A proof of the base case of the Emergence of Reason's Conjecture for CM^∞ and all s.

O.W. Barnett [1] and J.B. Paris
Department of Mathematics
University of Manchester
Manchester M13 9PL
UK
obarnett@maths.man.ac.uk, jeff@ma.man.ac.uk

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Abstract

In this paper we extend the support for a general 'Emergence of Reasons' principle in predicate uncertain reasoning by proving the base case of the 'Emergence of Reasons Conjecture', for all s, in the case of the inference process CM^∞.

Introduction

The purpose of this paper is to continue the line of research into the so called 'Emergence of Reason's Conjecture' as documented in ...... by showing that in the case of the inference process CM^∞ the base case of this conjecture holds for all positive natural numbers s.

Describe conjecture and CM^∞

The Main Theorem

Theorem 1 *Let K^r be the consistent set of constraints*

$$\{Bel(P(a_{i_1})) = b_1, Bel(P(a_{i_1}) \wedge P(a_{i_2})) = b_2, \ldots, Bel(P(a_{i_1}) \wedge \ldots \wedge P(a_{i_s})) = b_s \mid 1 \leq i_1, \ldots, i_s \leq r,\ i_j \neq i_k \quad for \quad j \neq k\}.$$

Then for any s

$$\lim_{r \to \infty} CM^\infty(K^r) \left(\bigwedge_{i=1}^{m} P^{\epsilon_i}(a_i) \right)$$

exists and agrees with the canonical solution for some complete set of reasons.

Proof. Let

$$Z_i = CM^\infty(K^r) \left(\bigvee_{\substack{\vec{\epsilon} \in \{0,1\}^r \\ \sum \epsilon_t = i}} \bigwedge_{1 \leq j \leq r} P^{\epsilon_j}(a_j) \right).$$

We begin by showing that for any r there exist constants $\mu_0, \mu_1, \ldots, \mu_s$ such that

$$Z_i = \frac{\binom{r}{i}}{\mu_s \left(\frac{i}{r}\right)^s + \ldots + \mu_1 \left(\frac{i}{r}\right) + \mu_0}.$$

Firstly, by the Renaming Principle, the value of $CM^\infty(K^r)\left(\bigwedge_{j=1}^r P^{\epsilon_j}(a_j)\right)$ depends only on $\sum_{j=1}^r \epsilon_j$ so if

$$W_i = CM^\infty(K^r)\left(\bigwedge_{1\leq j\leq r} P^{\epsilon_j}(a_j)\right) \quad \text{for} \quad \sum_{j=1}^r \epsilon_j = i \tag{1}$$

then

$$Z_i = \binom{r}{i} W_i.$$

Also, if we let $u_i = Bel(\bigwedge_{j=1}^r P^{\epsilon_j}(a_j))$ for $\sum_{j=1}^r \epsilon_j = i$ then, for any inference process satisfying the Renaming Principle, the system of constraints, K^r, is equivalent to

$$\sum_{i=0}^r \binom{r}{i} u_i = 1, \sum_{i=0}^r \frac{i}{r}\binom{r}{i} u_i = b_1, \ldots, \sum_{i=0}^r \frac{i(i-1)\ldots(i-s+1)}{r(r-1)\ldots(r-s+1)}\binom{r}{i} u_i = b_s. \tag{2}$$

According to (a special case of) de Finetti's representation theorem of exchangeable measures we have

$$u_i = \int x^i(1-x)^{r-i}\mathrm{d}\mu(x)$$

for $0 \leq i \leq r$ where μ is a measure on $[0,1]$. Now, for $0 < i < r$, since $x^i(1-x)^{r-i} > 0$ for $x \in (0,1)$, we have $u_i = 0$ if and only if $\mu(0,1) = 0$. Thus if $u_i = 0$ for any $0 < i < r$ only u_0 and u_r can possibly be non-zero. In this case (2) gives $b_1 = b_2 = \ldots = b_s = u_r$ and we can use the basic properties of probability functions to show that the only solution for K^r agrees with that for the complete set of reasons Q_1, Q_2 with $\lambda_1 = b_1$, $\lambda_2 = 1 - b_1$, $\beta_1 = 1$, $\beta_2 = 0$. Now we need only consider the case where $\mu(0,1) > 0$ and consequently $u_i > 0$ for $0 \leq i \leq r$. In this case, by the Open-mindedness Principle, the W_i are all non-zero and the $CM^\infty(K^r)$ belief function is that which maximizes the sum

$$\sum_{i=0}^{r} \binom{r}{i} \log u_i$$

with respect to the above constraints. Now, using the Lagrange Multipliers method, we can find constants $\nu_0, \nu_1, \ldots, \nu_s$ such that

$$\frac{1}{W_i} - \nu_s \frac{i(i-1)\ldots(i-s+1)}{r(r-1)\ldots(r-s+1)} - \ldots - \nu_1 \frac{i}{r} - \nu = 0.$$

Rearranging gives the required form. We will consider the case where $\mu_s = 0$ later but for now we shall assume that $\mu_s \neq 0$. We now prove some results about the behaviour of the Z_i for large values of r.

Lemma 2 *The function*

$$f(x) = x^x(1-x)^{(1-x)} \tag{3}$$

is strictly decreasing for $x \in (0, \frac{1}{2})$ and strictly increasing for $x \in (\frac{1}{2}, 1)$.

Proof. Differentiating gives

$$f'(x) = x^x(1-x)^{(1-x)} \log\left(\frac{x}{1-x}\right).$$

Now, if $x \in (0, \frac{1}{2})$ then $\frac{x}{1-x} < 1$ so $f'(x) < 0$ and if $x \in (\frac{1}{2}, 1)$ then $\frac{x}{1-x} > 1$ so $f'(x) > 0$ giving the required result. ■

Lemma 3 *Given any $\epsilon > 0$ there exists N such that for all $r \geq N$ if $0 < \beta < \beta + \epsilon \leq \alpha < \frac{1}{2}$ and the Z_i are increasing for $[\beta r] \leq i \leq \alpha r$ or if $\frac{1}{2} < \alpha < \alpha + \epsilon \leq \beta < 1$ and the Z_i are decreasing for $[\alpha r] \leq i \leq \beta r$ then $Z_{[\alpha r]} > r^2 Z_{[\beta r]}$.*

Proof. We use Stirling's formula which states that

$$r! = \sqrt{2\pi r}\left(\frac{r}{e}\right)^r e^{\theta(r)}$$

where $\theta(r) = \frac{1}{12r} + O(\frac{1}{r^2})$. From this we get

$$\binom{r}{xr} = \frac{e^{\theta(r)-\theta((1-x)r)-\theta(xr)}}{\sqrt{2\pi r}(x^{xr+\frac{1}{2}}(1-x)^{(1-x)r+\frac{1}{2}})}. \tag{4}$$

If we let

$$g(x) = \mu_s x^s + \ldots + \mu_1 x + \mu_0 \tag{5}$$

then using (4) we can show that (writing αr for $[\alpha r]$ etc.)

$$\frac{Z_{\left(\frac{\alpha+\beta}{2}\right)r}}{Z_{\alpha r}} = e^{\phi}\sqrt{\frac{\alpha(1-\alpha)}{\left(\frac{\alpha+\beta}{2}\right)\left(1-\frac{\alpha+\beta}{2}\right)}}\left(\frac{f(\alpha)}{f\left(\frac{\alpha+\beta}{2}\right)}\right)^{r}\left(\frac{g(\alpha)}{g\left(\frac{\alpha+\beta}{2}\right)}\right) \tag{6}$$

where $\phi \to 1$ as $r \to \infty$ and f is as in (3). Now, by lemma 2, we have $\frac{f(\alpha)}{f(\frac{\alpha+\beta}{2})} < 1$. Thus if we can show that $\frac{g(\alpha)}{g(\frac{\alpha+\beta}{2})}$ is bounded above by some constant then the right hand side of the above expression will be less than r^{-2} for r sufficiently large. Since the Z_i are increasing on $[\beta r, \alpha r]$ (or decreasing on $[\alpha r, \beta r]$) this would give us $r^2 Z_{\beta r} < r^2 Z_{\left(\frac{\alpha+\beta}{2}\right)r} < Z_{\alpha r}$ as required.

Since $Z_i \geq 0$ and $\binom{r}{i} > 0$ for $i = 1, 2, \ldots, r$ we must have $g(\frac{i}{r}) > 0$. Since g is a polynomial, thus continuous with finitely many turning points, for r sufficiently large we can consider a suitable subinterval of $[\beta, \alpha]$ ($[\alpha, \beta]$) on which $g(x) > 0$. By considering such a subinterval (if necessary) we can assume that $g(x) > 0$ for $x \in [\beta, \alpha]$ ($x \in [\alpha, \beta]$). Now we can write

$$g(x) = h_1(x)h_2(x)\ldots h_n(x)$$

where each $h_i(x)$ is an irreducible polynomial of degree less than or equal to 2 and since $g(x) > 0$ we can assume $h_i(x) > 0$ for $x \in [\beta, \alpha]$ ($x \in [\alpha, \beta]$) $i = 1, 2, \ldots n$.

We now wish to find an upper bound for each ratio $\frac{h_i(\alpha)}{h_i\left(\frac{\alpha+\beta}{2}\right)}$. First we consider the case where h_i is linear. Since $h_i(x) > 0$ for $x \in [\beta, \alpha]$ ($x \in [\alpha, \beta]$) we have

$$h_i(\alpha) < h_i(\alpha) + h_i(\beta) = 2h_i\left(\frac{\alpha+\beta}{2}\right)$$

so

$$\frac{h_i(\alpha)}{h_i\left(\frac{\alpha+\beta}{2}\right)} < 2.$$

Now we consider the case where h_i is quadratic, say

$$h_i(x) = \lambda x^2 + \mu x + \nu.$$

Since there are only finitely many h_i and each has at most one turning point we can assume (by considering a suitable subinterval if necessary) that each h_i is monotone on $[\beta, \alpha]$ ($[\alpha, \beta]$). Since the result follows immediately if $h_i(\frac{\alpha+\beta}{2}) > h_i(\alpha)$ we can assume that h_i is strinctly increasing on $[\beta, \alpha]$ (or decreasing on $[\alpha, \beta]$). We are assuming h_i is irreducible so $\lambda > 0$ and $\mu^2 - 4\lambda\nu < 0$ so we can write

$$\frac{h_i(\alpha)}{h_i(\frac{\alpha+\beta}{2})} = \frac{\left(\frac{\lambda\alpha+\frac{\mu}{2}}{\sqrt{\lambda\nu-\frac{\mu^2}{4}}}\right)^2 + 1}{\left(\frac{\lambda(\frac{\alpha+\beta}{2})+\frac{\mu}{2}}{\sqrt{\lambda\nu-\frac{\mu^2}{4}}}\right)^2 + 1}.$$

Since h_i is strictly increasing on $[\beta, \alpha]$ (or decreasing on $[\alpha, \beta]$) it must be the case that $\lambda\alpha + \frac{\mu}{2}$ and $\lambda\beta + \frac{\mu}{2}$ are both positive (or both negative respectively), so if we let

$$h(x) = \left| \frac{\lambda x + \frac{\mu}{2}}{\sqrt{\lambda\nu - \frac{\mu^2}{4}}} \right| \tag{7}$$

then we must have $0 < h(\beta) < h(\frac{\alpha+\beta}{2}) < h(\alpha)$ since $\lambda > 0$. Thus

$$h(\alpha) < h(\alpha) + h(\beta) = 2h\left(\frac{\alpha+\beta}{2}\right)$$

and

$$\frac{h_i(\alpha)}{h_i(\frac{\alpha+\beta}{2})} < \frac{h(\alpha)^2}{h\left(\frac{\alpha+\beta}{2}\right)^2} < 4$$

giving a suitable upper bound. Now $\sum \deg(h_i) = s$ so

$$\frac{g(\alpha)}{g(\frac{\alpha+\beta}{2})} < 2^s$$

and the result follows. ∎

Lemma 4 *Given $\epsilon > 0$ there exists $N > 0$ such that for all $r \geq N$ if $0 < \beta < \beta + \epsilon \leq \alpha < \frac{1}{2}$ then the Z_i cannot be monotone decreasing on $[\beta r] \leq i \leq \alpha r$ and if $\frac{1}{2} < \alpha < \alpha + \epsilon \leq \beta < 1$ then the Z_i cannot be monotone increasing on $[\alpha r] \leq i \leq \beta r$.*

Proof. Suppose the Z_i are decreasing on $[\beta r, \alpha r]$ (increasing on $[\alpha r, \beta r]$) where $0 < \beta < \beta + \epsilon \leq \alpha < \frac{1}{2}$ ($\frac{1}{2} < \alpha < \alpha + \epsilon \leq \beta < 1$) then by (6) we can write

$$\frac{Z_{\alpha r}}{Z_{\left(\frac{\alpha+\beta}{2}\right)r}} = e^{-\phi}\sqrt{\frac{\left(\frac{\alpha+\beta}{2}\right)\left(1-\frac{\alpha+\beta}{2}\right)}{\alpha(1-\alpha)}}\left(\frac{f\left(\frac{\alpha+\beta}{2}\right)}{f(\alpha)}\right)^r\left(\frac{g\left(\frac{\alpha+\beta}{2}\right)}{g(\alpha)}\right) \leq 1.$$

Now, by the argument in the proof of the previous lemma we have $\frac{g(\alpha)}{g(\frac{\alpha+\beta}{2})} < 2^s$ so

$$\frac{f\left(\frac{\alpha+\beta}{2}\right)}{f(\alpha)} < (2^s e^{\phi})^{\frac{1}{r}} \to 1 \quad \text{as} \quad r \to \infty$$

giving

$$\frac{f\left(\frac{\alpha+\beta}{2}\right)}{f(\alpha)} \leq 1.$$

But now, by lemma 2

$$\frac{f\left(\frac{\alpha+\beta}{2}\right)}{f(\alpha)} > 1$$

giving the required contradiction. ■

Lemma 5

- *If $s = 2n+1$ for some $n \in \mathbb{N}$ and $\mu_s > 0$ then the Z_i have at most $n+2$ local maxima. If the Z_i have precisely $n+2$ local maxima then one of these occurs at $i = 0$.*

- *If $s = 2n+1$ for some $n \in \mathbb{N}$ and $\mu_s < 0$ then the Z_i have at most $n+2$ local maxima. If the Z_i have precisely $n+2$ local maxima then one of these occurs at $i = r$.*

- *If $s = 2n$ for some $n \in \mathbb{N}$ and $\mu_s > 0$ then the Z_i have at most $n+1$ local maxima.*

- *If $s = 2n$ for some $n \in \mathbb{N}$ and $\mu_s < 0$ then the Z_i have at most $n+2$ local maxima. If the Z_i have precisely $n+2$ local maxima then one of these occurs at $i = 0$ and one at $i = r$. If the Z_i have $n+1$ local maxima then one of these occurs at either $i = 0$ or $i = r$.*

Proof. For $0 < i_0 < r$, the Z_i have a local maximum at i_0 if and only if

$$\frac{g(\frac{i_0-1}{r})}{g(\frac{i_0}{r})} \geq \frac{i_0}{r - i_0 + 1} \qquad \text{and} \qquad \frac{g(\frac{i_0}{r})}{g(\frac{i_0+1}{r})} \leq \frac{i_0 + 1}{r - i_0}.$$

Thus, by continuity, the Z_i can only have a local maximum at i_0 if there is some $x_0 \in [\frac{i_0-1}{r}, \frac{i_0}{r}]$ such that

$$\frac{g(x_0)}{g(x_0 + \frac{1}{r})} = \frac{rx_0 + 1}{r - rx_0}$$

or equivalently, x_0 is a root of the polynomial equation

$$p(x) = 0$$

where

$$p(x) = g(x)(r - rx) - g(x + \frac{1}{r})(rx + 1)$$

such that $p(x) > 0$ for $x \in (x_0 - \epsilon, x_0)$ and $p(x) < 0$ for $x \in (x_0, x_0 + \epsilon)$ for some $\epsilon > 0$. Now we can write

$$p(x) = -2\mu_s x^{s+1} + O(x^s)$$

so we see that $p(x)$ is a polynomial with degree $s + 1$ and with leading coefficient of opposite sign to that of $g(x)$. Note that the Z_i can only have a maxima at $i = 0$ if $p(0) < 0$ and at $i = r$ if $p(1 - \frac{1}{r}) > 0$. Now, if $s = 2n + 1$ and $\mu_s > 0$ then p has degree $2n + 2$ and a negative leading coefficient. Hence p can have at most $n + 1$ roots corresponding to maxima of the Z_i. Also $p(x) < 0$ for $x < x_0$ where x_0 is the first root of p and $p(x) < 0$ for $x > x_1$ where x_1 is its last root, so if the Z_i have $n+1$ local maxima in $(0, 1)$ they may also have a maxima at 0 but not 1. Similar arguments complete the proof for the other cases. ■

Suppose the Z_i have local maxima at $m_1, m_2, \ldots, m_q$ and let $e_j = \frac{m_j}{r}$ for $j = 1, 2, \ldots, q$. Let $\delta(r)$ be such that $r\delta(r)$ is the smallest positive real number satisfying

$$Z_i \leq \frac{1}{r^2} \max\{Z_{re_1}, Z_{re_2}, \ldots, Z_{re_q}\}$$

whenever $|i-re_1|$, $|i-re_2|$, ...,$|i-re_q| > r\delta(r)$. We now show that $\delta(r) \to 0$ as $r \to \infty$. By lemma 4, given $\epsilon > 0$ there exists $N_1 > 0$ such that for all $r \geq N_1$

- one of the local maxima must occur in the interval $[r(\frac{1}{2}-\epsilon), r(\frac{1}{2}+\epsilon)]$ (say $e_q \in [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$).
- the Z_i are increasing for $i < r(e_q - \epsilon)$ and decreasing for $i > r(e_q + \epsilon)$ except possibly in the intervals $[r(e_j-\epsilon), r(e_j+\epsilon)]$ for $j = 1, 2, \ldots, q-1$.

Now by lemma 3 there exists $N_2 > 0$ such that for all $r \geq N_2$, if the Z_i are increasing (decreasing) on $[(e_j - 2\epsilon)r, (e_j - \epsilon)r]$ ($[(e_j + \epsilon)r, (e_j + 2\epsilon)r]$) then $Z_{(e_j-2\epsilon)r} < r^{-2} Z_{(e_j-\epsilon)r} < r^{-2} Z_{e_j r}$ ($Z_{(e_j+2\epsilon)r} < r^{-2} Z_{(e_j+\epsilon)r} < r^{-2} Z_{e_j r}$). Thus for $r \geq \max\{N_1, N_2\}$ we have $\delta(r) < 2\epsilon$ so $\delta(r) \to 0$ as $r \to \infty$ as required. Let

$$Y_j = \sum_{\substack{|i-re_j| \leq r\delta(r) \\ |i-re_{j'}| > r\delta(r) \text{for} j' < j}} Z_i$$

then, since $e_q \to \frac{1}{2}$ as $r \to \infty$, we have

$$\begin{aligned} Y_1 + \ldots + Y_{q-1} + Y_q &= 1 + O\left(\frac{1}{r}\right) \\ e_1 Y_1 + \ldots + e_{q-1} Y_{q-1} + \frac{1}{2} Y_q &= b_1 + O(\delta(r)) \\ &\vdots \\ e_1^s Y_1 + \ldots + e_{q-1}^s Y_{q-1} + \frac{1}{2^s} Y_q &= b_s + O(\delta(r)) \end{aligned}$$

Lemma 6 *We can find a unique solution to the above equations of the form*

$$e_j = F_j(b_1, \ldots, b_s) + O(\delta(r))$$

$$Y_j = G_j(b_1, \ldots, b_s) + O(\delta(r))$$

for some functions F_j, G_j, $j = 1, 2, \ldots, q$, *(with* $F_q(\vec{b}) = \frac{1}{2}$*)*

Proof. Since the e_i^r, Y_i^r are bounded seequences they must have convergent subsequences. The limits of such subsequences give suitable solutions so we only need to prove uniqueness.

Suppose we have

$$\begin{aligned} Y_1 + \ldots + Y_q &= 1 = Y_{q+1} + \ldots + Y_{q+l} \\ e_1 Y_1 + \ldots + e_q Y_q &= b_1 = e_{q+1} Y_{q+1} + \ldots + e_{q+l} Y_{q+l} \\ &\vdots \\ e_1^s Y_1 + \ldots + e_q^s Y_q &= b_s = e_{q+1}^s Y_{q+1} + \ldots + e_{q+l}^s Y_{q+l} \end{aligned}$$

where $e_q = e_{q+l} = \frac{1}{2}$; the Y_i are all non-zero; $e_1, \ldots, e_q$ are pairwise distinct; $e_{q+1}, \ldots, e_{q+l}$ are pairwise distinct; both solutions satisfy the restrictions set by lemma 5 (so, for example, if $s = 2n+1$ and $q = l = n+2$ then there are $i \in \{1, \ldots, q-1\}$, $j \in \{q+1, \ldots, q+l-1\}$ such that $e_i = e_j = 0$ or 1). We wish to show that these are essentially the same solution in the sense that $q = l$ and for each $i \in \{1, \ldots, q\}$ there exists $j \in \{q+1, \ldots, q+l\}$ such that $e_i = e_j$ and $Y_i = Y_j$. Suppose t is the number of distinct e_i in the above equations. We see immediately that $t \leq q+l-1$ since $e_q = e_{q+l} = \frac{1}{2}$. However, depending on the parity of s and the sign of μ_s, lemma 5 places further restrictions. If $s = 2n+1$ and $q = l = n+2$ then for some $i \in \{1, \ldots, q-1\}$ and $j \in \{q+1, \ldots, q+l-1\}$ we have $e_i = e_j = 0$ or 1. Thus $t \leq q+l-2 = 2n+2 = s+1$. Similarly if q or $l \leq n+1$ we have $t \leq s+1$. If $s = 2n$ and $\mu_s > 0$ then $t \leq q+l-1 \leq 2n+1 = s+1$. If $s = 2n$ and $\mu_s < 0$ then if $q = l = n+2$ we have $t \leq q+l-3 = 2n+1 = s+1$. Similarly if q or $l \leq n+1$ we have $t \leq s+1$. Thus we always have $t-1 \leq s$ and we can rearrange the above equations to give

$$\begin{aligned} V_1 + \ldots + V_t &= 0 \\ e_{i_1} V_1 + \ldots + e_{i_t} V_t &= 0 \\ &\vdots \\ e_{i_1}^{t-1} V_1 + \ldots + e_{i_t}^{t-1} V_t &= 0 \\ &\vdots \\ e_{i_1}^{s} V_1 + \ldots + e_{i_t}^{s} V_t &= 0 \end{aligned}$$

where $e_{i_1}, \ldots, e_{i_t}$ are pairwise distinct and for $i = 1, \ldots, t$ either $V_i = \pm Y_j$ for some $j \in \{1, \ldots, q+l\}$ or $V_i = Y_{j_1} - Y_{j_2}$ for some $j_1 \in \{1, \ldots, q\}$,

$j_2 \in \{q+1, \ldots, q+l\}$. Thus if

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ e_{i_1} & e_{i_2} & \ldots & e_{i_t} \\ \vdots & \vdots & & \vdots \\ e_{i_1}^{t-1} & e_{i_2}^{t-1} & \ldots & e_{i_t}^{t-1} \end{pmatrix}$$

and

$$\vec{V} = (V_1, V_2, \ldots, V_t)$$

then

$$\mathbf{A}\vec{V}^T = \mathbf{0}.$$

However $\mathbf{A}$ is a van der Monde matrix and its determinant is given by

$$\det(\mathbf{A}) = \prod_{j<k} (e_{i_j} - e_{i_k})$$

so, by our assumption that the e_{i_j} are distinct, we have $\det(\mathbf{A}) \neq 0$ so $V_i = 0$ for $i = 1, \ldots, t$. Since we are assuming the Y_i's are non-zero we must have $0 = V_i = Y_{j_1} - Y_{j_2}$ for some $j_1 \in \{1, \ldots, q\}$, $j_2 \in \{q+1, \ldots, q+l\}$ for $i = 1, \ldots, t$. Thus $t = q = l$ and the result follows. ■

Now we can complete the proof as follows.

$$ME(K^r)\left(\bigwedge_{i=1}^{k} P(a_i) \wedge \bigwedge_{i=k+1}^{m} \neg P(a_i)\right)$$

$$= \sum_{i=0}^{r-m} \binom{r-m}{i} W_{k+i}$$

$$= \sum_{i=k}^{r-m+k} \binom{r-m}{i-k} W_i$$

$$= \sum_{i=0}^{r} \binom{r}{i} \frac{i(i-1)\ldots(i-k+1)}{r(r-1)\ldots(r-k+1)} \frac{(r-i)(r-i-1)\ldots(r-i-m+k+1)}{(r-k)(r-k-1)\ldots(r-m+1)} W_i$$

$$= \sum_{i=0}^{r} \frac{i(i-1)\ldots(i-k+1)}{r(r-1)\ldots(r-k+1)} \frac{(r-i)(r-i-1)\ldots(r-i-m+k+1)}{(r-k)(r-k-1)\ldots(r-m+1)} Z_i.$$

So

$$\lim_{r\to\infty} ME(K^r)\left(\bigwedge_{i=1}^{k} P(a_i) \wedge \bigwedge_{i=k+1}^{m} \neg P(a_i)\right) = \sum_{i=1}^{q} \lambda_i \beta_i^k (1-\beta_i)^{m-k}$$

where

$$\beta_i = F_i(b_1, \ldots, b_s)$$

$$\lambda_i = G_i(b_1, \ldots, b_s).$$

Throughout we have assumed that $\mu_s \neq 0$ but clearly if $\mu_s = 0$ then the above proof holds if we replace s everywhere by $s-1$. ■