^k(a) \leq p(a)D^k(1/2) \). The proof of this is by induction on \( s \) (as above). It is clearly true if \( s = 0 \) (since then \( a = 1/2 \)) so assume the result for \( b \) of the form \( r/2^{s+1} \) where \( q < s \) and \( 0 < r < 2^{s+1} \), \( r \) odd. Now it is straightforward, but rather tedious, to check that for \( a \neq 1/2 \), if \( a \) equals one of \( bc, b + c - bc, bc + (1 - b)(1 - c), b(1 - c) + c(1 - b) \), with \( b, c \) of the form \( r/2^{s+1} 0 < r < 2^{s+1}, r \) odd, then \( q \) must be less than \( s \). Hence there are only finitely many such pairs \( \langle b, c \rangle, r = 1, \ldots, j \) giving \( a \) (by at least one of these expressions) and for each of these both \( p(b), p(c) \) exist. Thus

\[ D^k_{n+1}(a) \leq \sum_{i=0}^{n} \mu_i^{n+1} \sum_{r=1}^{j} D^k_i(b_r)D^k_{n-i+1}(c_r) \]

\[ \leq 2 \sum_{r=1}^{j} p(b_r)p(c_r) \sum_{i=0}^{n} \mu_i^{n+1}D^k_i(1/2)D^k_{n-i+1}(1/2) \]

and we may take \( p(a) = 2 \sum_{r=1}^{j} p(b_r)p(c_r) \). Finally, by taking limits we see that by the analogue to Theorem 5.7 given in the previous section this \( p(a) \) also works for \( D^k_{n+1}(a) \).

We now turn to proving that the weak limit of the \( D^n \) exists, to which end the following rather technical lemma will be very useful.

**Lemma 1** Suppose that \( T_1, T_2, W \in \mathcal{H} \) and \( E_W(x^2) \leq 31/100 \). Then for \( k \geq 3 \),

\[ |E_{T_1 \cdot W}(x^k) - E_{T_2 \cdot W}(x^k)| \leq \frac{91}{200} |E_{T_1}(x^2) - E_{T_2}(x^2)| + P_k(T_1, T_2), \]

where \( P_k(T_1, T_2) \) is a linear polynomial in the \( |E_{T_1}(x^r) - E_{T_2}(x^r)| \), \( r = 0, \ldots, k - 1 \).

**Proof.** By (4),(5), and noticing that \( E_W(x^r) \leq 1 \) for all \( r \), we have

\[ |E_{T_1 \cdot W}(x^k) - E_{T_2 \cdot W}(x^k)| \leq \frac{1}{2} |E_{T_1}(x^k) - E_{T_2 \cdot W}(x^k)| \cdot E_W(x^k) \]

\[ + \frac{1}{2} |E_{T_1}(x^k) - E_{T_2}(x^k)| \cdot |E_W(2x - 1)^k| \]

\[ + P_k(T_1, T_2). \]

If \( E_W(x^2) \leq 31/100 \) then for \( k \geq 3 \),

\[ E_W(x^k) \leq E_W(x^3) = \frac{1}{2} E_W(x^3) + E_W((1 - x)^3) = \frac{3}{2} E_W(x^2) - \frac{3}{2} E_W(x) + \frac{1}{2} \leq 43/200, \]
since \( W \) is symmetric, and
\[
|E_W((2x - 1)^k)| \leq E_W((2x - 1)^2) = 4E_W(x^2) - 4E_W(x) + 1 \leq 48/200,
\]
from which the result follows.

\[\text{Theorem 2}\]
Let \( T^0_0 \in \mathcal{H} \) and \( E_{T^0_0}(x^2) < 1/2 \). Then the sequence \( E_{T^n_0}(x^2) \) (defined as in (11) with \( \mathcal{T} \) in place of \( D \)) is monotone increasing to limit 3/10 if \( E_{T^n_0}(x^2) \leq 3/10 \) and monotone decreasing to limit 3/10 if \( E_{T^n_0}(x^2) \geq 3/10 \).

\[\text{Proof.}\]
Let \( t_n = E_{T^n_0}(x^2) \), \( w_n = E_{W^n}(x^2) \) where \( W_n = \sum_{i=0}^{\infty} 2\mu_i T^n_i \). Then by using the identities (10), i.e.
\[
T^n_{n+1} = W^n \ast T^n_{n+1},
\]
\[
W^n \ast W^n = 2W^n - T^n_0,
\]
with (4) and simplifying we obtain that
\[
t_{n+1} = \frac{5t_{n+1}w_n}{2} - \frac{t_{n+1}}{2} - \frac{w_n + 3}{8},
\]
\[
5\frac{w_n^2}{2} - w_n + \frac{3}{2} = 2w_n - t_n.
\]
(14)

Hence \( w_n = \frac{3}{8} - \sqrt{\frac{21}{100} - \frac{2t_n}{5}} \) and a little elementary calculus shows that \( t_n \leq 3/10 \) just if \( w_n \leq 3/10 \). Now let \( u_n = 3/10 - w_n \), and \( s_n = 3/10 - t_n \). Notice that since \( w_n \) is a second moment \( w_n \leq 1/2 \) and \( u_n \geq -1/5 \). Then
\[
s_n = \frac{u_n}{2}(3 + 5u_n), \quad u_n = -\frac{3}{10} + \sqrt{\frac{9}{100} - \frac{2s_n}{5}}, \quad s_{n+1} = \frac{u_n}{3 + 10u_n}.
\]

If \( s_n \geq 0 \) (\( \leq 0 \)) then \( u_n \geq 0 \) (\( \leq 0 \)) so \( s_{n+1} \geq 0 \) (\( \leq 0 \)) and it follows that the \( s_n, u_n \) have the same sign, equivalently \( w_n, t_n \) both lie on the same side of 3/10 for all \( n \). By momentarily treating the \( t_n, w_n \) as continuous variables we obtain that \( dt_{n+1}/dw_n, dw_n/dt_n > 0 \) so that if \( t_{n-1} \leq t_n \) (\( t_{n-1} \geq t_n \)) then \( w_{n-1} \leq w_n \) (\( w_{n-1} \geq w_n \)) and \( t_n \leq t_{n+1} \) (\( t_n \geq t_{n+1} \)). Hence to show that the \( t_n \) and \( w_n \) are increasing if \( t_0 \leq 3/10 \) and decreasing if \( t_0 \geq 3/10 \) it is sufficient to notice that since \( t_0 \) is a second moment \( 1/4 \leq t_0 < 1/2 \) so \( t_1 \geq t_0 \), equivalently
\[
\frac{3}{20} + \sqrt{\frac{21}{100} - \frac{2t_0}{5}} \geq 5t_0 \sqrt{\frac{21}{100} - \frac{2t_0}{5}},
\]
holds just if \( t_0 \leq 3/10 \).

From these observations it follows that the limits of the \( t_n, u_n \), say \( t, w \), exist. Taking the limit of the identities in (14) shows that \( t = w = 3/10 \).

\[\text{Theorem 3}\]
Let \( T^0_0 \in \mathcal{H} \). Then for each \( k \), \( \lim_{n \to \infty} E_{T^n_0}(x_k) \) and \( \lim_{n \to \infty} E_{W^n}(x_k) \) exist.

\[\text{Proof.}\]
The result is clear if \( k < 2 \) and if \( E_{T^n_0}(x^2) = 1/2 \) so assume \( k \geq 2 \) and \( E_{T^n_0}(x^2) < 1/2 \) (the only remaining option since \( E_{T^n_0}(x^2) \) can be at most 1/2). For \( k = 2 \) the result now follows from Lemma 2 so we may assume that \( k \geq 3 \) and, by induction, that we already have the result for the \( E_{T^n_0}(x^r) \), \( E_{W^n}(x^r) \) with \( r < k \).

Since \( 3/10 < 31/100 \), by using Lemma 2 (and ignoring some initial terms if necessary), we may assume that for all the \( T^0_0, W^n \) we have that \( E_{T^n_0}(x^2), E_{W^n}(x^2) < 31/100 \). Then by Lemma 1 and (4),(5),(10),
\[ |E_{T_{n+1}}(x^k) - E_{T_n}(x^k)| = |E_{W^{*}R_{n+1}}(x^k) - E_{W^{*}R_n}(x^k)| \]
\[ \leq |E_{W^{*}R_{n+1}}(x^k) - E_{W^{*}R_n}(x^k)| + |E_{W^{*}R_n}(x^k) - E_{W^{*}T_n}(x^k)| \]
\[ \leq \frac{91}{500} |E_{W^{*}}(x^k) - E_{W^{*}}(x^k)| + \frac{91}{500} |E_{W^{*}}(x^k) - E_{W}(x^k)| \]
\[ + P_n(T_{n+1}, T_n) + P_n(W_n, W_m), \]

and similarly
\[ |E_{W^{*}}(x^k) - E_{W}(x^k)| \leq \frac{91}{500} |E_{W^{*}}(x^k) - E_{W}(x^k)| + \frac{91}{500} |E_{W}(x^k) - E_{W}(x^k)| \]
\[ \leq \frac{91}{500} |E_{T_{n+1}}(x^k) - E_{T_n}(x^k)| + \frac{91}{500} |E_{T_n}(x^k) - E_{T_n}(x^k)| \]
\[ + 2P_n(W_n, W_m). \]

Hence, with a little manipulating,
\[ |E_{T_{n+1}}(x^k) - E_{T_n}(x^k)| \leq \frac{9100}{11881} |E_{T_n}(x^k) - E_{T_n}(x^k)| + c_1P_n(T_{n+1}, T_n) + c_2P_n(W_n, W_m) \]

for some constants \( c_1, c_2 \). Since these last three terms tend, by inductive hypothesis, to zero as \( n, m \to \infty \) the usual trick of assuming that \( \beta = \text{LimSup}_n |E_{T_{n+1}}(x^k) - E_{T_n}(x^k)| > 0 \) and deriving a contradiction suffices to show that \( \lim_{n \to \infty} E_{T_n}(x^k) \) exists. A similar argument, using the second string of inequalities above, shows the the same result for the \( W_n \).

We are now ready to show the main result of this section.

**Theorem 4** Let \( T_0^R \in \mathcal{H}, T_0^R \neq 1/2(U_0 + U_1) \), so \( E_{T_0^R}(x^2) < 1/2 \). Then the weak limit of the \( T_n^R \) exists and is independent of the initial choice of \( T_0^R \) (subject to \( T_0^R \neq 1/2(U_0 + U_1) \)).

**Proof.** That the weak limit of the \( T_n^R \) exists follows from Lemma 3 and earlier remarks. To show that we reach the same limit no matter which starting distribution \( T_0^R \neq 1/2(U_0 + U_1) \) we take suppose that \( S_0^R \) was any other such choice and let the \( Y^n \) be defined from the \( S_0^R \) just as the \( W^n \) were defined from the \( T_0^R \). It suffices to show that for each \( k \),
\[ \lim_{n \to \infty} E_{T_n^R}(x^k) = \lim_{n \to \infty} E_{S_n^R}(x^k), \]
\[ \lim_{n \to \infty} E_{W^n}(x^k) = \lim_{n \to \infty} E_{Y^n}(x^k). \]

This is clear for \( k < 2 \) and also holds for \( k = 2 \) by Lemma 3 (where each limit is 3/10). So assume that \( k > 2 \) and that we already have the result for lower powers of \( x \). Again, by removing initial terms if necessary, we may assume that for all \( n \),
\[ E_{T_0^R}(x^2), E_{S_0^R}(x^2), E_{W}(x^2), E_{Y}(x^2) < 31/100. \]

Then, by an exactly analogous argument to that used in the proof of Lemma 3,
\[ |E_{T_{n+1}}(x^k) - E_{S_{n+1}}(x^k)| \leq \frac{9100}{11881} |E_{T_n}(x^k) - E_{S_n}(x^k)| + c_1P_n(T_{n+1}, S_{n+1}) + c_2P_n(W_n, Y_n), \]
\[ (1 - \frac{91}{200})|E_{W^n}(x^k) - E_{Y^n}(x^k)| \leq \frac{91}{500} |E_{T_n}(x^k) - E_{S_n}(x^k)| + 2P_n(W_n, Y_n), \]

from which the result follows.

We call this unique measure \( J \) (after Jiřka Villímová). Notice again that following this process any initial choice of \( T_0^R \) will give \( J \) as the weak limit with the sole exception of \( 1/2(U_0 + U_1) \), which simply gives itself again as the limit. \( J \) is (at this time) our favoured candidate for a 'natural prior probability distribution'.

We end this section by proving an attractive, and rather simpler, characterisation of \( J \). For the purpose of the next theorem let \( K_0 \in \mathcal{H}, K_0 \neq 1/2(U_0 + U_1) \) and let the \( K_n \) be defined by \( K_{n+1} = K_n \cdot K_n \).
Theorem 5 (i) $\lim_{n \to \infty} E_{K_n}(x^2) = 3/10$.
(ii) $\lim_{n \to \infty} E_{K_n}(x^k)$ exists for all $k$, so the $K_n$ have a weak limit, $K$ say.
(iii) $K \ast K = K$.
(iv) $K = J$, so $J \ast J = J$.

Proof. (i) Let $k_n = E_{K_n}(x^2)$. Then from (4), and the fact that $K_{n+1} = K_n \ast K_n$,

$$k_{n+1} = \frac{5k_n^2}{2} - k_n + \frac{3}{8}. \quad (15)$$

Putting $k_n = 3/10 + \delta_n$ we obtain $k_{n+1} = 3/10 + \delta_n/2 + 5\delta_n^2/2$. From this it follows that if $0 \leq \delta_0$ then, since $\delta_0 = k_0 - 3/10 < 1/5$, the $\delta_n$ are positive and decreasing, whilst if $0 \leq \delta_0$, the $\delta_n$ are all negative and increasing. In either case then the limit of the $k_n$ must exist and, from (15), its value must be $3/10$.

(ii) Applying Lemma 1, for $k > 2$,

$$|E_{K_{n+1}}(x^k) - E_{K_{n+1}}(x^k)| = |E_{K_{n+1}}(x^k) - E_{K_{n+1}}(x^k)|$$

$$\leq |E_{K_{n+1}}(x^k) - E_{K_{n+1}}(x^k)| + |E_{K_{n+1}}(x^k) - E_{K_{n+1}}(x^k)|$$

$$\leq \frac{91}{100}|E_{K_n}(x^k) - E_{K_n}(x^k)| + 2P_k(K_n, K_n)$$

and the result follows by induction on $k$ using the usual tricks.

(iii) For any $k$,

$$|E_{K \ast K}(x^k) - E_{K \ast K}(x^k)| = |E_{K \ast K}(x^k) - E_{K \ast K}(x^k)|$$

$$\leq |E_{K \ast K}(x^k) - E_{K \ast K}(x^k)| + |E_{K \ast K}(x^k) - E_{K \ast K}(x^k)|$$

$$\leq 2P_k(K, K_n) + \frac{91}{100}|E_{K}(x^k) - E_{K}(x^k)|$$

Since the right hand side tends to zero as $n \to \infty$ we have that

$$E_{K \ast K}(x^k) = \lim_{n \to \infty} E_{K \ast K}(x^k) = E_{K}(x^k)$$

and hence, since $K, K \ast K$ are countably additive, $K = K \ast K$.

(iv) Letting $K_0 = K$ we see that because $K \ast K = K$ each later $K_n = K$ also. Hence the limit of the $K_n$ is also $K$. But since $E_{K}(x^2) = 3/10$, $K \neq \frac{1}{2}(U_0 + U_1)$ so this must mean that $K = J$ by the previous theorem.

It should be clear that if we had started here with $K_0 = \frac{1}{2}(U_0 + U_1)$ then the $K_n$ would never have varied and this would also have been our final limit. As a consequence of this we have the following corollary.

Theorem 6 For $K \in H$ the only solutions to $K = K \ast K$ are $J$ and $\frac{1}{2}(U_0 + U_1)$.

This corollary provides us with an alternative justification for the choice of $J$ as a natural prior probability distribution. For consider some naturally encountered probability function (on $S_{L_1}$), equivalently random 0,1 process in the real world. As we have already argued our experience of such 'naturally encountered' probability functions is that this process is ultimately a very complex combination of other processes and that any true randomness is hidden deep down at the microscopic level. Now it seems reasonable to suppose that on closer inspection this process will be seen to be the result of a simple combination of some few other natural processes, where, just as in the initial process, the true randomness
in hidden deep down at the microscopic level. Clearly then, if we suppose that our initial natural probability function was distributed according to a natural prior $K$ then we would seem obliged to afford the same status to the other natural probability functions whose simple combination yielded our initial function. Hence, if we now further agree that the possible `simple combinations' are conjunction, disjunction, if and only if, and its dual (and all of these are equally likely), then we are led to conclude that $K$ will have to satisfy

$$K = \frac{1}{4}(K \land K + K \lor K + K \iff K + K \not\iff K) = K \ast K.$$ 

Hence, since $K = \frac{1}{2}(U_0 + U_1)$ is clearly unacceptable here, $K$ must be equal to $J$.

Notice that the identity $J \ast J = J$ allows us to calculate the moments of $J$ by using the recursive formula provided by (4), i.e.

$$E_J(x^0) = 1$$
$$E_J(x^k) = E_{J \ast J}(x^k)$$
$$= \frac{1}{4}(1 + (-1)^k)E_J(x^k)^2 + \sum_{r=0}^{k-1} \frac{(-1)^r}{r!}E_J(x^r)^2 \binom{k}{r} + \sum_{r=0}^{k} \frac{1}{2}E_J(x^r(1-x)^{2-r})^2 \binom{k}{r}.$$ 

By using the method of Bernstein Polynomials (see [1]) these $k$'th moments in turn allows us to compute histogram approximations to $J$. We are indebted to Marcus Hill for actually carrying out these computations for values of $k$ up to 203, and plotting the approximating histogram, which at this level has a sharp peak at 1/2, bumps at 1/4, 3/4, barely discernible bumps at 1/8, 3/8, 5/8, 7/8, and short flick-ups at 0 and 1.

**Theorem 7** $J$ is continuous.

**Proof.** We first show that $J(0) = J(1) = 0$ by using the identity

$$J = J \ast J = \frac{1}{4}(J \land J + J \lor J + J \iff J + J \not\iff J),$$

and considering how measure on 0 could have originated from the terms on the righthand side. Clearly since $x \cdot 0 = 0 \cdot x = 0$ for $0 \leq x \leq 1$, and $x + 0 - x \cdot 0 = 0$ just if $x = 0$, the contribution to $J(0)$ from

$$\frac{1}{4}(J \land J + J \lor J)$$

is

$$\frac{1}{4}(2J(0) - J(0)^2 + J(0)^2).$$

Also, since

$$xy + (1-x)(1-y) = 0 \iff (x = 0 \text{ and } y = 1) \text{ or } (x = 1 \text{ and } y = 0),$$

the measure on 0 arising from the last two terms is

$$\frac{1}{4}(2J(0)J(1) + J(0)^2 + J(1)^2) = J(0)^2.$$

Combining these and using (16) gives

$$J(0) = \frac{J(0)^2}{2} + J(0)^2,$$

whose only solutions are $J(0) = 0, \frac{1}{2}$. If $J(0) = \frac{1}{2}$ then $J = \frac{1}{2}(U_0 + U_1)$, which is not the case, so $J(0) = J(1) = 0$.

The next step is to show that $J$ is either continuous or completely discrete (atomic), that is $\sum_{J(x) > 0} J(x) = 1$. Let $m = \sum_{J(x) > 0} J(x)$ and let $\beta = J(\frac{1}{2})$. 

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Since discrete measure on a point \( x \neq 0, 1, \frac{1}{2} \) can never arise via the \( * \) operation from continuous measure, using (16), any non-zero measure, \( J(x) \), on a point \( x \) must arise from

\[
\frac{1}{4} \left( \sum_{b+c=x} + \sum_{b+c-bc=x} + \sum_{b+1-b)(1-c)=x} + \sum_{b(1-c)+1-bc=x} \right) J(b) J(c),
\]

(17)

where the \( b, c \) in these summations are such that \( J(b), J(c) > 0 \).

In the case \( x = \frac{1}{2} \) the contribution to \( J(x) \) (\( \beta \)) from \( \frac{1}{4} (J \land J + J \lor J) \) in (16) again arises purely from discrete measure

\[
\frac{1}{4} \left( \sum_{b+c=x} + \sum_{b+c-bc=x} \right) J(b) J(c).
\]

(18)

However, since

\[ xy + (1-x)(1-y) = \frac{1}{2} \iff x = \frac{1}{2} \text{ or } y = \frac{1}{2}, \]

the last two terms of (16) give a contribution to \( \beta = \frac{1}{2} \) from both the discrete and continuous measure of

\[
\frac{1}{4} (2\beta - \beta^2 + 2\beta - \beta^2) = \beta - \beta^2/2.
\]

(19)

Hence, by considering the contribution to \( m \) arising from the righthand side of (16), we have

\[
m \leq \beta - \frac{\beta^2}{2} + \frac{1}{4} \left( \sum_{b,c} J(b) J(c) + \sum_{b,c \neq \frac{1}{2}} J(b) J(c) \right),
\]

where, as usual from now on, the \( b, c \) in these summations are such that \( J(b), J(c) > 0 \). From this it follows that

\[
m \leq \beta - \beta^2/2 + \frac{1}{4} (2\beta + 2(m - \beta)^2) = m(m - \beta) + \beta,
\]

which is only possible if \( m = 0, 1 \) (since if 0 < \( m < 1 \) then \( \beta < m \), because from (16) the point \( \frac{1}{2} = (\frac{1}{2})(\frac{1}{2}) \) must certainly have non-zero measure if \( \frac{1}{2} \) does, so \( m(m - \beta) + \beta < (m - \beta) + \beta = m \). Our task now is to show that \( m = 0 \), i.e. that \( J \) is continuous.

Assume \( m = 1 \). Our plan now is to consider some of the points which must acquire non-zero measure and derive a contradiction.

First notice that if \( z, b \in [0, 1] \) with \( J(z), J(b) > 0 \) then there is at most one \( c \in [0, 1] \) such that \( J(c) > 0 \) and either \( bc = z \) or \( b + c - bc = z \) (since the former requires \( z \leq b \) and the latter \( z \geq b \)). Hence if \( \Lambda_z \) is a set of such pairs for \( z \) then

\[
\sum_{<b,c> \in \Lambda_z} J(b) J(c) \leq \max \left\{ J(c) | <b,c> \in \Lambda_z \right\} \left( \sum_{<b,c> \in \Lambda_z} J(b) \right).
\]

(20)

Similarly if \( z, b \in [0, 1], z \neq \frac{1}{2} \) and \( J(b), J(z) > 0 \) then there is at most \( c \in [0, 1] \) (and, necessarily, \( c \neq \frac{1}{2} \)) such that \( bc + (1-b)(1-c) = z \) (and similarly for \( b(1-c) + (1-b)c \) in place of \( bc + (1-b)(1-c) = z \)) and for a set \( \Lambda_z \) of such pairs the inequality (20) again holds. We shall use this inequality repeatedly in what follows in order to estimate the measure assigned to points.

Now let \( z_0 \in [0, 1] \) be such that \( J(z_0) = \max \left\{ J(z) | z \in [0, 1] \right\} \). Suppose that \( z_0 \neq \frac{1}{2} \). Then, from (16),

\[
J(z) = \frac{1}{4} \left( \sum_{<b,c> \in \Lambda_z^1} + \sum_{<b,c> \in \Lambda_z^2} + \sum_{<b,c> \in \Lambda_z^3} \right) J(b) J(c),
\]

(21)

where

\[
\Lambda_z^1 = \left\{ <b,c> | J(b), J(c) > 0 and bc = z or b + c - bc = z \right\},
\]

\[
\Lambda_z^2 = \left\{ <b,c> | J(b), J(c) > 0 and bc + (1-b)(1-c) = z \right\},
\]

\[
\Lambda_z^3 = \left\{ <b,c> | J(b), J(c) > 0 and b(1-c) + (1-b)c = z \right\}.
\]

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Hence, by (20),

\[ J(z_0) \leq \frac{1}{4} \left( J(z_0) \sum_b J(b) + 2J(z_0) \sum_b J(b) \right) = \frac{3}{4} J(z_0), \]

which is a contradiction since \( J(z_0) > 0 \). Hence \( z_0 = \frac{1}{2} \) and \( J(z_0) = \beta \).

Having shown that \( \frac{1}{2} \) must be the point with the most measure, we shall now obtain an estimate of the next largest measure on a point. [In fact with a little more work the proof would show that the points with the second largest measures are \( \frac{1}{2} \) and \( \frac{3}{4} \).] For let \( z_1 \) be such that \( J(z_1) = \max \{ J(z) | z \in [0, 1], z \neq \frac{1}{2} \} \).

Then, with the above notation, for \( <b, c> \in A^1_{z_1} \), \( J(b)J(c) = J(b_0)\beta \) for that unique \( b_0 \) (if any) such that \( b_0/2 = z_0 \) or \( b_0 + 1/2 - b_0/2 = z_0 \), whilst \( J(b)J(c) \leq J(b)J(z_0) \) otherwise. Hence, assuming as worst case that \( b_0 \) does exist,

\[
\sum_{<b,c> \in A^1_{z_1}} J(b)J(c) = J(b_0)\beta + \sum_{<b,c> \in A^1_{z_1}, b \neq b_0} J(b)J(c)
\leq J(b_0)\beta + (1 - J(b_0))J(z_1), \quad \text{since the only pair } <b, c> \\
\text{with } c = \frac{1}{2} \text{ has been removed from the summation,}
\leq J(z_1) + J(b_0)(\beta - J(z_1))
\leq J(z_1) + \beta(\beta - J(z_1)).
\]

As for the other two summations in (21) (for \( z = z_1 \)), since \( z_1 \neq \frac{1}{2} \), for \( <b, c> \in A^2_{z_1} \) or \( A^3_{z_1} \), neither \( b \) nor \( c \) can equal \( \frac{1}{2} \), so, by (20),

\[
\left( \sum_{<b,c> \in A^2_{z_1}} + \sum_{<b,c> \in A^3_{z_1}} \right) J(b)J(c) \leq 2(1 - \beta)J(z_1).
\]

Substituting these estimates into (21) gives

\[
4J(z_1) \leq J(z_1) + \beta(\beta - J(z_1)) + 2(1 - \beta)J(z_1),
\]

and hence

\[
J(z_1) \leq \frac{\beta^2}{1 + 3\beta} < \beta^2. \tag{22}
\]

Then plan now is to derive a contradiction by showing that there must be another point, not equal to \( \frac{1}{2} \), whose measure exceeds this upper bound of \( \beta^2 \).

Let

\[ B = \{ i/2^n | 0 < i < 2^n, n \in \mathbb{N} \}, \]

and let \( z_2 \in [0, 1] - B \) be such that

\[ J(z_2) = \max \{ J(b) | b \notin B \}. \]

[We do not discount the possibility that \( J(z_2) = 0 \).]

Now using (21), in the case \( z = \frac{1}{2} \), and the estimate (20), we obtain

\[
J(1/2) = \frac{1}{4} \left( \sum_{<b,c> \in A^1_{1/2}} J(b)J(c) + 4\beta - 2\beta^2 \right). \tag{23}
\]

Now for \( b, c \in B, bc \neq \frac{1}{2} \) and \( bc + (1 - b)(1 - c) \neq \frac{1}{2} \) so we can express the summation here as

\[
\sum_{<b,c> \in A^1_{1/2}} J(b)J(c) = \sum_{<b,c> \in A^1_{1/2}, b \in B} J(b)J(c) + \sum_{<b,c> \in A^1_{1/2}, c \in B} J(b)J(c) + \sum_{b,c \notin B} J(b)J(c).
\]

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Using (20) the first two sums on the right-hand side of this equation are bounded above by \( J(z_2) \sum_{b \in B} J(b) \) whilst the last sum is bounded by \( J(z_2) (1 - \sum_{b \in B} J(b)) \). Substituting these estimates into (23) gives
\[
4\beta \leq 2J(z_2) \sum_{b \in B} J(b) + J(z_2)(1 - \sum_{b \in B} J(b)) + 4\beta - 2\beta^2,
\]
from which we obtain
\[
2\beta^2 \leq J(z_2)(1 + \sum_{b \in B} J(b)) \leq 2J(z_2),
\]
contradicting (22), as required.

The problem of the differentiability of \( J \) remains to be investigated. [By analogy with Theorem 4.4 in [1] where the measure \( D \) is shown to non-differentiable at points of the form \( i/2^n \) one might suspect a similar failure for \( J \).]

This completes our study of the measure \( J \) for the present. In the next section we turn our attention to the question of multivariate natural priors.

The Extension of \( J \) to \( SL_n \)

In the previous sections we derived the prior probability distribution \( J \) on \([0, 1]\), equivalently on the probability functions on \( SL_1 \) (where \( L_n = \{q_1, ..., q_n\} \)). In short the justification for considering \( J \) as a prior on naturally encountered probability functions was based on the assumptions that randomness in nature is the result of a very deep combination of 'microscopic' random events and that such combinations can be adequately modelled as sentences in the propositional calculus.

In this section we turn our attention to extending \( J \) from probability functions on \( SL_1 \) to probability functions on \( SL_n \). In particular we shall be interested in the sort of probability functions on \( SL_n \) which knowledge engineers endeavour to approximate via expert systems. That is, probability functions on on \( SL_n \) in which the propositional variables \( q_i \) correspond to natural, and rather interrelated, features. For example the relevant features a doctors use in diagnosing, say, types of tumours.

One explanation for the dependencies we find between such observable features is that the features themselves are the result of combinations of certain other 'basic', independent, features and that the dependencies between the observable features arise as a result of having these 'basic' features in common. In this section we shall consider modelling this explanation by treating the (independent) basic features as distributed according to \( J \) and the observable features as corresponding to sentences in built up from these basic features, more precisely, as sentences in \( B^m_n \), where \( B^m_n \) is the set of those sentences of the language \( L'_m = \{p_1, ..., p_m\} \) built up, without repeated propositional variables, from \( \pm p_1, ..., \pm p_m \) using exactly \( k \) occurrences of the connectives \( \wedge, \vee, ++, +, \) (so necessarily \( k < m \)), and where the expected values of the distinct propositional variables in sentences in \( B^m_n \) are independently distributed according to \( J \). Thus for a 'natural' probability function, \( P \) say, on \( SL_n \) we are thinking of there being sentences \( \theta_1, ..., \theta_n \in B^m_n \) such that
\[
P(q_i) = \text{Expected truth value of } \theta_i
= \text{Probability that } \theta_i \text{ is true},
\]
and for an atom \( \bigwedge_{i=1}^n q_i^c \) of \( SL_n \),
\[
P(\bigwedge_{i=1}^n q_i^c) = \text{Expected truth value of } \bigwedge_{i=1}^n \theta_i^c
= \text{Probability that } \bigwedge_{i=1}^n \theta_i^c \text{ is true},
\]

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where the probabilities that the \( p_j \) are true \((j = 1, \ldots, m)\) have been randomly, and independently, chosen according to \( J \). We shall further assume here that all sentences \( \theta \in B_n^k \) are equally likely.

We now turn this into a formal definition of \( J_n \). First recall that we identify a probability function \( P \) on \( SL_n \) with the vector

\[
< P(a_1), \ldots, P(a_{2^n}) > \in \mathbb{D}_n = \{ < x_1, \ldots, x_{2^n} > | x_i \geq 0, \sum x_i = 1 \}
\]

where the \( a_1, \ldots, a_{2^n} \) are the distinct atoms of \( SL_n \), i.e. those sentences of the form \( \wedge_{i=1}^{n} q_i^{a_i} \). Thus we will define \( J_n \) as a probability function on \( (the \ Borel \ subsets \ of) \ \mathbb{D}_n \). For an atom \( \alpha = \wedge_{i=1}^{n} q_i^{a_i} \) and \( \theta_1, \ldots, \theta_n \in B_n^k \) we set \( \alpha(\theta_1, \ldots, \theta_n) = \alpha(\overrightarrow{\theta}) \) to be the sentence formed by replacing each \( q_i \) in \( \alpha \) by \( \theta_i \) (i.e. \( \alpha(\overrightarrow{\theta}) = \wedge_{i=1}^{n} \theta_i^{a_i} \), where as usual \( \theta^1 = \theta, \theta^0 = \neg \theta \)). For \( A \) a Borel subset of \( \mathbb{D}_n \) we now define

\[
J_n(A) = \frac{1}{|B_n^k|} \sum_{\theta_1, \ldots, \theta_n \in B_n^k} J_\overrightarrow{\theta}(A),
\]

where, for \( \phi_1, \ldots, \phi_n = \overrightarrow{\phi} \in SL_n^k \),

\[
J_\overrightarrow{\phi}(A) = J^m(\{ < a_1, \ldots, a_m > \in [0,1]^m | h(a, \alpha_1(\overrightarrow{\phi})), \ldots, h(a, \alpha_{2^n}(\overrightarrow{\phi})) \geq A \} ),
\]

\( J^m \) is the product measure \( J \times J \times \ldots \times J \ (m \ times) \) on \([0,1]^m\) (see, e.g. [5]), and \( h(\overrightarrow{a}, \overrightarrow{\phi}) \) is the expected truth value of \( \phi \in SL_n^k \) given that the \( p_1, \ldots, p_m \) are independently distributed with expected truth values \( a_1, \ldots, a_m \) respectively. Notice that if \( \overrightarrow{\phi} \equiv \phi \) then \( h(\overrightarrow{a}, \overrightarrow{\phi}) = h(\overrightarrow{a}, \phi) \) Notice also that if \( \phi \) has no repeated propositional variables, as happens if \( \phi \in B_n^k \), then \( h(\overrightarrow{a}, \phi) \) can easily be calculated by the recursion

\[
\begin{align*}
  h(\overrightarrow{a}, p_i) &= a_i \\
  h(\overrightarrow{a}, \neg p_i) &= 1 - a_i \\
  h(\overrightarrow{a}, \phi_1 \land \phi_2) &= h(\overrightarrow{a}, \phi_1) \cdot h(\overrightarrow{a}, \phi_2) \\
  h(\overrightarrow{a}, \phi_1 \lor \phi_2) &= h(\overrightarrow{a}, \phi_1) + h(\overrightarrow{a}, \phi_2) - h(\overrightarrow{a}, \phi_1) \cdot h(\overrightarrow{a}, \phi_2) \\
  h(\overrightarrow{a}, \phi_1 \leftrightarrow \phi_2) &= h(\overrightarrow{a}, \phi_1) \cdot (1 - h(\overrightarrow{a}, \phi_2)) + h(\overrightarrow{a}, \phi_2) \cdot (1 - h(\overrightarrow{a}, \phi_1))
\end{align*}
\]

Unfortunately in the case of repeated variables we cannot calculate \( h(\overrightarrow{a}, \phi) \) so simply.

There are, of course, a numbers of criticisms one might mount against this particular candidate \( J_n \) as a distribution on the naturally encountered probability functions of \( SL_n \). The first, perhaps, is how \( k, m \) are to be chosen. (Strictly these parameters should appear explicitly in \( J_n \) but to avoid a surfeit of subscripts and superscripts we suppress mention of them.) In short we view \( k \) and \( m \) as adjustable parameters. For fixed \( k \) having \( m \) small has the effect of making the correlations between the \( q_i \) more variable. The effect of \( k \) rather more subtle, basically having \( k \) large means that overall there are more dependencies between the \( q_i \).

A second, and more serious, criticism is why we chose the set of sentences \( B_n^k \). Would not \( \overrightarrow{A}_m^k \), (or even \( A_m^k \)) have been a more appropriate choice? It must be acknowledged that in part this choice was made in order to simplify the ensuing mathematics analysis! However there is some argument in favour of \( B_n^k \) in place of \( \overrightarrow{A}_m^k \). Namely, if we had used \( \overrightarrow{A}_m^k \) we would have been allowing the definite possibility of choosing \( \theta_i \) to be a tautology (or contradiction) in which case \( q_i \) would not be random at all. Now in the context of constructing expert systems we (apparently) never do this. That is, we do not include amongst our variables which are actually not variable all. In this sense then we preselect our features, and our choice of sentences should reflect this preselection. Of course not all sentences in \( \overrightarrow{A}_m^k \) with repeated propositional variables are tautologies or contradictions, but it seems rather difficult to find some workable compromise which allows repeated whilst avoiding the problem of tautologies and contradictions. For the present therefore we shall simply accept that this choice of \( B_n^k \) is not, perhaps, ideal, but nevertheless carry on to investigate some of its consequences for the \( J_n \).

Before doing this however it will be useful to have the following result connecting the operations \( \land, \lor, \neg, \rightarrow, \leftrightarrow \) on measures.
Lemma 2 Let $F_1, F_2 \in T, T_1, T_2 \in \mathcal{H}$. Then for a Borel set $A \subseteq [0,1],

\begin{align*}
F_1 \land F_2(A) &= F_1 \times F_2 \{ < a, b > | ab \in A \}, \\
F_1 \lor F_2(A) &= F_1 \times F_2 \{ < a, b > | (a + b - ab) \in A \}, \\
F_1 \iff F_2(A) &= F_1 \times F_2 \{ < a, b > | (ab + (1 - a)(1 - b)) \in A \}, \\
F_1 \uparrow F_2(A) &= F_1 \times F_2 \{ < a, b > | (a(1 - b) + b(1 - a)) \in A \}, \\
F_1 \uparrow F_2(A) &= \frac{1}{4} (F_1 \land F_2 + F_1 \lor F_2 + F_1 \iff F_2 + F_1 \uparrow F_2),
\end{align*}

and similarly for $T_1, T_2$.

Proof. For $F_1, F_2 \in T$ the identities follow directly from the definitions (2), (3). For $T_1, T_2 \in \mathcal{H}$ and $C \subseteq [0,1]$ a finite union of products of intervals the $T_1 \land T_2(C)$ etc are defined as as the limit of the $T_1^{(m)} \land T_2^{(n)}(C)$ etc. The analogues of Lemmas 5.1-5.3 of [1] shows that these are well defined. Clearly we also have, by the analogue of Lemma 5.1 of [1], that

$$
\lim_{n,m \to \infty} T_1^{(m)} \land T_2^{(n)}(C) = T_1 \times T_2(C')
$$

for $C' \subseteq [0,1]^2$ a countable union of products of intervals Taking limits of the above identities for $F_1, F_2$ and $A$ a finite union of intervals now shows that they also hold for $T_1, T_2$ on arguments of this form. Notice that for $0 \leq c \leq d \leq 1, \{ < a, b > | ab \in (c, d) \}$ etc is a countable union of products of intervals.

Finally since $T_1 \times T_2, T_1 \land T_2$ etc are countably additive the identities also hold for Borel $A \subseteq [0,1]$ as required.

Using this lemma we shall now give a useful alternative definition of $J_\theta$ in the case $n = 1$ (i.e. $\theta = \theta_1$). Recall that in the case $n = 1$, for $\phi \in SL_m$, $J_\theta(A) = J_\phi^m \{ < a_1, \ldots, a_m > | h(\bar{a}, \theta) \in A \}$.

Lemma 3 Let $\phi \in SL'_m$ have no repeated propositional variable and let $1 \leq 1 \leq m$. Then

$$
J_\phi = J \text{ if } \phi = \pm p_i
$$

$$
= J_{p_1} \circ J_{\phi_2} \text{ if } \phi = \phi_1 \circ \phi_2, \text{ some } \circ \in \{ \land, \lor, \iff, \uparrow \}.
$$

Proof. The proof is by induction on the number $r$ of propositional variables in $\phi$. Let $A$ be a Borel subset of $[0,1]$. In the case $r = 1$ we have

$$
J_{p_1}(A) = J_\theta^m \{ \bar{a} | h(\bar{a}, p_1) \in A \} = J(A),
$$

$$
J_{\neg p_1}(A) = J_\theta^m \{ \bar{a} | (1 - a_i) \in A \} = J(1 - a_i),
$$

$$
J_{p_1}(A) = J_\theta^m \{ \bar{a} | (1 - a_i) \in A \} = J(1 - a_i),
$$

as required.

Now assume that $r > 1$ and that we have proved the result for $\theta \in SL'_m$ with less than $r$ propositional variables (and no repeated propositional variable). Then taking the case $\circ = \land$ (the others are similar),

$$
J_{\phi_1 \land \phi_2}(A) = J_\theta^m \{ < a_1, \ldots, a_m > | h(\bar{a}, \phi_1 \land \phi_2) \in A \}
$$

$$
= J_\theta^m \{ < a_1, \ldots, a_m > | h(\bar{a}, \phi_1) \land h(\bar{a}, \phi_2) \in A \}.
$$

Now without loss of generality assume that all the propositional variables in $\phi_1$ are among $p_1, \ldots, p_s$ and all the propositional variables in $\phi_2$ are among $p_{s+1}, \ldots, p_m$. Then since $h(\bar{a}, \phi_1)$ only depends on
\( \phi_1 \wedge \phi_2 \), we may (temporarily) write it as \( \tilde{h}(a_1, \ldots, a_s, \phi_1) \) and similarly for \( \tilde{h}(a_{s+1}, \ldots, a_m, \phi_2) \). Thus, writing \( \tilde{a}' \) for \( \langle a_1, \ldots, a_s \rangle \) and \( \tilde{a}'' \) for \( \langle a_{s+1}, \ldots, a_m \rangle \), from (29) we obtain

\[
\begin{align*}
J_{\phi_1 \wedge \phi_2}(A) &= J^{m} \{ \langle \tilde{a}', \tilde{a}'' \rangle \mid \tilde{h}(\tilde{a}', \tilde{a}'' \cdot \tilde{h}(\tilde{a}', \phi_1) \cdot \tilde{h}(\tilde{a}'', \phi_2) \in A \} \\
&= J^{m} \{ \langle \tilde{a}', \tilde{a}'' \rangle \mid \tilde{h}(\tilde{a}', \phi_1), \tilde{h}(\tilde{a}'', \phi_2) \in E \{ \langle x, y \rangle \mid x \cdot y \in A \} \} \\
&= J^{s} \times J^{m-s} \{ \langle \tilde{a}', \tilde{a}'' \rangle \mid \tilde{h}(\tilde{a}', \phi_1), \tilde{h}(\tilde{a}'', \phi_2) \in E \{ \langle x, y \rangle \mid x \cdot y \in A \} \} \\
&= J_{\phi_1} \times J''_{\phi_2}(\langle x, y \rangle \mid x \cdot y \in A),
\end{align*}
\]  

(25)

where

\[
\begin{align*}
J'_{\phi_i}(X) &= J^{s} \{ \tilde{a}' \mid \tilde{h}(\tilde{a}', \phi_i) \in X \}, \\
J''_{\phi_i}(Y) &= J^{m-s} \{ \tilde{a}'' \mid \tilde{h}(\tilde{a}'', \phi_i) \in Y \}.
\end{align*}
\]

[Here we use the easily proved result from measure theory that if \( h, g \) are continuous, \( M_1, M_2 \) are countably additive measures and for Borel sets \( X, Y \),

\[
\begin{align*}
M'_1(X) &= M_1(h^{-1}X) = M_1\{x \mid h(x) \in X \}, \\
M'_2(Y) &= M_2(g^{-1}Y) = M_2\{x \mid g(x) \in Y \},
\end{align*}
\]

then for Borel \( Z \),

\[
M'_1 \times M'_2(Z) = M_1 \times M_2\{\langle x, y \rangle \mid h(x), g(y) \in Z \}.
\]

But clearly,

\[
J'_{\phi_i}(X) = J^{m} \{ \tilde{a}' \mid \tilde{h}(\tilde{a}', \phi_i) \in X \} = J_{\phi_i}(X), \quad i = 1, 2,
\]

so from (25) and Lemma 2,

\[
\begin{align*}
J_{\phi_1 \wedge \phi_2}(A) &= J_{\phi_1} \times J''_{\phi_2}(\langle x, y \rangle \mid xy \in A) \\
&= J_{\phi_1} \wedge J''_{\phi_2}(A)
\end{align*}
\]

as required.]

A consequence of this lemma is that for \( \phi \) without repeated propositional variables \( J_{\phi} \) is actually the same thing as replacing each \( p_i \), or \( \neg p_i \), throughout \( \phi \) by \( J \) and replacing each connective \( \wedge, \vee, \leftrightarrow, \vdash \) by the corresponding operation on the measures.

In our next three theorems we show that, given their intended modelling, the \( J_{\phi} \) satisfy some rather desirable properties. For the purpose of the first of these theorems recall (see, for example, [3]) that a measure \( M \) on \( D_n \) satisfies weak renaming if whenever \( 1 \leq i, j \leq n \) and \( \tau_i, \sigma_{ij} \) are the permutations of \( \{1, \ldots, 2^n\} \) such that for each atom \( \alpha_\kappa = \bigwedge q_\kappa^{m} \), \( \alpha_{\sigma_{ij}(\kappa)} \) is the atom resulting from replacing the exponent \( \epsilon_i \) in \( \alpha_\kappa \) by \( 1 - \epsilon_i \) and \( \alpha_{\sigma_{ij}(\kappa)} \) is the atom resulting from transposing the exponents \( \epsilon_i, \epsilon_j \) in \( \alpha_\kappa \) then

\[
M(A) = M(\tau_i A) = M(\sigma_{ij} A),
\]

where,

\[
\begin{align*}
\tau_i A &= \{ \langle x_1, x_2, \ldots, x_{2^n} \rangle \mid \langle x_{\tau_i(1)}, x_{\tau_i(2)}, \ldots, x_{\tau_i(2^n)} \rangle \in A \} \\
\sigma_{ij} A &= \{ \langle x_1, x_2, \ldots, x_{2^n} \rangle \mid \langle x_{\sigma_{ij}(1)}, x_{\sigma_{ij}(2)}, \ldots, x_{\sigma_{ij}(2^n)} \rangle \in A \}.
\end{align*}
\]  

(26)

In other words \( M \) is invariant under renaming the individual propositional variables or transposing a propositional variable with its negation and. This is clearly a desirable property in this context because we would not expect the likelihood of encountering a particular probability function to depend on what names (i.e. what \( \pm q_j \)) we had used to denote the features in question.

The notion of full renaming is similar except that we now allow \( \tau_i \), say, in (26), to be any permutation of \( \{1, \ldots, 2^n\} \). The \( J_n \) do not satisfy full renaming. To see this notice that
$J_n \{ < x_1, x_2, ..., x_{2^n} > \in \mathbb{D}_n \left| \sum_{i=1}^{2^n-1} x_i = 0 \right\} = 0,$

because for $\theta \in B^k_m$, $J^m \{ \bar{a} | h(\bar{a}, \theta) = 0 \} = 0$, whereas

$J_n \{ < x_1, x_2, ..., x_{2^n} > \in \mathbb{D}_n \left| \sum_{i=2}^{2^{n-1}+1} x_i = 0 \right\} \neq 0,$

since whenever $\theta_1 = \theta_2 = ... = \theta_n \in B^k_m$, $\sum_{i=2}^{2^{n-1}+1} h(\bar{a}, \alpha_i(\bar{\theta})) = 0$ for any $\bar{a}$.

**Theorem 8** The $J_n$ satisfy weak renaming.

**Proof.** As usual let $A$ be a Borel subset of $\mathbb{D}_n$ and let $\sigma = \sigma_{ij}$ be as above. Then,

$$J_n(\sigma A) = \frac{1}{|B^k_m|^n} \sum_{\theta_1, ..., \theta_n \in B^k_m} J^m \{ \bar{a} | < h(\bar{a}, \alpha_1(\bar{\theta})), ..., h(\bar{a}, \alpha_{2^n}(\bar{\theta})) > \in \sigma A \}$$

$$= \frac{1}{|B^k_m|^n} \sum_{\theta_1, ..., \theta_n \in B^k_m} J^m \{ \bar{a} | < h(\bar{a}, \alpha_{\sigma(1)}(\bar{\theta})), ..., h(\bar{a}, \alpha_{\sigma(2^n)}(\bar{\theta})) > \in A \}$$

$$= \frac{1}{|B^k_m|^n} \sum_{\theta_1, ..., \theta_n \in B^k_m} J^m \{ \bar{a} | < h(\bar{a}, \alpha_1(\bar{\theta})), ..., h(\bar{a}, \alpha_{2^n}(\bar{\theta})) > \in A \}$$

where $\phi_1, ..., \phi_n$ is just $\theta_1, ..., \theta_n$ with $\theta_i, \theta_j$ transposed. But in this last expression $\theta_1, ..., \theta_n$ can obviously be replaced under the summation by $\phi_1, ..., \phi_n$ to give $J_n(A)$ as required.

The other case we need to consider is where (dropping the subscript $i$) $\alpha_{\tau(i)}$ is the result of transposing $q_i, \neg q_i$ in $\alpha_r$. We again obtain

$$J_n(\tau A) = \frac{1}{|B^k_m|^n} \sum_{\theta_1, ..., \theta_n \in B^k_m} J^m \{ \bar{a} | < h(\bar{a}, \alpha_1(\bar{\theta})), ..., h(\bar{a}, \alpha_{2^n}(\bar{\theta})) > \in A \}$$

where $\phi_j = \theta_j$ for $j \neq i$ and $\phi_i = (\neg)\theta_i$ where $\neg$ is the result of replacing $\land, \lor, \leftrightarrow, \neg, \pm p_1, ..., \pm p_m$ in $\theta_i$ by $\lor, \land, \neg, \leftrightarrow, \pm p_1, ..., \pm p_m$ respectively. Clearly $(\neg)$ just permutes $B^k_m$ so summing over $\theta_1, ..., \theta_n \in B^k_m$ is the same as summing over $\phi_1, ..., \phi_n \in B^k_m$ and the result follows.

**Theorem 9** The $J_n$ marginalise.

**Proof.** Without loss of generality we may assume that the atoms $\beta_1, ..., \beta_{2^n+1}$ of $SL_{n+1}$ are such that $\beta_{2^n+1} = \alpha \land q_{n+1}, \beta_1 = \alpha \land \neg q_{n+1}$ where, as usual, $\alpha_1, ..., \alpha_{2^n}$ are the atoms of $SL_n$. Let $A$ be a Borel subset of $\mathbb{D}_n$ and let

$$A^+ = \{ < x_1, x_2, ..., x_{2^n+1} > \mid < x_1 + x_2, x_3 + x_4, ..., x_{2^n+1} + x_{2^n+1} > \in A \}.$$

Then for $\theta_1, ..., \theta_{n+1} \in B^k_m$,

$$J_{\theta_1, ..., \theta_{n+1}}(A^+) = \sum_{\theta_1, ..., \theta_n \in B^k_m} J^m \{ \bar{a}, \beta_1(\bar{\theta}_1, ..., \bar{\theta}_n), ..., h(\bar{a}, \beta_{2^n+1}(\bar{\theta}_1, ..., \bar{\theta}_{n+1})) > \in A^+ \}$$

$$= \sum_{\theta_1, ..., \theta_n \in B^k_m} J^m \{ \bar{a}, \alpha_1(\bar{\theta}_1, ..., \bar{\theta}_n), ..., h(\bar{a}, \alpha_{2^n}(\bar{\theta}_1, ..., \bar{\theta}_n)) > \in A \},$$

since $h(\bar{a}, \alpha_1(\bar{\theta}_1, ..., \bar{\theta}_n)) = h(\bar{a}, \beta_{2^n+1}(\bar{\theta}_1, ..., \bar{\theta}_{n+1})) + h(\bar{a}, \beta_{2^n+1}(\bar{\theta}_1, ..., \bar{\theta}_{n+1}))$.

Hence,

$$J_{\theta_1, ..., \theta_{n+1}}(A) = 21.$$
\[ J_{n+1}(A^+) = \frac{1}{|B_m|^{n+1}} \sum_{\theta_1, \ldots, \theta_{n+1} \in B_m^k} J_{\theta_1, \ldots, \theta_{n+1}}(A^+) \]
\[ = \frac{1}{|B_m|^{n+1}} \sum_{\theta_1, \ldots, \theta_n \in B_m^k} J_{\theta_1, \ldots, \theta_n}(A) \]
\[ = \frac{1}{|B_m|^n} \sum_{\theta_1, \ldots, \theta_n \in B_m^k} J_{\theta_1, \ldots, \theta_n}(A) \]
\[ = J_n(A), \]
as required.

Clearly this result would also hold if we replaced \( B_m^k \) by \( \tilde{A}_m^k \) or \( A_m^k \). It is largely the following result which encourages us however to prefer \( B_m^k \) to \( \tilde{A}_m^k \) or \( A_m^k \).

**Theorem 10** \( J_1 = J \).

**Proof.** We prove the result by induction on \( k \), simultaneously for all \( m > k \) (which for once we will need to make explicit, writing \( J_1^{[k]} \) in place of \( J_1 \)). For \( k = 0 \),

\[
J_1^{[0]} = \frac{1}{2m} \sum_{i=1}^{m} (J_{p_i} + J_{\neg p_i})
\]
\[ = \frac{1}{2m} \sum_{i=1}^{m} 2J_i \text{ by Lemma 3}, \]
\[ = J. \]

Now assume the result for \( r < k \) (where \( k < m \)). Then,

\[
J_1^{[k]}(A) = \frac{1}{|B_m|^k} \sum_{\phi_1 \in B_m^k} J_{\phi_1}(A)
\]
\[ = \frac{1}{|B_m|^k} \sum_{\phi_1 \land \phi_2 \in B_m^k} (J_{\phi_1 \land \phi_2}(A) + J_{\phi_1 \lor \phi_2}(A) + J_{\phi_1 \land \phi_2}(A) + J_{\phi_1 \lor \phi_2}(A))
\]
\[ = \frac{1}{|B_m|^k} \sum_{\phi_1 \land \phi_2 \in B_m^k} (J_{\phi_1} \land J_{\phi_2}(A) + J_{\phi_1} \lor J_{\phi_2}(A) + J_{\phi_1} \leftrightarrow J_{\phi_2}(A) + J_{\phi_1} \triangleleft J_{\phi_2}(A)) \text{ by Lemma 3},
\]
\[ = \frac{1}{|B_m|^k} \sum_{\phi_1 \land \phi_2 \in B_m^k} 4J_{\phi_1} \ast J_{\phi_2}(A). \tag{27} \]

Now for each \( \emptyset \neq S \subseteq \{p_1, \ldots, p_m\} \) let \( B(S) \) be the set of those \( \phi \in B_m^{[S]-1} \) for which all the propositional variables in \( \phi \) are in \( S \). Clearly by the inductive hypothesis, for such \( S \),

\[
\frac{1}{|B(S)|} \sum_{\theta_i \in B(S)} J_{\theta_i} = J_1^{[S]-1} = J.
\]

Hence, from (27), by partitioning the \( \phi_1, \phi_2 \) with \( \phi_1 \land \phi_2 \in B_m^k \),

\[
J_1^{[k]} = \frac{4}{|B_m|^k} \sum_{S_1, S_2} \sum_{\phi_1 \in B(S_1), \phi_2 \in B(S_2)} J_{\phi_1} \ast J_{\phi_2},
\]

22
where the first summation is over all $S_1, S_2$ such that $\emptyset \neq S_1, S_2 \subset \{p_1, \ldots, p_m\}$, $S_1 \cap S_2 = \emptyset$, $|S_1 \cup S_2| = k + 1$. Factorising gives

$$J^{[k]}_J = \frac{4}{|B_m^k|} \sum_{S_1, S_2} \left( \sum_{\phi_1 \in B(S_1)} J_{\phi_1} \right) \left( \sum_{\phi_2 \in B(S_2)} J_{\phi_2} \right) = \frac{4}{|B_m^k|} \sum_{S_1, S_2} |B(S_1)| \cdot |B(S_2)| J \ast J.$$

But since $J \ast J = J$ and $\sum_{S_1, S_2} |B(S_1)| \cdot |B(S_2)|$ is precisely the number of sentences $\phi_1 \wedge \phi_2 \in B_m^k$, this last expression is exactly $J$, as required. 

It is interesting to note that this theorem is independent of the particular choice of $k$ and $m$. Clearly this is an important result as far as our modelling of natural probability distributions is concerned. For it would have been unfortunate (to say the least) if having settled on $J$ as our prior probability distribution on natural probability functions on $SL_n$ we were then to argue for a prior for $SL_n$ which was different in the case $n = 1$. [In the light of this it is unfortunate perhaps that Theorem 10 fails if we use $A_m^k$ or $\hat{A}_m^k$ in place of $B_m^k$.]

We now turn to investigating the variability of $(P(q_1) - 1/2)^2$ and $(P(q_1 \wedge q_2) - P(q_1)P(q_2))^2$ for random $P$ according to $J_n$. Recall that the figures given in section 1 for these quantities for the uniform distribution were argued to be too low, certainly in the context of $P$ representing the underlying probabilities in a specialised area such as expert systems frequently attempt to approximate.

**Theorem 11** For $P$ random according to $J_n$,

$$E((P(q_1)) = 1/2, \quad E((P(q_1) - 1/2)^2) = 1/20,$$

and

$$E((P(q_1 \wedge q_2) - P(q_1)P(q_2))^2) =$$

$$\frac{1}{16} \left( \begin{array}{c} m \\ k+1 \end{array} \right)^{-1} \sum_{j=0}^{q} \left( \begin{array}{c} m - k - 1 \\ j \end{array} \right) \left( 1 + \frac{6f}{5} \right)^{2j} \left\{ \left( 1 - \frac{8f}{5} + \frac{120 - 8\sqrt{55}}{85} f^2 \right)^{k+1-j} - 2 \left( 1 - \frac{8f}{5} + \frac{4f^2}{5} \right)^{k+1-j} + \left( 1 - \frac{8f}{5} + \frac{8f^2}{5} \right)^{k+1-j} \right\},$$

where $q = \min (k + 1, m - k - 1)$ and $f = f_k$ is defined recursively by $f_0 = 1$ and

$$f_{s+1} = \sum_{t=0}^{s} \frac{3\mu_{s+1}^{t+1}}{4} \left( \frac{t+1}{s+2} \cdot f_t + \frac{s-t+1}{s+2} \cdot f_{s-t} \right).$$

**Proof.** By marginality the expected values $E(P(q_1))$ and $E((P(q_1) - 1/2)^2)$ are the same for $J_n$ as for $J$. But in the case of $J$ we already know that $E_J(x) = 1/2$ and $E_J(x - 1/2)^2 = E_J(x^2) - E_J(x) + 1/4 = 1/20$ respectively.

Having quickly dealt with these two cases we now go on to the considerably more complicated calculation of $E((P(q_1 \wedge q_2) - P(q_1)P(q_2))^2)$.

By marginality this equals

$$E_{J_n}((P(q_1 \wedge q_2) - P(q_1)P(q_2))^2),$$

equivalently,

$$|B_m^k|^{-2} \sum_{\theta_1, \theta_2 \in B_m^k} \int (h(\bar{a}, \theta_1 \wedge \theta_2) - h(\bar{a}, \theta_1) \cdot h(\bar{a}, \theta_2))^2 dJ^m(\bar{a}). \quad (28)$$
Expanding the square in (28) yields three distinct terms and we shall calculate these separately. We first consider

$$|B_m|^{-2} \sum_{\theta_1, \theta_2 \in B_m} \int h(\bar{a}, \theta_1)^2 h(\bar{a}, \theta_2)^2 dJ^m(\bar{a}).$$  \hspace{1cm} (29)$$

In order to start this calculation notice that for $\alpha \in AtL_{m}^\prime$ (i.e. the set of atoms of the language $L_{m}^\prime$), say $\alpha = \bigwedge_{i=1}^{m} p_i^\epsilon_i$,

$$h(\bar{a}, \alpha) = \prod_{i=1}^{m} (\epsilon_i a_i + (1 - \epsilon_i)(1 - a_i)),$$  \hspace{1cm} (30)

and for $\theta \in SL_{m}^\prime$,

$$h(\bar{a}, \theta) = \sum_{\alpha \in S_\theta} h(\bar{a}, \alpha),$$

where $S_\theta = \{ \alpha \in AtL_{m}^\prime \mid \alpha \models \theta \}$. Thus the expression $h(\bar{a}, \theta_1)^2 h(\bar{a}, \theta_2)^2$ consists of a sum of terms of the form

$$h(\bar{a}, \alpha_1) h(\bar{a}, \alpha_2) h(\bar{a}, \alpha_3) h(\bar{a}, \alpha_4)$$  \hspace{1cm} (31)

where $\alpha_1, \alpha_2 \in S_{\theta_1}, \alpha_3, \alpha_4 \in S_{\theta_2}$.

Now for $\alpha, \alpha' \in AtL_{m}^\prime$, say $\alpha = \bigwedge_{i=1}^{m} p_i^\epsilon_i, \alpha' = \bigwedge_{i=1}^{m} p_i^{\epsilon'_i}$, let

$$U(\alpha, \alpha') = \{ s \mid \epsilon_s \neq \epsilon'_s \},$$

and for $\theta \in SL_{m}^\prime$ let

$$L(\theta) = \{ s \mid p_s \text{ appears in } \theta \}.$$

Clearly for $\theta$ a randomly chosen sentence from $B_m^k$, the probability that both $\alpha$ and $\alpha'$ are in $S_\theta$ is a fixed function, $C$ say, of just $|L(\theta) \cap U(\alpha, \alpha')|$.

Thus for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in AtL_{m}^\prime$ the coefficient of the term (31) in

$$\sum_{\theta_1, \theta_2 \in B_m^k} h(\bar{a}, \theta_1)^2 h(\bar{a}, \theta_2)^2$$  \hspace{1cm} (32)

is

$$\beta = |B_m^k|^2 \cdot \left( \sum_{r=0}^{u_1} C(r) \cdot G(u_1, r) \right) \cdot \left( \sum_{r=0}^{u_2} C(r) \cdot G(u_2, r) \right)$$  \hspace{1cm} (33)

where $u_1 = |U(\alpha_1, \alpha_2)|, u_2 = |U(\alpha_3, \alpha_4)|$ and $G(u, r)$ is the probability that for a random $\theta \in B_m^k$,

$$|L(\theta) \cap U| = r$$

when $U$ is a fixed subset of $\{1, \ldots, m\}$ of size $u$.

Clearly for $r \leq u, k + 1 - r \leq m - u$,

$$G(u, r) = \binom{u}{r} \binom{m - u}{k + 1 - r} \binom{m}{k + 1}^{-1}$$  \hspace{1cm} (34)

In order to compute $C(r)$ we first need to define an auxiliary number $f$. Let $\alpha, \alpha' \in AtL_{m}^\prime$ with $|U(\alpha, \alpha')| = 1$, say $U(\alpha, \alpha') = \{ j \}$, and let $f_s$ be the probability that for $\theta$ randomly chosen from those sentences in $B_m^k$ mentioning $p_j$, $|\{\alpha, \alpha'\} \cap S_{\theta}| = 1$.

Clearly $f_0 = 1$ (since in that case $\theta$ must be $\pm p_j$). Furthermore for $s > 0$,

$$f_s = \sum_{\phi \in \{ \land, \lor, \rightarrow, \bot \}} \text{Prob} \{ \theta = \phi \circ \psi, \text{ some } \phi, \psi \} \cdot \text{Prob} \{|S_{\theta} \cap \{\alpha, \alpha'\}| = 1 \mid \theta = \phi \circ \psi, \text{ some } \phi, \psi \}$$  \hspace{1cm} (35)
Now in the case $\odot = \land$, for (otherwise) random $\theta \in B_m^2$, 

$$\text{Prob}\{ |S_\phi \cap \{\alpha, \alpha'\}| = 1 \mid \theta = \phi \land \psi \} =$$

$$\text{Prob}\{ p_j \text{ mentioned in } \phi, |S_\phi \cap \{\alpha, \alpha'\}| = 1, |S_\psi \cap \{\alpha, \alpha'\}| = 2 \} + \text{Prob}\{ p_j \text{ mentioned in } \psi, |S_\phi \cap \{\alpha, \alpha'\}| = 1, |S_\psi \cap \{\alpha, \alpha'\}| = 2 \}$$

(Since if $p_j$ is not mentioned in $\psi$ then $|S_\psi \cap \{\alpha, \alpha'\}| = 0$ or 2). A similar expression, with 0 in place of 2, is obtained for the case $\odot = \lor$. However for the cases $\odot = \leftrightarrow, \Rightarrow$ this last condition disappears.

Hence we see that for $\odot = \land, \lor$ they provide a contribution of

$$\frac{1}{2} \left( \frac{t + 1}{s + 1} \cdot f_t + \frac{s - t}{s + 1} \cdot f_{s-t-1} \right)$$

to the last factor in (35) whilst the contributions from $\odot = \leftrightarrow, \Rightarrow$ are twice this. Summing over the $\odot, \phi, \psi$ now gives

$$f_s = \sum_{t=0}^{s-1} \frac{3\mu_t^s}{4} \left( \frac{t + 1}{s + 1} \cdot f_t + \frac{s - t}{s + 1} \cdot f_{s-t-1} \right). \quad (36)$$

Returning now to the calculation of $C(r)$, if we imagine transforming an atom $\alpha$ into an atom $\alpha''$, where $|U(\alpha, \alpha') \cap L(\theta)| = r$, by successively changing the sign of some $p_s$ in $\alpha$ for $s \in U(\alpha, \alpha') \cap L(\theta)$ then

$$C(r) = \text{Prob}\{ \alpha', \alpha'' \in S_\phi, |S_\phi \cap \{\alpha, \alpha''\}| \neq 1 \}$$

$$+ \text{Prob}\{ \alpha' \in S_\phi, \alpha'' \notin S_\phi, |S_\phi \cap \{\alpha, \alpha''\}| = 1 \} = (1 - f) \cdot C(r + 1) + 1/2(1 - 2C(r + 1)) \cdot f$$

$$= f/2 + (1 - 2f)C(r - 1), \quad (37)$$

where $f = f_{r+1}$. Since $C(0) = 1/2$ it is straightforward to check that the solution to (37) is

$$C(r) = \frac{1}{4} \left( 1 + (1 - 2f)^r \right). \quad (38)$$

Having determined $C(r)$ now notice that the contribution of the term (31) to the integral (29) is, after using natural symmetry properties of $J^m$, on average, the integral with respect to $J^m(\bar{u})$ of

$$\frac{\beta}{|B_m^2|^2} \prod_{i=1}^{v} a_i^2(1 - a_i)^2 \prod_{i=v+1}^{u_1} a_i^2(1 - a_i) \prod_{i=v+1}^{u_2-v} a_i^2(1 - a_i) \prod_{i=u_1+2-v}^{u_2-2v} a_i^2(1 - a_i)^2 \quad (39)$$

where $v = |U(\alpha_1, \alpha_2) \cap U(\alpha_3, \alpha_4)|, u_1 = |U(\alpha_1, \alpha_3)|, u_2 = |U(\alpha_3, \alpha_4)|$ and by the expression ‘on average’ we mean that from all the $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ giving these values $v, u_1, u_2$ for each $s \notin U(\alpha_1, \alpha_2) \cup U(\alpha_3, \alpha_4)$, 1 \leq s \leq m half of these 4-tuples will give a contribution $a_i^2$ to the last factor in (39) (corresponding to when $p_s$ has the same sign in each of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and half will give a contribution of $a_i^2(1 - a_i)^2$ (corresponding to when $p_s$ has one sign in $\alpha_1, \alpha_2$ and the opposite sign in $\alpha_3, \alpha_4$).

Integrating out (39) gives

$$\frac{\beta}{|B_m^2|^2} \cdot E_J(x^2(1 - x)^2)^v E_J(x^2(1 - x))^{u_1+u_2-2v} \left( \frac{1}{2} E_J(x^4)^2 + \frac{1}{2} E_J(x^2(1 - x)^2)^2 \right)^{m-v-u_1-u_2} \quad (40)$$

Since for given $v, u_1, u_2$ there are $\binom{m}{v, u_1 - v, u_2 - v, m + v - u_1 - u_2}$ such $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (the $4^m$ arising because for each $i = 1, \ldots, m$ just the signs of $p_i$ in $\alpha_1, \alpha_3$ (say) can be chosen freely), summing overall these gives

$$\sum_{v, u_1, u_2} \frac{\beta_4^m}{|B_m^2|^2} \cdot \binom{m}{v, u_1 - v, u_2 - v, m + v - u_1 - u_2} E_J(x^2(1 - x)^2)^v E_J(x^2(1 - x))^{u_1+u_2-2v} \cdot \left( \frac{1}{2} E_J(x^4) + \frac{1}{2} E_J(x^2(1 - x)^2)^2 \right)^{m-v-u_1-u_2}. \quad (41)$$
Now by substituting (38) into (33) it can be checked that

\[
\frac{\beta}{|B_m^k|^2} = \frac{1}{16} \left(1 + \frac{1}{(k+1)!}\right) \left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-1} \frac{d^{k+1}}{dy^{k+1}}(1 + y(1 - 2f))u_1(1 + y)^{m-u_1} \left(1 + \frac{1}{(k+1)!}\right) \left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-1} \frac{d^{k+1}}{dz^{k+1}}(1 + z(1 - 2f))u_2(1 + z)^{m-u_2} \bigg|_{y=0} \bigg|_{z=0} \quad (42)
\]

Multiplying out the right-hand side product in (42) we obtain four terms. We shall calculate (41) for each of these in turn. The first term we consider is

\[
\frac{4^{m-2}}{(k+1)!} \left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-1} \frac{d^{k+1}}{dy^{k+1}}(1 + y(1 - 2f))u_1(1 + y)^{m-u_1} \bigg|_{y=0} \bigg|_{z=0} \quad (43)
\]

In this case we observe that if we set \( t = u_1 - v, s = u_2 - v, A_1 = E_J(x^2(1-x)^2), A_2 = E_J(x^2(1-x)), A_3 = (\frac{1}{2}E_J(x^2) + \frac{1}{2}E_J(x^2(1-x)^2)) \) then the contribution to (41) arising from (43) is

\[
\frac{4^{m-2}}{(k+1)!} \left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-1} \frac{d^{k+1}}{dy^{k+1}} \sum_{t,s,v} \left(\begin{array}{c} m \\ t,s,v \end{array}\right)(1 + y(1 - 2f))^{i+y}(1 + y)^{m-t-v} \bigg|_{y=0} \bigg|_{z=0} \quad (44)
\]

\[
= \frac{4^{m-2}}{(k+1)!} \left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-1} \frac{d^{k+1}}{dy^{k+1}} (A_1(1 + y(1 - 2f)) + A_2(1 + y(1 - 2f) + (1+y)) + A_3(1 + y))^m \bigg|_{y=0} \bigg|_{z=0} \quad (45)
\]

The corresponding term in (42) arising from \( z \) gives the same value, of course. As for the 'constant' term in (42), its contribution is

\[
4^{m-2} \sum_{t,s,v} \left(\begin{array}{c} m \\ t,s,v \end{array}\right)(1 + y(1 - 2f))^{i+y}(1 + z(1 - 2f))^{s+v}(1 + y)^{m-t-v}(1 + z)^{m-t-v} \bigg|_{y=0} \bigg|_{z=0} \quad (46)
\]

For the final term, the one involving both \( y \) and \( z \), its contribution is

\[
\left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-2} \frac{4^{m-2}}{((k+1)!)^2} \frac{d^{k+1}}{dy^{k+1}} \frac{d^{k+1}}{dz^{k+1}} \sum_{t,s,v} \left(\begin{array}{c} m \\ t,s,v \end{array}\right)(1 + y(1 - 2f))^{i+y}(1 + z(1 - 2f))^{s+v}(1 + y)^{m-t-v}(1 + z)^{m-t-v} \bigg|_{y=0} \bigg|_{z=0} \quad (47)
\]

\[
= \left(\begin{array}{c} m \\ k + 1 \end{array}\right)^{-2} \frac{4^{m-2}}{((k+1)!)^2} \frac{d^{k+1}}{dy^{k+1}} \frac{d^{k+1}}{dz^{k+1}} (A_1(1 + y(1 - 2f)) + A_2(1 + y(1 - 2f) + (1+y)) + A_3(1 + y))^m \bigg|_{y=0} \bigg|_{z=0} \quad (48)
\]

We now have then that (29) equals 2 \times (45) + (46) + (48). We now turn to the calculation of the 'middle' term of (28) that is

\[
-2 |B_m^k|^{-2} \sum_{\theta_1, \theta_2 \in B_m^k} \int h(\tilde{\sigma}, \theta_1) h(\tilde{\sigma}, \theta_2) h(\tilde{\sigma}, \theta_1 \wedge \theta_2) dJ^m(\tilde{\sigma}) \quad (49)
\]

As in the previous case let \( \alpha_1, \alpha_2, \alpha_3 \in AtL^c \). Then for random \( \theta_1, \theta_2 \in B_m^k \),

\[
\text{Prob}(\alpha_1, \alpha_3 \in S\theta_1) = \sum_{r=0}^{u_1} C(r)G(u_1, r),
\]

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\[
\text{Prob} \left( \alpha_2, \alpha_3 \in S_{\theta_2} \right) = \sum_{r=0}^{u_2} C(r) G(u_2, r),
\]

where \( u_1 = |U(\alpha_1, \alpha_3)|, \) \( u_2 = |U(\alpha_2, \alpha_3)|. \) (Notice that the two 'events' given are clearly independent.) Hence, as before, the coefficient of \( h(\alpha_1, \alpha_3) h(\alpha_2, \alpha_3) h(\alpha_1 \land \alpha_2) \)

\[
\sum_{\theta_1, \theta_2 \in B_m^*} h(\alpha_1, \theta_1) h(\alpha_2, \theta_2) h(\alpha_1 \land \theta_2)
\]

is again

\[
\beta = |B_m| \cdot \left( \sum_{r=0}^{u_1} C(r) G(u_1, r) \right) \cdot \left( \sum_{r=0}^{u_2} C(r) G(u_2, r) \right),
\]

giving a contribution to the integral (48) of

\[
- \frac{2\beta}{|B_m|} \int \prod_{i=1}^{v} a_i^2(1 - a_i) \prod_{i=v+1}^{u_1+u_2-2v} a_i^2(1 - a_i) \prod_{i=v+1}^{m} a_i^3 \, dJ^m(\alpha)
\]

(49)

where \( v = |U(\alpha_1, \alpha_3) \cap U(\alpha_2, \alpha_3)|. \) Summing over all possible \( v, u_1, u_2 \) and evaluating the integral gives a value for (48) of

\[
-2^{m+1} \sum_{v, u_1, u_2} \frac{\beta}{|B_m|} \left( \sum_{t, s, v, m - t - s - v}^{m} E_J(x^2(1-x))^{u_1+u_2-v} E_J(x^3)^{m-u_1-u_2+v} \right)
\]

\[
= -2^{m+1} \sum_{t, s, v} \frac{\beta}{|B_m|} \left( \sum_{m - t - s - v}^{m} E_J(x^2(1-x))^{t+s+v} E_J(x^3)^{m-t-s-v} \right)
\]

(50)

where \( t = u_1 - v, \) \( s = u_2 - v. \)

As in the previous case, by using (42), we can express (50) as the sum of three terms (one of them duplicated). Of these the term corresponding to (43) is

\[
- \frac{2^{m-3}}{(k+1)!} \left( \frac{m}{k+1} \right)^{-1} \frac{d^{k+1}}{d^k y + 1} \left( \sum_{t, s, v, m - t - s - v}^{m} A_4^{t+s+v} A_5^{m-t-s-v} \right) \frac{(1 + y(1 - 2f))^{t+s+v} (1 + y)^{m-t-s-v}}{y=0}
\]

\[
= - \frac{2^{m-3}}{(k+1)!} \left( \frac{m}{k+1} \right)^{-1} \frac{d^{k+1}}{d^k y + 1} (2A_4(1 + y(1 - 2f)) + A_4(1 + y) + A_5(1 + y))^m |_{y=0}
\]

\[
= -2^{m-3} (3A_4 + A_5)^{m-k-1} (3A_4 - 4A_4 f + A_3)^{k+1} t
\]

(51)

where \( A_4 = E_J(x^2(1-x)), \) \( A_5 = E_J(x^3). \)

Corresponding to the 'constant' term (46) we now have

\[
-2^{m-3} \sum_{t, s, v}^{m} \left( t, s, v, m - t - s - v \right) A_4^{t+s+v} A_5^{m-t-s-v} = -2^{m-3} (3A_4 + A_5)^m
\]

(52)

whilst the term corresponding to (47) now becomes

\[
- \left( \frac{m}{k+1} \right)^{-2} \frac{2^{m-3}}{((k+1)!)^2} \frac{d^{k+1}}{d^k y + 1} \left( \sum_{t, s, v, m - t - s - v}^{m} A_4^{t+s+v} A_5^{m-t-s-v} \right) \frac{(1 + y(1 - 2f))^{t+s+v}(1 + z(1 - 2f))^{m-t-s-v}(1 + z)^{m-t-s-v}}{z=0}
\]

\[
= - \left( \frac{m}{k+1} \right)^{-2} \frac{2^{m-3}}{((k+1)!)^2} \frac{d^{k+1}}{d^k y + 1} \left( A_4(1 + y(1 - 2f))(1 + z(1 - 2f)) \right)
\]

(53)
\[ + A_4(1 + y(1 - 2f))(1 + z) + A_4(1 + y)(1 + z(1 - 2f)) + A_6(1 + y)(1 + z) )^m \mid_{y=0}^{z=0} \]  

(53) 

Together then (51), (52) and (53) sum to (50).

Turning finally to the last remaining term,

\[ |B_m^k|^{-2} \sum_{\theta_1, \theta_2 \in B_m^k} \int h(\bar{a}, \theta_1 \land \theta_2)^2 dJ^m(\bar{a}), \]  

(54) 

let \( \alpha_1, \alpha_2 \in \text{At}^L_u \) and let \( \theta_1, \theta_2 \) be random sentences from \( B_m^k \).

Then

\[ \text{Prob}(\alpha_1, \alpha_2 \in S_{\theta_1}, \alpha_1, \alpha_2 \in S_{\theta_2}) = \left( \sum_{r=0}^{u} C(r)G(u, r) \right)^2, \]

where \( u = |U(\alpha_1, \alpha_2)| \), and the coefficient of \( h(\bar{a}, \alpha_1)h(\bar{a}, \alpha_2) \) in

\[ \sum_{\theta_1, \theta_2 \in B_m^k} h(\bar{a}, \theta_1 \land \theta_2)^2 \]

is

\[ \gamma = |B_m^k|^2 \cdot \left( \sum_{r=0}^{u} C(r)G(u, r) \right)^2 \]  

(55) 

Again this gives a contribution to (54) of

\[ \frac{\gamma^2}{|B_m^k|^2} \cdot \int \prod_{i=1}^{u} a_i(1 - a_i) \prod_{i=u+1}^{m} a_i^2 dJ^m(\bar{a}), \]

and summing over \( u \) and evaluating gives

\[ \sum_{u=0}^{m} \frac{\gamma^2}{|B_m^k|^2} \cdot \binom{m}{u} \cdot \text{E}_J(x(1-x))^u \text{E}_J(x^2)^{m-u}. \]  

(56) 

From (55), (34), (38)

\[ \frac{\gamma}{|B_m^k|^2} = \left( \frac{1}{4} \sum_{r=0}^{u} (1 + (1 - 2f)^r) \binom{u}{r} \binom{m-u}{k+1} \binom{m}{k+1}^{-1} \right)^2 \]

(57) 

Noticing that

\[ \sum_{r=0}^{u} \binom{u}{r} \binom{m-u}{k+1-r} = \frac{d^{k+1}}{d y^{k+1}} (1 + y)^{m-u} \bigg|_{y=0}^{y=0} \]

\[ = \binom{m}{k+1}, \]

we see that (57) simplifies to

\[ 2^{m-u}(1 + \binom{m}{k+1})^{-1} \frac{d^{k+1}}{d y^{k+1}} (1 + y)^{m-u} \]

\[ \cdot (1 + \binom{m}{k+1})^{-1} \frac{d^{k+1}}{d z^{k+1}} (1 + z)^{m-n} \bigg|_{z=0}^{z=0} \]

(58) 

Substituting into (56) we see that the ‘constant’ term gives

\[ \sum_{u=0}^{m} 2^{m-u} \binom{m}{u} A_6^u A_7^{m-u} = 2^{m-u}(A_6 + A_7)^m \]

(59)
where $A_6 = E_J(x(1-x))$, $A_7 = E_J(x^2)$.
Similarly one of the 'middle' terms in (58) gives

$$
\frac{d^{k+1}}{dy^{k+1}} \sum_{u=0}^{m} \left( \begin{array}{c} m \\ k + 1 \end{array} \right) y^{-1} \binom{m}{u} A_6^u A_7^{m-u} (1+y(1-2f))^{m-u} \bigg|_{y=0} 
$$

$$
= \frac{d^{k+1}}{dy^{k+1}} \left( \begin{array}{c} m \\ k + 1 \end{array} \right)^{-1} \left( \frac{2^{m-4}}{(k+1)!} \right)^2 (A_6(1+y(1-2f)) + A_7(1+y))^{m-1} \bigg|_{y=0} 
$$

$$
= 2^{m-4}(A_6 + A_7)^{m-2}(A_6(1-2f) + A_7)^{k+1} \quad (60)
$$

Finally the term in both $y$ and $z$ gives

$$
\frac{d^{k+1}}{dy^{k+1}} \frac{d^{k+1}}{dz^{k+1}} \sum_{u=0}^{m} \left( \begin{array}{c} m \\ k + 1 \end{array} \right) z^{-2} \frac{2^{m-4}}{(k+1)!} A_6^u A_7^{m-u} (1+y(1-2f))^{m-u} (1+z)^{m-u} \bigg|_{y=0} \bigg|_{z=0} 
$$

$$
= \frac{d^{k+1}}{dy^{k+1}} \frac{d^{k+1}}{dz^{k+1}} \left( \begin{array}{c} m \\ k + 1 \end{array} \right)^{-2} \frac{2^{m-4}}{(k+1)!} A_6(1+y(1-2f)) \bigg|_{y=0} 
$$

$$
\cdot (1+z(1-2f))(1+z)^{m-u} \bigg|_{z=0} 
$$

$$
= 2^{m-4}(A_6 + A_7)^{m-2}(A_6(1-2f) + A_7)^{k+1} \quad (61)
$$

and the sum of (59), $2 \times (60)$, (62) gives (54).

We are now (almost) ready to calculate (28). First notice that $E_J(x) = 1/2$ and since $J$ is symmetric

$$
E_J(x^3) = E_J((1-x)^3).
$$

Expanding both sides and simplifying gives

$$
E_J(x^3) = \frac{3}{2} E_J(x^2) - \frac{1}{4}, \quad (63)
$$

and from this we obtain

$$
A_1 = E_J(x^2) - 2E_J(x^3) + E_J(x^4),
$$

$$
A_2 = E_J(x^3) - E_J(x^4),
$$

$$
A_3 = E_J(x^4) - E_J(x^2) + \frac{1}{2} E_J(x^2),
$$

$$
A_1 + A_2 = \frac{1}{4} - \frac{1}{2} E_J(x^2),
$$

$$
A_2 + A_3 = -\frac{1}{2} E_J(x^2),
$$

$$
A_1 + 2A_2 + A_3 = \frac{1}{4},
$$

$$
A_4 = E_J(x^2) - E_J(x^3) = \frac{1}{4} - \frac{1}{2} E_J(x^2),
$$

$$
A_5 = E_J(x^3),
$$

$$
3A_4 + A_5 = \frac{1}{2},
$$

$$
A_6 = E_J(x) - E_J(x^2) = \frac{1}{2} - E_J(x^2),
$$

$$
A_7 = E_J(x^3),
$$

$$
A_6 + A_7 = \frac{1}{2}.
$$
Using these figures we now obtain

\begin{align*}
(45) & \quad = \frac{1}{16} (1 + 2f(2E_J(x^2) - 1))^{k+1}, \\
(51) & \quad = \frac{1}{8} (1 + 2f(2E_J(x^2) - 1))^{k+1}, \\
(60) & \quad = \frac{1}{16} (1 + 2f(2E_J(x^2) - 1))^{k+1}, \\
(46) & \quad = \frac{1}{16}, \\
(52) & \quad = \frac{1}{8}, \\
(59) & \quad = \frac{1}{16},
\end{align*}

from which we conclude that (28) reduces to just (48) + (53) + (62). In order to calculate these notice that

\begin{align*}
\frac{d^{k+1}}{dy^{k+1}} \frac{d^{k+1}}{dx^{k+1}} (a + by + cz + dyz)^m \bigg|_{y=0}^{x=0} &= \frac{d^{k+1}}{dy^{k+1}} (c + dy)^{k+1} (a + by)^{m-k-1} \bigg|_{y=0}^{x=0} \\
&= \sum_{j=0}^{q} \frac{m!(k+1)!}{(m-k-1-j)!j!} a^{m-k-1-j} b^j c^j d^{k+1-j},
\end{align*}

where \( q = \min (k+1, m-k-1) \).

Substituting in this identity gives that (48),(53),(62) equal, respectively,

\begin{align*}
\frac{1}{16} \left( \begin{array}{c} m \\ k+1 \end{array} \right)^{-1} & \sum_{j=0}^{q} \left( \begin{array}{c} m-k-1 \\ j \end{array} \right) (1 + 2f(2E_J(x^2) - 1))^{2j} (1 + 4f(2E_J(x^2) - 1) + 16f^2 A_1)^{k+1-j}, \\
\frac{1}{8} \left( \begin{array}{c} m \\ k+1 \end{array} \right)^{-1} & \sum_{j=0}^{q} \left( \begin{array}{c} m-k-1 \\ j \end{array} \right) (1 + 2f(2E_J(x^2) - 1))^{2j} (1 + 4f(2E_J(x^2) - 1) + 2f^2 (1 - 2 E_J(x^2)))^{k+1-j}, \\
\frac{1}{16} \left( \begin{array}{c} m \\ k+1 \end{array} \right)^{-1} & \sum_{j=0}^{q} \left( \begin{array}{c} m-k-1 \\ j \end{array} \right) (1 + 2f(2E_J(x^2) - 1))^{2j} (1 + 4f(2E_J(x^2) - 1) + 4f^2 (1 - 2 E_J(x^2)))^{k+1-j},
\end{align*}

Hence (28) equals

\begin{align*}
\frac{1}{16} \left( \begin{array}{c} m \\ k+1 \end{array} \right)^{-1} & \sum_{j=0}^{q} \left( \begin{array}{c} m-k-1 \\ j \end{array} \right) (1 + 2f(2E_J(x^2) - 1))^{2j} \left\{ (1 + 4f(2E_J(x^2) - 1) + 16f^2 A_1)^{k+1-j} \\
&- 2(1 + 4f(2E_J(x^2) - 1) + 2f^2 (1 - 2 E_J(x^2)))^{k+1-j} \\
&+ (1 + 4f(2E_J(x^2) - 1) + 4f^2 (1 - 2 E_J(x^2)))^{k+1-j} \right\}
\end{align*}

(64)

where \( q = \min (k+1, m-k-1) \), \( A_1 = E_J(x^2 (1-x^2)) \), \( f = f_a \).

If we now substitute the actual values of 3/10 of \( E_J(x^2) \) and \( \frac{3}{34} - \frac{\sqrt{85}}{170} \) of \( A_1 \) (calculated using (16)) into (64) we obtain,

\begin{align*}
\frac{1}{16} \left( \begin{array}{c} m \\ k+1 \end{array} \right)^{-1} & \sum_{j=0}^{q} \left( \begin{array}{c} m-k-1 \\ j \end{array} \right) (1 + \frac{6f}{5})^{2j} \left\{ \left( 1 - \frac{8f}{5} + \left( \frac{120 - 8\sqrt{85}}{85} \right) f^2 \right)^{k+1-j} \\
&- \left( 1 - \frac{8f}{5} + \left( \frac{120 - 8\sqrt{85}}{85} \right) f^2 \right)^{k+1-j} \right\}
\end{align*}
\[-2 \left( 1 - \frac{8f}{5} + \frac{4f^2}{5} \right)^{k+1-j} + \left( 1 - \frac{8f}{5} + \frac{8f^2}{5} \right)^{k+1-j} \}\]

as required. \[\blacksquare\]

The value here for \(E((P(q_1) - 1/2)^2)\) of 1/20 is a considerable 'improvement' on that obtained for the uniform distribution on \(SL_n\) for all but very small \(n\), since it does allow a reasonable variation away from 1/2 of the probabilities of the \(q_i\). Of course we could have obtained a 'better' value for a family of Dirichlet distributions by taking the family corresponding to the uniform distribution at level 1 i.e. on the probability functions on \(SL_1\). However this choice would seem, if anything, even harder to justify than assuming the uniform distribution on probability functions on \(SL_n\), for large \(n\).

In the particular case where \(q = m - k - 1\) (i.e. \(m \leq 2k + 2\)) the expression (64) for \(E((P(q_1 \wedge q_2) - P(q_1)P(q_2))^2)\) simplifies to

\[
\frac{1}{16} \left( \begin{array}{c} m \\ k + 1 \end{array} \right)^{-1} (B_1 - 2B_2 + B_3) \tag{65}
\]

where \(B_1, B_2, B_3\) are, respectively,

\[
(1 + 4f(2E_J(x^2) - 1) + 16f^2A_1)^{k+1} \left( 1 + \frac{(1 + 2f(2E_J(x^2) - 1))^2}{(1 + 4f(2E_J(x^2) - 1) + 16f^2A_1)^{m-k-1}} \right)
\]

\[
(1 + 4f(2E_J(x^2) - 1) + 2f^2(1 - 2E_J(x^2)))^{k+1} \left( 1 + \frac{(1 + 2f(2E_J(x^2) - 1))^2}{(1 + 4f(2E_J(x^2) - 1) + 2f^2(1 - 2E_J(x^2)))^{m-k-1}} \right)
\]

\[
(1 + 4f(2E_J(x^2) - 1) + 4f^2(1 - 2E_J(x^2)))^{k+1} \left( 1 + \frac{(1 + 2f(2E_J(x^2) - 1))^2}{(1 + 4f(2E_J(x^2) - 1) + 4f^2(1 - 2E_J(x^2)))^{m-k-1}} \right)
\]

Substituting the actual values of 3/10 of \(E_J(x^2)\) and \(\frac{3}{34} - \frac{\sqrt{55}}{179}\) of \(A_1\) for \(B_1, B_2, B_3\) in gives

\[
B_1 = \left( 1 - \frac{8f}{5} + \frac{120 - 8\sqrt{55}}{85} f^2 \right)^{2k+2-m} \left( 2 - \frac{16f}{5} + \frac{872 - 40\sqrt{55}}{425} f^2 \right)^{m-k-1}
\]

\[
B_2 = \left( 1 - \frac{8f}{5} + \frac{4f^2}{5} \right)^{2k+2-m} \left( 2 - \frac{16f}{5} + \frac{36f^2}{25} \right)^{m-k-1}
\]

\[
B_3 = \left( 1 - \frac{8f}{5} + \frac{8f^2}{5} \right)^{2k+2-m} \left( 2 - \frac{16f}{5} + \frac{46f^2}{25} \right)^{m-k-1}
\]

Sample values of (65) are:

\[
\begin{array}{lllll}
  k = 0 & k = 1 & k = 2 & k = 3 & k = 4 \\
  m = 1 & 0.0446 & & & \\
  m = 2 & 0.0223 & 0.02535 & & \\
  m = 3 & 0.006507 & 0.01279 & & \\
  m = 4 & 0.002059 & 0.003313 & 0.006856 & \\
  m = 5 & 0.001267 & 0.001687 & 0.003941 & \\
  m = 6 & 0.0005101 & 0.0006644 & 0.0009071 & \\
  m = 7 & 0.0003129 & 0.0003469 & & \\
  m = 8 & 0.0001479 & 0.0001675 & & \\
  m = 9 & 0.00008973 & & & \\
  m = 10 & 0.00004762 & & & \\
\end{array}
\]

The corresponding values given by the uniform distribution on probability functions on \(SL_n\) are:
\[
\begin{align*}
\text{\(n = 1\)} & \quad \text{\(n = 2\)} & \quad \text{\(n = 3\)} & \quad \text{\(n = 4\)} & \quad \text{\(n = 5\)} & \quad \text{\(n = 6\)} \\
0.00833 & 0.007143 & 0.005051 & 0.003096 & 0.001732 & 0.0009185 \\
\text{\(n = 7\)} & \quad \text{\(n = 8\)} & \quad \text{\(n = 9\)} & \quad \text{\(n = 10\)} & \quad \text{\(n = 11\)} & \quad \text{\(n = 12\)} \\
0.0004734 & 0.0002432 & 0.0001211 & 0.00006080 & 0.00003046 & 0.00001524
\end{align*}
\]

What sort of meaningful conclusions can be drawn from these figures we shall not consider further here, except to say that they clearly do confirm our expectation that for small values of \(m\) and \(k\) propositional variables distributed according to the \(J_n\) are more interdependent than in the family of uniform distributions, at least for moderately large \(n\), and do therefore provide (in addition to their explanatory power) a better general model of such specialised areas as expert systems frequently endeavour to approximate.

We finally remark that it is particularly interesting to note here that in deriving (64) and (65) we have used no special properties of \(J\) except symmetry.

**Conclusions and Further Research**

We have demonstrated in this paper that the distribution \(J\) and its multivariate extensions \(J_n\) clearly have a number of desirable properties as candidates for 'natural' priors, not least that they are based on a clear model of random processes in the real world. However as the similar attempt in [1] clearly forewarned many of these properties depend quite critically on our initial choice \(\Lambda, \vee, \oplus\) of connectives. As we have already argued this produces in \(J\) a superior final distribution to the distribution \(\frac{1}{2}(U_0 + U_1)\) produced by using just the connectives \(\Lambda, \vee\). But are there other choices of connectives which, for whatever reason, are preferable to the one we have made in this paper?

In addition to the properties of the \(J_n\) that we have considered there are several further properties of priors on natural probability functions which we certainly might like to hold. For example inferential monotonicity (see [4]). Whether or not this holds for the \(J_n\) remains to be resolved.

One other property of \(J_n\) that we might like to hold is that if we use it as a prior for making inductive inferences in the manner described in chapter 12 of [3] then a sufficiently large (but finite) number of confirming instances \(\theta(a_1), \theta(a_2), \ldots, \theta(a_k)\) (and no counter-examples) of a quantifier free formula \(\theta(x)\) not mentioning any \(a_i\) should suffice to conclude a non-zero belief (i.e. subjective probability) that \(\forall x \theta(x)\), even if \(\theta(x)\) is not a tautology. As shown in [3] this property does not hold for the Dirichlet distributions. However we show in Appendix A that this does hold for \(J_n\) with \(n > 1\). The property does not hold for \(n = 1\) for the \(J_n\) as defined but would hold if we had used \(A_{\infty}^k\) or \(A_k^\infty\) in place of \(B_k\) in their definition. Unfortunately in that case, as we have already remarked, \(J_1\) would not equal \(J\) which seems too high a price to pay. [The idea of constructing a new \(J, J'\) say, with the property that \(J_1 = J'\) even though the \(J'_n\) were defined in terms of the \(A_{\infty}^k\) or \(A_k^\infty\) instead of the \(B_k\) would seem (even if it were possible) rather artificial since \(J'\) would necessarily have to be a function of the \(k, m\).]

Clearly then several problems and lines of enquiry remain to be explored. Nevertheless we are hopeful that in view of the pleasing properties that the \(J_n\) do possess, and their underlying motivation, they represent a possible contender for natural prior probability distributions.

**Appendix A**

Theorem 12.10 of [3] gives that in the case of a predicate language \(\mathcal{L}\) with unary predicate symbols \(P_1(x), \ldots, P_n(x)\) and constant symbols \(a_0, a_1, a_2, \ldots\) if a probability function \(\text{Bel}\) is defined on the sentences \(S\mathcal{L}\) of \(\mathcal{L}\) by

\[
\text{Bel}(Q_{i_0}(a_0) \land Q_{i_1}(a_1) \land \ldots \land Q_{i_k}(a_k)) = \int x_{i_0}x_{i_1}\ldots x_{i_k}d\mu(x),
\]

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where the \( Q_j(x), \ j = 1, \ldots, 2^k, \) run through the state descriptions \( \bigwedge_{i=1}^k P_i^k(x), \) \( \mu \) is a countably additive measure on \([0,1]\) satisfying weak renaming, and

\[
\sum_k \text{Bel}(Q_1(a_k)) \bigwedge_{i=0}^{k-1} \neg Q_1(a_i) = \infty,
\]

then for any non-tautological sentence \( \forall x \theta(x) \) with \( \theta(x) \) quantifier free and not mentioning any \( a_i, \)
\( \text{Bel}(\forall x \theta(x)) = 0. \) [This is because the additional requirements of that theorem are automatically fulfilled for \( \text{Bel} \) defined from such a \( \mu \) in this way.]

Since (66) is equivalent to

\[
\prod_k (1 - \text{Bel}(Q_1(a_k)) \bigwedge_{i=0}^{k-1} \neg Q_1(a_i)) = 0,
\]

and,

\[
\text{Bel}(Q_1(a_k)) \bigwedge_{i=0}^{k-1} \neg Q_1(a_i) = \frac{E_\mu((1-x_1)^k)}{E_\mu((1-x_1)^k)} - \frac{E_\mu((1-x_1)^{k+1})}{E_\mu((1-x_1)^k)},
\]

(66) is equivalent to

\[
\prod_k \frac{E_\mu((1-x_1)^{k+1})}{E_\mu((1-x_1)^k)} = 0, \quad \text{i.e.} \lim_{k \to \infty} E_\mu((1-x_1)^k) = 0.
\]

Now if

\[
\mu \{ <x_1, x_2, \ldots, x_2^s \in \mathbb{D}_s \mid x_1 = 0 \} = 0
\]

then for any \( \epsilon > 0 \) we can find (by the assumed countable additivity of \( \mu \)) \( \delta > 0 \) such that

\[
\mu \{ <x_1, x_2, \ldots, x_2^s \in \mathbb{D}_s \mid |x_1| > \delta \} \geq 1 - \epsilon.
\]

Hence \( E_\mu((1-x_1)^k) \leq (1-\epsilon)(1-\delta)^k + \epsilon \leq 2\epsilon \) for large \( k \) and (67) gives that \( \text{Bel}(\forall x \theta(x)) = 0. \)

On the other hand if (68) fails, say

\[
\mu \{ <x_1, x_2, \ldots, x_2^s \in \mathbb{D}_s \mid x_1 = 0 \} = b > 0,
\]

then

\[
\text{Bel}(\neg Q_1(a_0) \land \neg Q_1(a_1) \land \ldots \land \neg Q_1(a_{k-1})) = \int (1-x_1)^k \, d\mu(\bar{x})
\]

\[
\geq 1 \cdot \mu \{ <x_1, x_2, \ldots, x_2^s \in \mathbb{D}_s \mid x_1 = 0 \}
\]

\[
= b > 0,
\]

for all \( k, \) so \( \text{Bel}(\forall x \neg Q_1(x)) > 0, \) despite \( \neg Q_1(x) \) not being a tautology.

The aforementioned results concerning the \( J_n \) and Dirichlet distributions now follow since (68) holds for \( J_1 \) and all the Dirichlet distributions but fails for the \( J_{n} \) with \( n > 1 \) since if \( \theta_1, \ldots, \theta_n \in B^m_\mu \) and \( \theta_2 \equiv \neg \theta_1 \) (assuming without loss of generality that the atom \( \alpha_1 = \bigwedge_{i=1}^n \theta_i \) then

\[
J_{\bar{g}} \{ <x_1, x_2, \ldots, x_2^s \in \mathbb{D}_n \mid x_1 = 0 \} =
\]

\[
= J^m \{ <a_1, \ldots, a_m \mid h(\bar{a}, \alpha_1(\bar{\theta})) = 0 \} = 1,
\]

and

\[
J_n \{ <x_1, x_2, \ldots, x_2^s \in \mathbb{D}_n \mid x_1 = 0 \} \geq |B^k_m|^{-1} > 0,
\]

as required.
Appendix B

In subsequent work in this area, [6], Paul Watton generalised the results in the main body of this paper to the case where we start with a weighted combination \( \gamma : (1/2 - \gamma) \) of \( \wedge/v \) connectives to \( \leftrightarrow/\subseteq \) connectives, where \( 1/4 \leq \gamma \leq 1/2 \). Thus the corresponding value of \( \gamma \) in the early papers [1],[2], where only \( \wedge/v \) connectives were considered was 1/2, whilst in the main body of this paper \( \gamma = 1/4 \). Notice that if we treat all binary connectives which genuinely depend on both arguments as equally likely then \( \gamma = 2/5 \), a value which we have lately came to favour.

Setting aside the case \( \gamma = 1/2 \) Watton shows that the results we obtain in this paper for \( \gamma = 1/4 \) are essentially replicated for \( 1/4 \leq \gamma < 1/2 \). Again the intermediate measures \( D^{n+1} \) come out to be discrete (recall that on the contrary they were continuous for \( \gamma = 1/2 \)) while the limit of these, which Watton calls \( J^\alpha \), is, as in the case of \( J = J^{1/4} \), continuous. Furthermore we again reach this same \( J^\gamma \) no matter what initial \( T_0 \in H \) we start from, provided \( T_0 \neq 1/2(U_0 + U_1) \).

Theorem 5 continues to hold in this wider context, again yielding a characterisation of \( J^\gamma \) as the unique non-trivial solution of \( K = K * K \) (for the corresponding \( * \) operation) and, in turn, a recursive expression for the moments,

\[
E_{J^\gamma}(x^0) = 1,
\]

\[
E_{J^\gamma}(x^k) = \gamma(1 + (-1)^k)E_{J^\gamma}(x^k)^2 + \sum_{r=0}^{k-1} \gamma(-1)^r E_{J^\gamma}(x^r)^2 \binom{k}{r} + (1 - 2\gamma) \sum_{r=0}^{k} E_{J^\gamma}(x^r(1-x)^{k-r})^2 \binom{k}{r}.
\]

By using the method of Berstein polynomials with these moments Watton produced approximating histograms of \( J^\gamma \) for a range of values of \( \gamma \in [1/4,1/2] \). These all show the characteristic bump at 1/2 and sharp rises at 0 and 1. For \( \gamma = 1/4 \) the bump at 1/2 is certainly the dominant feature, the rises at the end points being hardly noticeable. There are also discernible hillocks at 1/8,1/4,3/4,7/8. As we increase \( \gamma \) however these hillocks flatten out and have all but disappeared by \( \gamma = 2/5 \). At the same time the bump at 1/2 is falling and the rises at the end points are becoming more pronounced so that by the time \( \gamma = 7/15 \) the approximating histogram is roughly U shaped with just a gradual swelling at 1/2.

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