Some Limit Theorems for ME, MD and CM^{∞}

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Abstract

We apply methods of abduction derived from propositional probabilistic reasoning to predicate probabilistic reasoning, in particular inductive logic, by treating *finite* predicate knowledge bases as potentially *infinite* propositional knowledge bases. Full and detailed proofs are given to show that for a range of predicate knowledge bases (such as those typically associated with inductive reasoning) and several key propositional inference processes (in particular the Maximum Entropy Inference Process) this procedure is well defined, and furthermore yields an explanation for the validity of the induction in terms of 'reasons'.

Motivation

Consider the following situation. I am sitting by a bend in a road and I start to wonder how likely it is that the next car which passes will skid on this bend. I have some knowledge which seems relevant, for example I know that if there is ice on the road then there is a good chance of a skid, and similarly if the bend is unsigned, the camber adverse, etc.. I possibly also have some knowledge of how likely it is that there is ice on the road, how likely it is that the bend is unsigned (possibly conditioned on the iciness of the road) etc.. Notice that this is generic knowledge which applies equally to any potential passing car.

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Armed with this knowledge base I may now form some opinion as to the likely outcome when the next car passes. Subsequently several cars pass by. I note the results and in consequence possibly revise my opinion as to the likelihood of the next car through skidding.

Clearly we are all capable of forming opinions, or beliefs, in this way, but is it possible to formalize this inductive process, this process of *uncertain reasoning* about a general population (of potential passing cars in this case) from basic generic knowledge (of ice etc.) and possibly some knowledge of a finite number of previous instances (of passing cars)?

In [11] we announced such a formalization for a limited class of knowledge bases by extending ideas on abductive reasoning about *finite* propositional probabilistic knowledge bases to such predicate, and potentially *infinite*, knowledge bases. The purpose of this paper is somewhat generalize the results of [11] and to provide full and detailed proofs.

Throughout we shall adopt the nomenclature of [11]. Thus we shall assume that we are working in a predicate language L with 0-ary predicates (i.e. propositional variables) $Q_1, Q_2, ..., Q_q$ (e.g. standing for 'ice on the road' etc.), a single unary predicate $P(x)$ (e.g. standing for car x skids) and a denumerable list of constants $a_1, a_2, a_3, ...$ (e.g. standing for the sequence of passing cars). Let SL denote the set of closed, quantifier free sentences of this language using, say, the connectives $\neg, \wedge, \vee$. Essentially then SL may be thought of as the set of sentences of the *propositional* language with the infinite set of propositional variables $Q_1, Q_2, ..., Q_q, P(a_1), P(a_2), P(a_3), ...$. For future use let $SL^{(r)} \subset SL$ be the set of sentences of the finite propositional language with propositional variables $Q_1, Q_2, ..., Q_q, P(a_1), P(a_2), P(a_3), ..., P(a_r)$. Let $\rho_1, \ ..., \ \rho_{2^q}$ denote the atoms of $\{Q_1, \ ..., \ Q_q\}$, i.e the conjunctions $\bigwedge_{j=1}^q Q_j^{\delta_j}$ ($\delta_j \in \{0,1\}$ for $j = 1, 2, ..., 2^q$) where $Q^1 = Q$ and $Q^0 = \neg Q$).

We shall further assume that our (generic) knowledge base is of the form

$$\bigcup_{i=1}^{\infty} K(a_i),$$

where $K(a_1)$ consists of a (satisfiable) finite set of linear constraints (over the reals)

$$c_1 Bel(\theta_1) + c_2 Bel(\theta_2) + ... + c_m Bel(\theta_m) = d,$$

on a subjective probability function $Bel : SL \to [0,1]$, with $\theta_1, \theta_2, ..., \theta_m$ sentences from $SL^{(1)}$ and $K(a_i)$ is the result of replacing a_1 everywhere in $K(a_1)$ by a_i. So, for example, with the above interpretation of $Q_1, P(a_1)$, etc. my knowledge that given the road is icy car a_1 will skid with (subjective) probability 1/5 might be reformulated as the linear constraint

$$Bel(P(a_1) \wedge Q_1) - 1/5 \cdot Bel(Q_1) = 0,$$

on my assigning subjective probability function Bel. Thus in this note we are identifying knowledge with a satisfiable set of linear constraints on a probability function Bel, where, as usual (see [6]), a function $Bel : SL \to [0,1]$ is a *probability function* if it satisfies that for all $\theta, \phi \in SL$:

($P1$) If $\models \theta$ then $Bel(\theta) = 1$.
($P2$) If $\models \neg(\theta \wedge \phi)$ then $Bel(\theta \vee \phi) = Bel(\theta) + Bel(\phi)$.

Associated with a probability function $Bel()$ is a *conditional probability function* $Bel(|) : SL \times SL \to [0,1]$ satisfying that for $\theta, \phi \in SL$,

$$Bel(\theta|\phi) Bel(\phi) = Bel(\theta \wedge \phi).$$

In view of the ambiguity here when $Bel(\phi) = 0$ we shall throughout treat expressions such as $Bel(\theta|\phi) = c$ as shorthand for the unambiguous $Bel(\theta \wedge \phi) = cBel(\phi)$ etc..

Note the following:

Lemma 1 *If $K(a_1)$ is satisfiable then so is $\bigcup_{i=1}^{\infty} K(a_i)$, and conversely.*

Proof. Let $Bel : SL^{(1)} \to [0, 1]$ be a solution of $K(a_1)$ and for $r \in \mathbb{N}$, $\theta \in SL^{(r)}$ define

$$Bel^{\infty}(\theta) = \sum_{\left(\rho_k \wedge \bigwedge_{i=1}^{r} P^{\epsilon_i}(a_i)\right) \models \theta} = \frac{\prod_{i=1}^{r} Bel(\rho_k \wedge P^{\epsilon_i}(a_1))}{(Bel(\rho_k))^{r-1}},$$

so in particular,

$$Bel^{\infty}\left(\rho_k \wedge \bigwedge_{i=1}^{r} P^{\epsilon_i}(a_i)\right) = \frac{\prod_{i=1}^{r} Bel(\rho_k \wedge P^{\epsilon_i}(a_1))}{(Bel(\rho_k))^{r-1}}.$$

The consistency of this definition is easily checked as well as the fact that Bel^{∞} satisfies $(P1)$ and $(P2)$. Furthermore, for any $k \in \{1, ..., 2^q\}$ and $\epsilon_i \in \{0, 1\}$,

$$\begin{aligned} Bel^{\infty}\left(\rho_k \wedge P^{\epsilon_m}(a_m)\right) &= \sum_{\substack{\epsilon_i \in \{0,1\} \\ i=1,\ldots,m-1}} Bel^{\infty}\left(\rho_k \wedge \bigwedge_{i=1}^{m} P^{\epsilon_i}(a_i)\right) \\ &= \sum_{\substack{\epsilon_i \in \{0,1\} \\ i=1,\ldots,m-1}} \frac{\prod_{i=1}^{m} Bel(\rho_k \wedge P^{\epsilon_i}(a_1))}{(Bel(\rho_k))^{m-1}} = Bel\left(\rho_k \wedge P^{\epsilon_m}(a_1)\right) \end{aligned}$$

so Bel satisfies $K(a_m)$ for each m, i.e. Bel^{∞} satisfies $\bigcup_{j=1}^{\infty} K(a_j)$. The converse is trivial. ■

Notice that the generic nature of the knowledge is captured by the fact that the knowledge base is invariant under renaming, i.e. permutation, of the a_i. Notice also that any constraint in this knowledge base only mentions at most one a_i. In other words we are assuming that any relation between the $P(a_i)$ is entirely accounted for by, or mediated through, their individual relationships to the $Q_1, Q_2, ..., Q_q$. In terms of our skidding example this amounts to the assumption that the action of one car does not directly influence the actions of any other car.

The question of induction $(\mathcal{Q})$ that we are interested in here then is:-

> Given my knowledge base $\bigcup_{i=1}^{\infty} K(a_i)$ what belief (i.e. subjective probability), $Bel(P(a_i))$, should I assign to $P(a_i)$? More generally what belief should I assign to $Bel(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i}))$ where the $\epsilon_i \in \{0, 1\}$?

[In such expressions we take it as read that the n_i are distinct.] It is important to point out here that in asking this question we assume that the knowledge base *sums up all my knowledge* (the so called *Watt's Assumption* of [6]).

Ideally (in our view) the answer to this question should follow from considerations of rationality and common sensicality. We shall consider that point shortly. For the moment however we notice that there is one situation in which the 'right' answer *seems* abundantly clear. Namely, suppose $K(a_1)$ consists of the (consistent) set of constraints:

$$
\begin{array}{ll}
\text{(i)} & Bel(P(a_1) \wedge Q_j) - \beta_j Bel(Q_j) = 0, \quad j = 1, 2, ..., q, \\
\text{(ii)} & Bel(Q_j \wedge Q_k) = 0, \quad 1 \leq j < k \leq q, \\
\text{(iii)} & Bel(Q_j) = \lambda_j, \quad j = 1, 2, ..., q, \ \text{where } \sum_{j=1}^{q} \lambda_j = 1.
\end{array} \tag{1}
$$

In this case the Q_j form a *complete set of reasons*, in that they are (i) 'reasons' (*for* $P(a_1)$ if $\beta_j > 1/2$, *against* if $\beta_j < 1/2$), (ii) disjoint, and (iii) exhaustive. Given a knowledge base of this special form there is an evident solution based on the implicit assumption that the $P(a_i)$ are, modulo the knowledge base, independent of each other, namely:-

$$Bel(P(a_i)) = \sum_{j=1}^{q} \lambda_j \beta_j,$$

and more generally

$$Bel(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})) = \sum_{j=1}^{q} \lambda_j \beta_j^l (1 - \beta_j)^{r-l},$$

where $l = \sum \epsilon_i$. We shall call this the *canonical solution* based on this complete set of reasons. It is interesting to note at this point that if two reasons, Q_i and Q_j say, have the same strength, that is $\beta_i = \beta_j$, then as far as the canonical solution is concerned they may be combined into a single reason with this common strength and weight $\lambda = \lambda_i + \lambda_j$. From this point of view then reasons are characterized purely by their strengths.

The Maximum Entropy solution

We now turn to considering solutions Bel to $\bigcup_{i=1}^{\infty} K(a_i)$ based on principles of common sense. Common sense principles were introduced explicitly in [8] (several of these had appeared earlier, especially in [12], a paper drawing similar conclusions albeit from rather stronger initial assumptions) as *constraints on the process of assigning beliefs* from (finite, linear) probabilistic knowledge bases. In that paper (subsequently improved in [6], [10] and [7]) it was shown that the *Maximum Entropy Inference Process, ME,* is the only inference process which satisfies all these common sense principles.

To expand on this result and its context, in [8] we defined an *inference process* N to be a function which for any finite, linear, satisfiable, set K of constraints on a probability function Bel on the sentences $S\mathcal{L}$ of a finite propositional language $\mathcal{L}$, *selects* a particular probability function $Bel = N(K)$ satisfying K[1]. Thus N corresponds to a process for (consistently) assigning probabilities on the basis of such knowledge bases K. The common sense principles referred to above arise by considerations of the consistency (in its informal, everyday, sense) of this process (see in particular [7]).

As far as the inference process ME is concerned, whilst it could be defined as the unique solution to these principles it has an alternate, older, and much more

[1]Strictly we should also include the language $\mathcal{L}$ as an argument of N. However for this paper we shall only consider *language invariant* inference processes, that is inference processes which are independent of the overlying language insofar as assigning probabilities to a particular sentence is concerned. For a further explanation of this point see [6]

practical characterization. Namely $ME(K)$ is that solution Bel to K for which the entropy

$$-\sum_{i=1}^{2^n} Bel(\alpha_i)log(Bel(\alpha_i))$$

is maximal, where the α_i run over the *atoms* of $S\mathcal{L}$.

Given this privileged status of ME it would seem natural to argue that the answer to our question $\mathcal{Q}$ should be that provided by ME. Indeed, it could be claimed that to do otherwise would be to contradict common sense. Unfortunately however, we cannot mechanically apply ME here because our knowledge base $\bigcup_{i=1}^{\infty} K(a_i)$ and the overlying language are infinite. Nevertheless, the nature of the original problem points clearly to the direction we should take. To illustrate this in the example of the passing cars, we remark that the idea that there are actually infinitely many of them queuing up to negotiate this bend is (despite the daily impression left by the rush hour!) clearly an idealization[2]. In truth there is only a 'potential infinity', and this being the case we would argue that the correct application of common sense in $\mathcal{Q}$ would be to assign

$$Bel\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right)$$

the value

$$\lim_{n\to\infty} ME\left(\bigcup_{j=1}^{n} K(a_j)\right)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right), \tag{2}$$

assuming that this limit exits. Notice that if all such limits do exist (indeed this applies to any inference process, not just ME) then the property that they satisfy $(P1)$, $(P2)$ is also preserved in the limit. In other words these limiting values determine a probability function on sentences of the language with propositional variables $P(a_1), P(a_2), P(a_3), \ldots$.

Our first goal is to show that in the case where $K(a_1)$ is a complete set of reasons as in (1) the limit in (2) exists and equals the canonical solution. This will be demonstrated in the following subsection once we have shown that ME satisfies a certain *Strong Independence Principle* (Theorem 8). For the record, and because of providing a reference for results appearing elsewhere, we shall give a somewhat lengthy derivation this result directly from the principles. In fact this result can readily be obtained straight from the definition of ME (indeed it is essentially *System Independence* in Shore and Johnson's axiomatization of ME, see [12]). Thus the reader who is familiar with the basic properties of ME may well wish at this point to skip forward to Theorem 9 and only refer back to the omitted material as/when/if necessary.

A Strong Independence Principle for ME

Throughout, we use the properties that any $ME(K)$ has on account of being a probability function, and the language invariance of ME without explicit mention. *Irrelevant Information*, *Relativization*, *Independence*, *Obstinacy*, *Equivalence*, *Renaming* and *Open Mindedness* refer to the principles defined in [6].

[2] Though such a word can hardly be considered appropriate in this case!

Lemma 2 *(i) Let* $K = \{Bel(p_1) = a,\ Bel(p_2) = b\}$. *Then* $ME(K)(p_1 \wedge p_2) = ab$. *(ii) Let* $K = \{Bel(p_1) = a,\ Bel(p_2) = b, Bel(p_3) = c\}$. *Then* $ME(K)(p_1 \wedge p_2 \wedge p_3) = abc$.

Proof. (i) Let

$$K_1 = \{Bel(p_1) = a,\ Bel(p_2) = b,\ Bel(p_3) = 1\},$$

$$K_2 = \{Bel(p_1|p_3) = a,\ Bel(p_2|p_3) = b,\ Bel(p_3) = 1\}.$$

By *Irrelevant Information*, $ME(K)(p_1 \wedge p_2) = ME(K_1)(p_1 \wedge p_2)$. By *Equivalence*, $ME(K_1) = ME(K_2)$ and by *Independence*, $ME(K_2)(p_1 \wedge p_2) = ME(K_2)(p_1 \wedge p_2|p_3) = ab$, as required.

(ii) If $c = 0$ then the result is obvious. Assume $c > 0$. By *Irrelevant Information* and (i),

$$ME(K)(p_1 \wedge p_3) = ac \quad \textit{and} \quad ME(K)(p_2 \wedge p_3) = bc$$

so

$$ME(K)(p_1 \wedge \neg p_3) = a - ac \quad \textit{and} \quad ME(K)(p_2 \wedge \neg p_3) = b - bc.$$

Let

$$K_1 = \{Bel(p_3) = c,\ Bel(p_1 \wedge p_3) = ac,\ Bel(p_2 \wedge p_3) = bc\},$$

$$K_2 = \{Bel(p_1 \wedge \neg p_3) = a - ac,\ Bel(p_2 \wedge \neg p_3) = b - bc\}.$$

By *Obstinacy* and *Equivalence*,

$$ME(K) = ME(K \cup K_1 \cup K_2) = ME(K_1 \cup K_2). \tag{3}$$

By *Relativization* (with $\phi = p_3$),

$$ME(K_1 \cup K_2)(p_1 \wedge p_2 \wedge p_3) = ME(K_1)(p_1 \wedge p_2 \wedge p_3). \tag{4}$$

By *Equivalence*, $ME(K_1) = ME(Bel(p_1|p_3) = a,\ Bel(p_2|p_3) = b,\ Bel(p_3) = c)$ so $ME(K_1)(p_1 \wedge p_2|p_3) = ab$ by *Independence* and $ME(K_1)(p_1 \wedge p_2 \wedge p_3) = abc$. The result now follows by (3) and (4). ■

Lemma 3 *Let*

$$K = \{Bel(p_1) = a_1, ...,\ Bel(p_n) = a_n,\ Bel(p_{n+1}) = a_{n+1}\}$$

and

$$\begin{aligned} K_1 \quad = \quad & \{Bel(p_n \wedge p_{n+1}) = a_n a_{n+1},\ Bel(p_1 \wedge p_n \wedge p_{n+1}) = a_1 a_n a_{n+1}, \\ & Bel(p_2 \wedge p_n \wedge p_{n+1}) = a_2 a_n a_{n+1},\ ... \\ & ...,\ Bel(p_{n-1} \wedge p_n \wedge p_{n+1}) = a_{n-1} a_n a_{n+1}\}. \end{aligned}$$

Then

$$ME(K)(p_1 \wedge ... \wedge p_n \wedge p_{n+1}) = ME(K_1)(p_1 \wedge ... \wedge p_n \wedge p_{n+1}).$$

Proof. By Irrelevant Information and Lemma 2,

$$\begin{array}{rcl} ME(K)(p_n \wedge p_{n+1}) & = & a_n a_{n+1}, \\ ME(K)(p_1 \wedge p_n \wedge p_{n+1}) & = & a_1 a_n a_{n+1}, \\ ME(K)(p_2 \wedge p_n \wedge p_{n+1}) & = & a_2 a_n a_{n+1}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & .. & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ ME(K)(p_{n-1} \wedge p_n \wedge p_{n+1}) & = & a_{n-1} a_n a_{n+1}, \end{array}$$

and thus also

$$\begin{array}{rcl} ME(K)(p_n \wedge \neg p_{n+1}) & = & a_n - a_n a_{n+1}, \\ ME(K)(\neg p_n \wedge p_{n+1}) & = & a_{n+1} - a_n a_{n+1}, \\ ME(K)(p_1 \wedge \neg(p_n \wedge p_{n+1})) & = & a_1 - a_1 a_n a_{n+1}, \\ ME(K)(p_2 \wedge \neg(p_n \wedge p_{n+1})) & = & a_2 - a_2 a_n a_{n+1}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & .. & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ ME(K)(p_{n-1} \wedge \neg(p_n \wedge p_{n+1})) & = & a_{n-1} - a_{n-1} a_n a_{n+1}. \end{array}$$

Let K_2 consist of the constraints

$$\begin{array}{rcl} Bel(p_1 \wedge \neg(p_n \wedge p_{n+1})) & = & a_1 - a_1 a_n a_{n+1}, \\ Bel(p_2 \wedge \neg(p_n \wedge p_{n+1})) & = & a_2 - a_2 a_n a_{n+1}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & .. & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ Bel(p_{n-1} \wedge \neg(p_n \wedge p_{n+1})) & = & a_{n-1} - a_{n-1} a_n a_{n+1}, \\ Bel(p_n \wedge \neg p_{n+1}) & = & a_n - a_n a_{n+1}, \\ Bel(\neg p_n \wedge p_{n+1}) & = & a_{n+1} - a_n a_{n+1}. \end{array}$$

By *Obstinacy* and *Equivalence*, $ME(K) = ME(K \cup K_1 \cup K_2) = ME(K_1 \cup K_2)$ Finally, by *Relativization* (with $\phi = p_n \wedge p_{n+1}$),

$$ME(K_1 \cup K_2)(p_1 \wedge ... \wedge p_n \wedge p_{n+1}) = ME(K_1)(p_1 \wedge ... \wedge p_n \wedge p_{n+1})$$

and the result follows.

Corollary 4 *Let*

$$K = \{Bel(p_1) = a_1,\ ..., Bel(p_n) = a_n,\ Bel(p_{n+1}) = a_{n+1}\}$$

and

$$K_0 = \{Bel(p_1) = a_1,\ ..., Bel(p_n) = a_n a_{n+1},\ Bel(p_{n+1}) = 1\}.$$

Then

$$ME(K)(p_1 \wedge ... \wedge p_n \wedge p_{n+1}) = ME(K_0)(p_1 \wedge ... \wedge p_n \wedge p_{n+1}).$$

Proof The proof follows by the above lemma used twice (once for K and once for K_0 in place of K). ■

Lemma 5 *Let*

$$K = \{Bel(p_1) = a_1,\ Bel(p_2) = a_2,\ ...,\ Bel(p_n) = a_n,\ Bel(p_{n+1}) = 1\}$$

and

$$K_1 = \{Bel(p_1) = a_1,\ Bel(p_2) = a_2,\ ...,\ Bel(p_n) = a_n\}.$$

Then

$$ME(K)(p_1 \wedge p_2 \wedge\ ...\ \wedge p_{n+1}) = ME(K_1)(p_1 \wedge p_2 \wedge\ ...\ \wedge p_n).$$

Proof. $ME(K)(p_1 \wedge\ ...\ \wedge p_n \wedge \neg p_{n+1}) = 0$ so

$$ME(K)(p_1 \wedge\ ...\ \wedge p_n) = ME(K)(p_1 \wedge\ ...\ \wedge p_n \wedge p_{n+1}).$$

By *Irrelevant Information* $ME(K)(p_1 \wedge\ ...\ \wedge p_n)$ equals $ME(K_1)(p_1 \wedge\ ...\ \wedge p_n)$ and the result follows. ■

Theorem 6 *Let* $K = \{Bel(p_1) = a_1,\ Bel(p_2) = a_2,\ ...,\ Bel(p_n) = a_n\}$. *Then* $ME(K)(p_1 \wedge p_2 \wedge\ ...\ \wedge p_n) = a_1 a_2 ... a_n$.

Proof. The result is true for $n = 2$ and $n = 3$ by Lemma 2. Assume that it is true for some $n \geq 3$ and consider

$$K = \{Bel(p_1) = a_1,\ ...,\ Bel(p_{n-1}) = a_{n-1},\ Bel(p_n) = a_n,\ Bel(p_{n+1}) = a_{n+1}\}.$$

Let

$$K_1 = \{Bel(p_1) = a_1,\ ...,\ Bel(p_{n-1}) = a_{n-1},\ Bel(p_n) = a_n a_{n+1},\ Bel(p_{n+1}) = 1\}$$

and

$$K_2 = \{Bel(p_1) = a_1,\ ...,\ Bel(p_{n-1}) = a_{n-1},\ Bel(p_n) = a_n a_{n+1}\}$$

By Corollary 4,

$$ME(K)(p_1 \wedge\ ...\ \wedge p_n \wedge p_{n+1}) = ME(K_1)(p_1 \wedge\ ...\ \wedge p_n \wedge p_{n+1})$$

and by Lemma 5,

$$ME(K_1)(p_1 \wedge\ ...\ \wedge p_n \wedge p_{n+1}) = ME(K_2)(p_1 \wedge\ ...\ \wedge p_n).$$

By the inductive hypothesis,

$$ME(K_2)(p_1 \wedge\ ...\ \wedge p_n) = a_1 a_2 ... a_{n-1}(a_n a_{n+1})$$

so

$$ME(K)(p_1 \wedge\ ...\ \wedge p_n \wedge p_{n+1}) = a_1 a_2 ... a_n a_{n+1},$$

and the result now follows by induction. ■

Corollary 7 *Let*

$$K = \{Bel(p_1 \wedge p_0) = ab_1,\ ...,\ Bel(p_n \wedge p_0) = ab_n,\ Bel(p_0) = a\}$$

where $0 < a \leq 1,\ 0 \leq b_i \leq 1$. *Then*

$$ME(K)(p_1 \wedge\ ...\ \wedge p_n \wedge p_0) = ab_1 ... b_n.$$

Proof. Let

$$K_1 = \{Bel(p_1) = b_1,\ ...,\ Bel(p_n) = b_n,\ Bel(p_0) = a\}.$$

By Theorem 6

$$ME(K_1)(p_1 \wedge\ ...\ \wedge p_n \wedge p_0) = ab_1 ... b_n \tag{5}$$

By *Irrelevant Information* and Lemma 2

$$ME(K_1)(p_1 \wedge p_0) = ab_1,\ ...,\ ME(K_1)(p_n \wedge p_0) = ab_n$$

and thus also

$$ME(K_1)(p_1 \wedge \neg p_0) = b_1 - ab_1,\ ...,\ Bel(p_n \wedge \neg p_0) = b_n - ab_n.$$

Let

$$K_2 = \{Bel(p_1 \wedge \neg p_0) = b_1 - ab_1,\ ...,\ Bel(p_n \wedge \neg p_0) = b_n - ab_n\}.$$

By *Obstinacy* and *Equivalence*

$$ME(K_1) = ME(K_1 \cup K \cup K_2) = ME(K \cup K_2) \tag{6}$$

and by *Relativization* (with $\phi = p_0$),

$$ME(K \cup K_2)(p_1 \wedge\ ...\ \wedge p_n \wedge p_0) = ME(K)(p_1 \wedge\ ...\ \wedge p_n \wedge p_0). \tag{7}$$

The result now follows by (5), (6) and (7). ■

Theorem 8 *Let*

$$K = \{Bel(p_1 \wedge p_0) = ab,\ ...,\ Bel(p_n \wedge p_0) = ab, Bel(p_0) = a\}$$

where $0 < a \leq 1,\ 0 \leq b \leq 1$. *Then*

$$ME(K)(p_{n_1}^{\epsilon_1} \wedge\ ...\ \wedge p_{n_r}^{\epsilon_r} \wedge p_0) = ab^l(1-b)^{r-l}$$

where $l = \sum_{i=1}^r \epsilon_i$ *and* $1 \leq n_1 < n_2 < ... < n_r \leq n$.

Proof. For $\delta = 0, 1$ let $b(\delta) = b$ if $\delta = 0$ and $b(\delta) = 1 - b$ if $\delta = 1$. By *Renaming* (strictly a consequence *Renaming*, see Theorem 7.7 of [6])

$$ME(K)(p_1^{\delta_1} \wedge\ ...\ \wedge p_n^{\delta_n} \wedge p_0) = ME(K_1)(q_1 \wedge\ ...\ \wedge q_n \wedge p_0) \tag{8}$$

where

$$K_1 = \{Bel(q_1 \wedge p_0) = ab(\delta_1),\ ...,\ Bel(q_n \wedge p_0) = ab(\delta_n), Bel(p_0) = a\}.$$

By Corollary 7, (8) has value $ab^m(1-b)^{n-m}$ where $m = \sum_{i=1}^n \delta_i$, so the result follows by 'summing out' the p_js which are not amongst $p_{n_1}, ..., p_{n_r}$. (We have

$$ME(K)(p_{n_1}^{\epsilon_1} \wedge\ ...\ p_{n_r}^{\epsilon_r} \wedge p_0) = \sum_{\substack{\delta_{n_i} = \epsilon_{n_i} \\ \delta_j \in \{0,1\}\ otherwise}} ME(K)\left(p_1^{\delta_1} \wedge\ ...\ \wedge p_n^{\delta_n} \wedge p_0\right),$$

so fixing $j_0 \neq n_i$ $(i = 1, ..., r)$, the terms in the sum above split into pairs differing only in the value of δ_{j_0} each of which has sum $ab^{m'}(1-b)^{n-1-m'}$ where $m' = \sum_{j \neq j_0} \delta_j$. We can arrive at the result by eliminating the $j \neq n_i$ one by one.) ■

Theorem 9 *Let* $K(a_1)$ *be as in (1). Then*

$$\lim_{n\to\infty} ME\left(\bigcup_{j=1}^n K(a_j)\right)\left(\bigwedge_{i=1}^r P^{\epsilon_i}(a_{n_i})\right) = \sum_{k=1}^q \lambda_k \beta_k^l (1-\beta_k)^{r-l}$$

where $l = \sum_{i=1}^r \epsilon_i$, *i.e.* $\lim_{n\to\infty} ME\left(\bigcup_{j=1}^n K(a_j)\right)$ *exists and equals the canonical solution.*

Proof. We have

$$ME\left(\bigcup_{j=1}^{n} K(a_j)\right)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right) = \sum_{k=1}^{q} ME\left(\bigcup_{j=1}^{n} K(a_j)\right)\left(Q_k \wedge \bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right).$$

Assume $n > n_1, ..., n_r$. By relativization(with $\phi = Q_k$) and by Theorem 8,

$$ME\left(\bigcup_{j=1}^{n} K(a_j)\right)\left(Q_k \wedge \bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right) = \lambda_k \beta_k^l (1-\beta_k)^{r-l}$$

where $l = \sum_{i=1}^{r} \epsilon_i$. The result follows. ■

In the above case the situation is particularly simple because each sequence

$$ME(\bigcup_{j=1}^{n} K(a_j))(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})),$$

is eventually constant. More interesting is the case of a general (finite, satisfiable, of course) $K(a_1)$ discussed below.

The ME answer for a general $K(a_1)$

Theorem 10 *The limits*

$$\lim_{n\to\infty} ME\left(\bigcup_{j=1}^{n} K(a_j)\right)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right)$$

exist and agree with the canonical solution for some complete set of reasons.

This is perhaps initially a rather surprising result. It says that no matter what our generic knowledge $K(a_1)$ is, if we assign values to the $\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})$ according to $ME(\bigcup_{j=1}^{n} K(a_j))$ (i.e. according to common sense on the basis of knowledge $\bigcup_{j=1}^{n} K(a_j)$) then in the limit these assignments look as if they have been based, canonically, on some complete set of reasons. In other words, in the limit 'reasons' have *emerged* to explain our answers!

Of course the proof itself provides some de-mystification, these 'reasons' in fact correspond in the limit to the atoms of the language with propositional variables $Q_1, Q_2, ..., Q_q$, so in particular there are just 2^q of them. Thinking of these atoms as specifying the background world, or state of the world, in which the experiments are to be conducted leads then in turn to identifying these 'reasons' with the 'possible worlds'.

Proof of Theorem 10. Let

$$K^n = \bigcup_{j=1}^{n} K(a_j) \qquad (0 < n \in \mathbb{N}).$$

Note that K^n is consistent by Lemma 1 since we assume that $K(a_1)$ is consistent. Let

$$ME(K^n)(\rho_k) = S_k^n \quad (1 \le k \le 2^q).$$

By *Renaming*, $ME(K^n)(\rho_k \wedge P(a_i))$ $(1 \le i \le n)$ is independent of i. Let

$$ME(K^n)(\rho_k \wedge P(a_1)) = \ ... \ = ME(K^n)(\rho_k \wedge P(a_n)) = \ S_k^n X_k^n.$$

Now let

$$K_1^n = \{Bel(\rho_k) = S_k^n,\ Bel(\rho_k \wedge P(a_i)) = S_k^n X_k^n \mid k = 1, ..., 2^q,\ i = 1, ..., n\}.$$

By *Obstinacy* and *Equivalence* (using the fact that $K^n \cup K_1^n$ is equivalent to K_1^n),

$$ME(K^n) = ME(K^n \cup K_1^n) = ME(K_1^n).$$

Using *Relativization* (with $\phi = \rho_k$), Theorem 8 and Theorem 7.10 from [6] (to be able to treat ρ_k as if it was a (new) propositional variable, as needed in Theorem 8) we can conclude

$$ME(K_1^n)\left(\rho_k \wedge \bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)\right) = S_k^n\, (X_k^n)^m\, (1 - X_k^n)^{n-m}$$

where $m = \sum_{i=1}^n \epsilon_i$. Consequently, the probability function Bel on $SL^{(n)}$ defined by

$$Bel\left(\rho_k \wedge \bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)\right) = S_k^n\, (X_k^n)^{\sum_{i=1}^n \epsilon_i}\, (1 - X_k^n)^{n-\sum_{i=1}^n \epsilon_i}$$

($k = 1, ..., 2^q$ and $\epsilon_i \in \{0, 1\}$), satisfies K^n and since it is the solution of K^n which maximizes the entropy, no other solution of K^n determined analogously by $\vec{x}$, $\vec{s}$ ($\vec{x} = (x_1, ..., x_{2^q})$, $\vec{s} = (s_1, ..., s_{2^q})$) has as great an entropy. In other words, X_k^n, S_k^n is the point for which

$$f_n(\vec{x}, \vec{s}) = -\sum_{k,\epsilon_1,\ldots,\epsilon_n} s_k x_k^{\sum_{i=1}^n \epsilon_i}(1 - x_k)^{(n-\sum_{i=1}^n \epsilon_i)} \log\left(s_k x_k^{\sum_{i=1}^n \epsilon_i}(1 - x_k)^{(n-\sum_{i=1}^n \epsilon_i)}\right)$$

is maximal amongst all $1 \geq \vec{x} \geq 0$, $\vec{s} \geq 0$, $\sum_k s_k = 1$ for which Bel defined by

$$Bel\left(\rho_k \wedge \bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)\right) = s_k\, (x_k)^{\sum_{i=1}^n \epsilon_i}\, (1 - x_k)^{n-\sum_{i=1}^n \epsilon_i} \tag{9}$$

($k = 1, ..., 2^q$ and $\epsilon_i \in \{0, 1\}$), gives a solution of K^n, i.e.

$$Bel(\rho_k \wedge P(a_1)) = s_k x_k, \quad Bel(\rho_k \wedge \neg P(a_1)) = s_k(1 - x_k)$$

gives a solution of K^1. Now notice that

$$\begin{aligned}
&f_n(\vec{x}, \vec{s}) \ = \\
&= -\sum_{k,\epsilon_1,\ldots,\epsilon_n} s_k x_k^{\left(\sum_{i=1}^n \epsilon_i\right)}(1 - x_k)^{\left(n-\sum_{i=1}^n \epsilon_i\right)} \log\left(s_k x_k^{\left(\sum_{i=1}^n \epsilon_i\right)}(1 - x_k)^{\left(n-\sum_{i=1}^n \epsilon_i\right)}\right) \\
&= -\sum_k s_k \sum_{j=0}^{n} \binom{n}{j} x_k^j (1 - x_k)^{n-j} \log\left(s_k x_k^j (1 - x_k)^{n-j}\right) \\
&= -\sum_k s_k \log s_k - \sum_k s_k \sum_{j=0}^{n} \binom{n}{j} x_k^j (1 - x_k)^{n-j} (j \log x_k + (n - j) \log(1 - x_k)).
\end{aligned}$$

Since

$$\sum_{j=0}^{n} \binom{n}{j} j x^j (1 - x)^{n-j} = nx$$

(as can be seen from differentiating $(1-y+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j (1-y)^{n-j}$ with respect to x, letting $y = x$ and multiplying by x), and similarly

$$\sum_{j=0}^{n} \binom{n}{j} (n-j) x^j (1-x)^{n-j} = n(1-x),$$

we have

$$f_n(\vec{x}, \vec{s}) = -\sum_k s_k \log s_k - \sum_k n s_k (x_k \log x_k + (1-x_k)\log(1-x_k)).$$

Now let

$$y_k = s_k x_k \quad and \quad z_k = s_k(1-x_k).$$

Substituting $y_k + z_k = s_k$ and $\frac{y_k}{y_k+z_k} = x_k$ we get (even if $s_k = 0$, since $0 \log 0 = 0$)

$$\begin{aligned} f_n(\vec{x}, \vec{s}) = g_n(\vec{y}, \vec{z}) = \\ = & -\sum_k (y_k + z_k) \log(y_k + z_k) \\ & -n \sum_k (y_k + z_k) \left(\frac{y_k}{y_k+z_k} \log \frac{y_k}{y_k+z_k} + \frac{z_k}{y_k+z_k} \log \frac{z_k}{y_k+z_k} \right) \\ = & -\sum_k (y_k + z_k) \log(y_k + z_k) \\ & +n \left(-\sum_k y_k \log y_k - \sum_k z_k \log z_k + \sum_k (y_k + z_k) \log(y_k + z_k) \right) \end{aligned}$$

Maximizing f_n subject to Bel (on $SL^{(n)}$, as in (9)) being a solution of K^n thus amounts to maximizing g_n subject to Bel (on $SL^{(1)}$) defined by

$$Bel(\rho_k \wedge P(a_1)) = y_k, \quad Bel(\rho_k \wedge \neg P(a_1)) = z_k$$

being a solution of K^1 i.e. of $K(a_1)$. If $\vec{Y^n}$, $\vec{Z^n}$ is the solution of K^1 which maximizes g_n then

$$ME(K^n)\left(\rho_k \wedge \bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)\right) = \begin{cases} \frac{(Y_k^n)^{\Sigma\epsilon_i}(Z_k^n)^{n-\Sigma\epsilon_i}}{(Y_k^n+Z_k^n)^{n-1}} & if\ Y_k^n + Z_k^n \neq 0, \\ 0 & if\ Y_k^n + Z_k^n = 0. \end{cases} \tag{10}$$

We will show that the points $\vec{Y^n}$, $\vec{Z^n}$ which maximize g_n subject to K^1 tend to the point which maximizes

$$h(\vec{y}, \vec{z}) = -\sum_k y_k \log y_k - \sum_k z_k \log z_k + \sum_k (y_k + z_k) \log(y_k + z_k)$$

subject to K^1 (and, in case when there are several such points, that point amongst them for which

$$h_0(\vec{y},\ \vec{z}) = -\sum_k (y_k + z_k) \log(y_k + z_k)$$

is maximal). Let us denote this point $(\vec{Y}, \vec{Z})$. Note that $(\vec{Y}, \vec{Z})$ is uniquely defined since the function h is concave on $\{(\vec{y}, \vec{z}) \mid y_k \geq 0,\ z_k \geq 0,\ k = 1, ..., 2^q\}$. (To show the concavity of h, differentiate

$$h(t\vec{y} + (1-t)\vec{y'},\ t\vec{z} + (1-t)\vec{z'})$$

twice with respect to t, where $y_k, y'_k, z_k, z'_k \geq 0,\ t \in (0,1)$ (leaving out terms involving $y_k,\ y'_k$ when $y_k = y'_k = 0$, $z_k,\ z'_k$ when $z_k = z'_k = 0$ and $(y_k + z_k),\ (y'_k + z'_k)$ when $y_k = y'_k = z_k = z'_k = 0$), to obtain

$$-\sum_k \left(\frac{(y_k - y'_k)^2}{ty_k + (1-t)y'_k} + \frac{(z_k - z'_k)^2}{tz_k + (1-t)z'_k} - \frac{(y_k - y'_k + z_k - z'_k)^2}{(ty_k + (1-t)y'_k) + (tz_k + (1-t)z'_k)} \right) =$$

$$= \ -\sum_k \frac{((tz_k + (1-t)z'_k)(y_k - y'_k) - (ty_k + (1-t)y'_k)(z_k - z'_k))^2}{(tz_k + (1-t)z'_k)(ty_k + (1-t)y'_k)(t(y_k + z_k) + (1-t)(y'_k + z'_k))} \ \leq \ 0.)^3$$

Consequently, the set $Max(K^1, h)$ of solutions of K^1 which maximize h is convex, so the set of $((y_1 + z_1), ..., (y_{2^q} + z_{2^q}))$ such that $(\vec{y},\ \vec{z}) \in Max(K^1, h)$ is convex and thus there is a unique $((y_1 + z_1), ..., (y_{2^q} + z_{2^q}))$ amongst them which maximizes h_0 (entropy). The convexity of $Max(K^1, h)$ also guarantees that $((y_1 + z_1), ..., (y_{2^q} + z_{2^q}))$ determines $(\vec{y},\ \vec{z}) \in Max(K^1, h)$ uniquely, since adding $\{y_k + z_k = c_k :\ k = 1, ..., 2^q\}$ to K^1 produces a set of linear constraints with a convex set of solutions, for which h_0 is constant, and amongst which there is a unique one maximizing entropy of $(\vec{y},\ \vec{z})$, that is $h(\vec{y},\ \vec{z}) + h_0(\vec{y},\ \vec{z})$.

First assume that there is a unique point which maximizes h subject to K^1 (denoted $\vec{Y}, \vec{Z}$). Suppose that the sequence $(\vec{Y^n}, \vec{Z^n})$ does not tend to it. Then there is a subsequence $\vec{Y}^{n_r}, \vec{Z}^{n_r}$ which tends to $(\vec{Y'}, \vec{Z'}) \neq (\vec{Y}, \vec{Z})$. By the choice of $(\vec{Y}, \vec{Z})$,

$$h(\vec{Y}, \vec{Z}) - h(\vec{Y'}, \vec{Z'}) = \eta > 0$$

so by continuity of h, for sufficiently large n_r

$$h(\vec{Y}, \vec{Z}) - h(\vec{Y}^{n_r}, \vec{Z}^{n_r}) \geq \frac{\eta}{2}$$

and

$$\begin{aligned} & g_{n_r}(\vec{Y}, \vec{Z}) - g_{n_r}(\vec{Y}^{n_r}, \vec{Z}^{n_r}) = \\ = & \ -\textstyle\sum_k (Y_k + Z_k)\log(Y_k + Z_k) + \sum_k (Y_k^{n_r} + Z_k^{n_r}) \log((Y_k^{n_r} + Z_k^{n_r}) \\ & \ +n_r \left(h(\vec{Y}, \vec{Z}) - h\left(\vec{Y}^{n_r}, \vec{Z}^{n_r}\right)\right) \\ \geq & \ -q \log 2 + n_r \tfrac{\eta}{2} \end{aligned}$$

(since $\sum_k (Y_k + Z_k) = 1,\ \sum_k (Y_k^{n_r} + Z_k^{n_r}) = 1$ and $-\sum w_k \log w_k$ is positive and less or equal to $-\sum_k \frac{1}{2^q} \log \frac{1}{2^q} = q \log 2$ whenever $\sum_k w_k = 1$). This shows that for large n_r,

$$g_{n_r}(\vec{Y},\ \vec{Z}) > g_{n_r}\left(\vec{Y}^{n_r}, \vec{Z}^{n_r}\right)$$

contradicting the choice of $\left(\vec{Y}^{n_r}, \vec{Z}^{n_r}\right)$.

In case when h does not have a unique maximum subject to K^1 (and thus $(\vec{Y}, \vec{Z})$ is that (unique) point in $Max(K^1, h)$ for which h_0 is maximal), an argument similar to the one above shows that if the sequence $(\vec{Y^n},\ \vec{Z^n})$ does not tend to $(\vec{Y}, \vec{Z})$, then there is a subsequence $\vec{Y}^{n_r},\ \vec{Z}^{n_r}$ which tends to $(\vec{Y'},\ \vec{Z'}) \neq (\vec{Y},\ \vec{Z})$, where $(\vec{Y'}, \vec{Z'}) \in Max(K^1, h)$. We have

$$h(\vec{Y}, \vec{Z}) \geq h\left(\vec{Y}^{n_r}, \vec{Z}^{n_r}\right)$$

so since $g_{n_r}(\vec{y},\ \vec{z}) = h_0(\vec{y},\ \vec{z}) + n_r h(\vec{y},\ \vec{z})$

$$h_0\left(\vec{Y}^{n_r},\ \vec{Z}^{n_r}\right) \geq h_0(\vec{Y},\ \vec{Z}).$$

[3] Taking $k = 1$ this method shows in particular that the function $f(u, v) = -u \log u - v \log v + (u + v)\log(u + v)$ $(u \geq 0,\ v \geq 0)$ is concave, a fact which we will use later.

Consequently,

$$h_0(\vec{Y'},\ \vec{Z'}) \geq h_0(\vec{Y},\ \vec{Z}),$$

but that is impossible.
From (10) it follows (by 'summing out' the $P(a_j)$ which are not involved) that if $0 < n_1 < n_2 < ... < n_r \leq n$ $(i = 1, ..., r)$ then

$$ME(K^n)\left(\rho_k \wedge \bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right) = \begin{cases} \frac{(Y_k^n)^{\Sigma\epsilon_i}(Z_k^n)^{r-\Sigma\epsilon_i}}{(Y_k^n+Z_k^n)^{r-1}} & if\ Y_k^n + Z_k^n \neq 0, \\ 0 & if\ Y_k^n + Z_k^n = 0, \end{cases} \tag{11}$$

and hence

$$\lim_{n\to\infty} ME(K^n)\left(\rho_k \wedge \bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right) = \begin{cases} \frac{(Y_k)^{\Sigma\epsilon_i}(Z_k)^{r-\Sigma\epsilon_i}}{(Y_k+Z_k)^{r-1}} & if\ Y_k + Z_k \neq 0, \\ 0 & if\ Y_k + Z_k = 0. \end{cases} \tag{12}$$

In other words these limit values agree with those given by a canonical set of reasons $Q_1, ..., Q_{2^q}$ with strengths $\beta_j = Y_j/(Y_j + Z_j)$ and weights $\lambda_j = Y_j + Z_j$, proving Theorem 10.

■

To give a specific, albeit completely contrived, example suppose you are sitting in the waiting room of a driving test centre waiting for your nephew to return from taking his test. You know nothing at all about your nephew's prowess at the wheel. However you do have some fragments of knowledge about the arrangements for the test itself. Namely; on any one day all tests are carried out by the same examiner (one of L or S) around the same circuit (one of A or B); on average 30% of drivers pass; a driver who passes is twice as likely to have been tested by L than by S; if circuit A is chosen then the driver has only a 20% chance of passing; S prefers circuit B 70% of the time. The secretary now tells you with a malicious grin that all 3 previous tests that morning have resulted in failure. Based on this limited information what belief, as subjective probability, should you give to your nephew breaking the pattern?
Denoting the event of a successful test for driver a_i by $P(a_i)$ etc. and your personal probability function by Bel your knowledge might reasonably be captured by the following $K(a_1)$:

$$\begin{aligned} Bel(P(a_1)) &= 3/10 \\ Bel(P(a_1) \wedge \neg S) &= 2Bel(P(a_1) \wedge S) \\ Bel(P(a_1)|A) &= 1/5 \\ Bel(\neg A|S) &= 7/10 \end{aligned}$$

In this case the probability function,

$$Bel(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})) = \lim_{n\to\infty} ME\left(\bigcup_{j=1}^{n} K(a_j)\right)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right)$$

given by Theorem 10 is the canonical solution of the complete set of reasons given by (1) when $q = 4$ (corresponding to the 4 atoms $A \wedge S, A \wedge \neg S, \neg A \wedge S, \neg A \wedge \neg S$) and the β_i, λ_i are given by

i	1	2	3	4
β_i	0.554	0.666	0.282	0.275
λ_i	0.0824	0.000868	0.192	0.7243

In particular then, on learning of the 3 previous failures that morning Bel dictates that you should only give probability

$$Bel(P(a_4)|\neg P(a_1) \wedge \neg P(a_2) \wedge \neg P(a_3)) = \frac{\sum_{i=1}^{4} \lambda_i \beta_i (1-\beta_i)^3}{\sum_{i=1}^{4} \lambda_i (1-\beta_i)^3} = 0.167$$

to your nephew passing.

It is important to emphasize in this example that, as with the rest of this paper, we are dealing here with beliefs as *subjective probabilities*. Were we to consider the above example in terms of objective probabilities, say as given by the long term frequencies of the various combinations of circuits, examiners and outcomes, then surely most of us would be reasonably happy to assign a probability to $\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})$ of

$$\sum RF(\pm A \wedge \pm S) \cdot RF(P(a_1)| \pm A \wedge \pm S)^l (1 - RF(P(a_1)| \pm A \wedge \pm S))^{r-l},$$

where as usual $l = \sum \epsilon_i$ and $RF(A \wedge S)$ is the relative frequency of $A \wedge S$ etc.. In other words, to accept these $\pm A \wedge \pm S$ as 'reasons' with the λ_i, β_i given by the corresponding relative frequencies. What we show in this paper is that, if we accept the arguments for using ME in this way then complete sets of reasons emerge naturally also in the case where probabilities are subjective degrees of belief (and, as far as this paper is concerned, the knowledge base has this rather restricted form).

To those familiar with the popular image of ME the conclusion that the atoms are the 'reasons' may, after some brief consideration, appear a not unexpected *artifice* of ME. After all, a common view of ME is that it tries to avoid introducing unnecessary dependencies. In this case (i.e. $K(a_1)$) it seems that the only way knowledge of the outcome of one experiment can provide information about the outcome of another experiment is through the mediation of its effect on the possible worlds. Conditioning on a fixed world then should leave ME free to treat the experiments as entirely independent.
Attractive as this explanation may appear in this simple case further investigations would seem to suggest that this is not quite the whole story. Firstly, as we shall see later, this behavior is not simply an artifice of ME. Secondly, this behavior continues to be manifest in more general cases (than the simple $K(a_1)$ we looked at here), in particular where finitely many predicates $P(x)$ are allowed and where the above mentioned conditional independence is lacking. The next subsection deals with one possible generalization, namely considering $K(a_1)$ where in place of a single P, $P_1, ..., P_p$ are allowed (continuing the original example, $P_1(x)$ may stand for car x skids, $P_2(x)$ for car x has ABS, $P_3(x)$ for driver of car x is drunk etc.).

The ME answer for $K(a_1)$ involving $P_1, ..., P_p$

Let $C\mathcal{L}$ denote the set of all finite, linear and satisfiable sets K of constraints on a probability function Bel on the sentences $S\mathcal{L}$ of a finite propositional language $\mathcal{L}$. We shall need the following theorem which generalizes Theorem 6[4]:

[4]Again this result could be proved directly from the definition of ME. We shall not do so however because for future applications it will be useful to have a proof which hinges simply on the fact that ME satisfies certain principles

Theorem 11 *Let* $\mathcal{L}_1 = \{p_1, ..., p_m\}$, $\mathcal{L}_2 = \{q_1, ..., q_s\}$ *(*$\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$, $m \geq 1,\ s \geq 1$*), and let* $K_1 \in C\mathcal{L}_1$ *and* $K_2 \in C\mathcal{L}_2$*. Then*

$$ME(K_1 \cup K_2)\left(\bigwedge_{i=1}^{m} p_i^{\epsilon_i} \wedge \bigwedge_{j=1}^{s} q_j^{\delta_j}\right) = ME(K_1)\left(\bigwedge_{i=1}^{m} p_i^{\epsilon_i}\right) ME(K_2)\left(\bigwedge_{j=1}^{s} q_j^{\delta_j}\right). \tag{13}$$

To prove this theorem, we first derive a lemma.

Lemma 12 *Let* $\mathcal{L}_1 = \{p\}$, $\mathcal{L}_2 = \{q_1, ..., q_s\}$ *(*$p \notin \mathcal{L}_2$*) and let* $\beta(\vec{\delta}) = \bigwedge_j q_j^{\delta_j}$ *be the atoms of* $\mathcal{L}_2$*. Let* $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ *and let* $K_1, K_2 \in C\mathcal{L}$ *be, respectively,* $\{Bel(p) = e\}$ *and* $\{Bel(\beta(\vec{\delta})) = b(\vec{\delta}) :\ \delta_j \in \{0, 1\}\}$*. Then*

$$ME(K_1 \cup K_2)(p \wedge \beta(\vec{\delta})) = eb(\vec{\delta}),$$

$$ME(K_1 \cup K_2)(\neg p \wedge \beta(\vec{\delta})) = (1 - e)b(\vec{\delta}).$$

Proof. Let $ME(K_1 \cup K_2)\left(p \wedge \bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = f(j_0, \vec{\delta})$ and $ME(K_1 \cup K_2)\left(\bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = g(j_0, \vec{\delta})$, so $g(j_0, \vec{\delta})$ is the sum of $b(\vec{\delta})$ and $b(\vec{\delta}')$, where δ'_j differs from δ_j just if $j = j_0$, but we do not know at this stage what $f(j_0, \vec{\delta})$ is. Now fix j_0 and $\vec{\delta}$ and denote the set of constraints

$$K_1 \cup K_2 \cup \left\{Bel\left(p \wedge \bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = f(j_0, \vec{\delta})\right\}$$

by K. By *Obstinacy*,

$$ME(K) = ME(K_1 \cup K_2). \tag{14}$$

Note that K is equivalent to

$$K_2 \cup \left\{Bel\left(p \wedge \bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = f(j_0, \vec{\delta}),\ Bel\left(p \wedge \neg \bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = e - f(j_0, \vec{\delta})\right\}$$

and thus by *Relativization* (with $\phi = \bigwedge_{j \neq j_0} q_j^{\delta_j}$) and *Equivalence*

$$ME(K)(p \wedge \beta(\vec{\delta})) = ME(K')(p \wedge \beta(\vec{\delta}))$$

where

$$\begin{aligned} K' \ = \ \Big\{ & Bel\left(\textstyle\bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = g(j_0, \vec{\delta}),\ Bel\left(p \wedge \textstyle\bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = f(j_0, \vec{\delta}), \\ & Bel\left(q_{j_0}^{\delta_{j_0}} \wedge \textstyle\bigwedge_{j \neq j_0} q_j^{\delta_j}\right) = b(\vec{\delta})\Big\} \end{aligned}$$

so by *Independence* and theorem 7.10 from [6],

$$ME(K)(p \wedge \beta(\vec{\delta})) = \frac{f(j_0, \vec{\delta})b(\vec{\delta})}{g(j_0, \vec{\delta})}. \tag{15}$$

This shows that $\frac{f(j_0,\vec{\delta})}{g(j_0,\vec{\delta})}$ is independent of j_0, and consequently also of $\vec{\delta}$ since $\frac{f(j_0,\vec{\delta})}{g(j_0,\vec{\delta})}$ is, by definition of $f(j_0, \vec{\delta})$ and $g(j_0, \vec{\delta})$, independent of δ_{j_0}. Denoting their common value by a we have

$$ME(K)\left(p \wedge \beta(\vec{\delta})\right) = ab(\vec{\delta}).$$

Summing the above over $\vec{\delta}$ yields $ME(K)(p) = a$ so $a = e$ and the first result follows by (15) and (14). The dual result follows immediately. ■

Proof of Theorem 11 The proof is by induction on $m + s$. If $m = 1$, $s = 1$ then the claim follows by Lemma 2 (i). Assume that it is true for $m + s = k \geq 2$ and consider $m + s = k + 1$. Without loss of generality we can assume that $m > 1$. Let $\alpha_1, ..., \alpha_{2^m}$ denote the atoms of $\mathcal{L}_1$ and assume that they are arranged so that so that the first quarter of them contain $p_1 \wedge p_2$, the second quarter contain $p_1 \wedge \neg p_2$, the third quarter contain $\neg p_1 \wedge p_2$ and the last quarter contain $\neg p_1 \wedge \neg p_2$, in conjunction with atoms involving the remaining variables, in the same order for each quarter. Let

$$ME(K_1)(\alpha_i) = a_i, \quad ME(K_2)(\beta_j) = b_j, \quad (i = 1, ..., 2^m, \ j = 1, ..., 2^s).$$

By *Irrelevant Information*,

$$ME(K_1 \cup K_2)(\alpha_i) = a_i, \quad ME(K_1 \cup K_2)(\beta_j) = b_j, \quad (i = 1, ..., 2^m, \ j = 1, ..., 2^s).$$

so by *Obstinacy* and *Equivalence*,

$$ME(K_1 \cup K_2) = ME(K) \tag{16}$$

where

$$K = \{Bel(\alpha_i) = a_i, \ Bel(\beta_j) = b_j \mid i = 1, ..., 2^m, \ j = 1, ..., 2^s\}.$$

If for some i, $ME(K)(p_i)$ was 0 or 1 then (13) would either obviously amount to $0 = 0$ or it could be reduced (using *Irrelevant Information*) to involve one variable less and the inductive assumption would give us the result, so assume $ME(K)(p_i) \neq 0$, $ME(K)(p_i) \neq 1$ for $i = 1, ..., m$. Let us denote

$$\sum_{i=1}^{2^{m-1}} a_i = ME(K)(p_1) = e_{p_1}, \quad \sum_{i=1}^{2^{m-2}} (a_i + a_{i+2^{m-1}}) = ME(K)(p_2) = e_{p_2},$$

$$\sum_{i=1}^{2^{m-2}} a_i = ME(K)(p_1 \wedge p_2) = e_{p_1 \wedge p_2}, \quad \sum_{i=3\times 2^{m-2}+1}^{2^m} a_i = ME(K)(\neg p_1 \wedge \neg p_2) = e_{\neg p_1 \wedge \neg p_2}$$

$$\sum_{i=2^{m-1}+1}^{3\times 2^{m-2}} a_i = ME(K)(\neg p_1 \wedge p_2) = e_{\neg p_1 \wedge p_2}, \quad \sum_{i=2^{m-2}+1}^{2^{m-1}} a_i = ME(K)(p_1 \wedge \neg p_2) = e_{p_1 \wedge \neg p_2}.$$

Let us further denote $ME(K)(p_1 \wedge \beta_j) = c_j$, and let α'_i $(i = 1, ..., 2^{m-1})$ stand for the atoms of $\{p_2, ..., p_m\}$ such that $\alpha_i = p_1 \wedge \alpha'_i$ (and $\alpha_{i+2^{m-1}} = \neg p_1 \wedge \alpha'_i$). By *Obstinacy* and *Equivalence*,

$$ME(K) = ME(K') \tag{17}$$

where

$$\begin{aligned} K' = \{ \ & Bel(p_1) = e_{p_1}, \quad Bel(p_1 \wedge \alpha'_i) = a_i, \ Bel(\beta_j \wedge p_1) = c_j, \\ & Bel(\neg p_1 \wedge \alpha'_i) = a_{i+2^{m-1}}, \ Bel(\neg p_1 \wedge \beta_j) = b_j - c_j \\ & \mid i = 1, ..., 2^{m-1}, \ j = 1, ..., 2^s \ \}. \end{aligned}$$

By *Relativization* (with $\phi = p_1$), we have

$$ME(K')(p_1 \wedge \alpha'_i \wedge \beta_j) = ME(K'')(p_1 \wedge \alpha'_i \wedge \beta_j) \tag{18}$$

where

$$K'' = \{Bel(p_1) = e_{p_1},\quad Bel(p_1 \wedge \alpha_i') = a_i,\ Bel(p_1 \wedge \beta_j) = c_j,$$
$$Bel(\neg p_1 \wedge \alpha_i') = (1-e_{p_1})\tfrac{a_i}{e_{p_1}},\ Bel(\neg p_1 \wedge \beta_j) = (1-e_{p_1})\tfrac{c_j}{e_{p_1}}$$
$$\mid i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s\ \}.$$

Let

$$K''' = \{Bel(p_1) = e_{p_1},\ Bel(\alpha_i') = \frac{a_i}{e_{p_1}},\ Bel(\beta_j) = \frac{c_j}{e_{p_1}} \mid i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s\}.$$

By *Irrelevant Information* and Lemma 12, $ME(K''')(p_1 \wedge \alpha_i') = a_i$ and $ME(K''')(p_1 \wedge \beta_j) = c_j$ so by *Obstinacy* and *Equivalence*,

$$ME(K''') = ME(K''). \tag{19}$$

By *Irrelevant Information* and the inductive assumption,

$$ME(K''')(\alpha_i' \wedge \beta_j) = \frac{a_i}{e_{p_1}}\frac{c_j}{e_{p_1}} \quad i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s.$$

By *Obstinacy* and *Equivalence*,

$$ME(K''') = ME(K^{iv}) \tag{20}$$

where

$$K^{iv} = \{Bel(p_1) = e_{p_1},\ Bel(\alpha_i' \wedge \beta_j) = \frac{a_i}{e_{p_1}}\frac{c_j}{e_{p_1}} \mid i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s\}.$$

By Lemma 12,

$$ME(K^{iv})(p_1 \wedge \alpha_i' \wedge \beta_j) = \frac{a_i c_j}{e_{p_1}}, \quad i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s.$$

Consequently (using (17), (18), (19), (20))

$$ME(K)(p_1 \wedge \alpha_i' \wedge \beta_j) = ME(K)(\alpha_i \wedge \beta_j) = \frac{a_i c_j}{e_{p_1}}, \quad i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s. \tag{21}$$

Similarly (using Relativization with $\phi = \neg p_1$ we can prove that

$$ME(K)(\neg p_1 \wedge \alpha_i' \wedge \beta_j) = ME(K)(\alpha_{i+2^{m-1}} \wedge \beta_j) = \frac{a_{i+2^{m-1}}(b_j - c_j)}{1-e_{p_1}} \tag{22}$$

for $i = 1, ..., 2^{m-1},\ j = 1, ..., 2^s$. Consequently,

$$\begin{aligned} ME(K)(p_2 \wedge \beta_j) = &\ \textstyle\sum_{i=1}^{2^{m-2}} (ME(K)(\alpha_i \wedge \beta_j) + ME(K)(\alpha_{2^{m-1}+i} \wedge \beta_j)) \\ = &\ \tfrac{e_{p_1 \wedge p_2}}{e_{p_1}} c_j + \tfrac{e_{\neg p_1 \wedge p_2}}{1-e_{p_1}}(b_j - c_j). \end{aligned}$$

Let $c_j' = \frac{e_{p_1 \wedge p_2}}{e_{p_1}} c_j + \frac{e_{\neg p_1 \wedge p_2}}{1-e_{p_1}}(b_j - c_j)$. Treating p_2 as we did p_1 before, we can now add $Bel(p_2 \wedge \beta_j) = c_j'$ to K obtaining this time that

$$ME(K)(\alpha_i \wedge \beta_j) = \frac{a_i c_j'}{e_{p_2}}, \quad i = 1, ..., 2^{m-2}, 2^{m-1}+1, ..., 3 \times 2^{m-2},\ j = 1, ..., 2^s$$

and

$$ME(K)(\alpha_{i+2^{m-2}} \wedge \beta_j) = \frac{a_i(b_j - c_j')}{1-e_{p_2}}, \quad i = 1, ..., 2^{m-2}, 2^{m-1}+1, ..., 3 \times 2^{m-2},\ j = 1, ..., 2^s.$$

Assume that there an i in $\{1, ..., 2^{m-2}\}$ such that $a_i \neq 0$ (i.e. assume that $e_{p_1 \wedge p_2} \neq 0$). Then we have that $\frac{a_i c_j}{e_{p_1}} = \frac{a_i c'_j}{e_{p_2}}$, for $j = 1, ..., 2^s$ i.e.

$$\frac{c_j}{e_{p_1}} = \frac{\frac{e_{p_1 \wedge p_2}}{e_{p_1}} c_j + \frac{e_{\neg p_1 \wedge p_2}}{1 - e_{p_1}} (b_j - c_j)}{e_{p_2}}.$$

Using $e_{p_2} - e_{p_1 \wedge p_2} = e_{\neg p_1 \wedge p_2}$ this yields

$$c_j(e_{\neg p_1 \wedge p_2}((1 - e_{p_1}) + e_{p_1}) = e_{\neg p_1 \wedge p_2} e_{p_1} b_j,$$

i.e. (provided $e_{\neg p_1 \wedge p_2} \neq 0$)

$$c_j = e_{p_1} b_j \quad j = 1, ..., 2^s.$$

This, together with (21), (22) and (16), proves the theorem except if there is a problem with zeros. The above works when $e_{p_1 \wedge p_2} \neq 0$ and $e_{\neg p_1 \wedge p_2} \neq 0$. Similarly, swapping the roles of p_2 and $\neg p_2$, we get a proof also when $e_{p_1 \wedge \neg p_2} \neq 0$ and $e_{\neg p_1 \wedge \neg p_2} \neq 0$. Given that e_{p_1} and e_{p_2} cannot be 0 or 1, if both of the above fail to hold then either $e_{p_1 \wedge p_2} = 0$ and $e_{\neg p_1 \wedge \neg p_2} = 0$ or $e_{p_1 \wedge \neg p_2} = 0$ and $e_{\neg p_1 \wedge p_2} = 0$. Then we can consider from the beginning (in place of K_1, written in terms of the atoms) the set of constraints $\overline{K}_1$ obtained from K_1 by swapping each atom α_i with $\alpha_{i+2^{m-2}}$ for $i \in \{1, ..., 2^{m-2}\}$ and leaving the atoms α_i for $i \in \{2^{m-1} + 1, ..., 2^m$. By *Renaming*,

$$ME(K_1)(\alpha_i) = ME(\overline{K}_1)(\alpha_{i+2^{m-2}}), \; ME(K_1)(\alpha_{i+2^{m-2}}) = ME(\overline{K}_1)(\alpha_i)$$

for $i = 1, ..., 2^{m-2}$ and

$$ME(K_1)(\alpha_i) = ME(\overline{K}_1)(\alpha_i), \quad i = 2^{m-1} + 1, ..., 2^m.$$

Our proof above now works for $\overline{K}_1$ and K_2 since we swapped the roles of $p_1 \wedge p_2$ and $p_1 \wedge \neg p_2$. By *Renaming* again

$$ME(K_1 \cup K_2)(\alpha_i \wedge \beta_j) = ME(\overline{K}_1 \cup K_2)(\alpha_{i+2^{m-2}} \wedge \beta_j),$$

$$ME(K_1 \cup K_2)(\alpha_{i+2^{m-2}} \wedge \beta_j) = ME(\overline{K}_1 \cup K_2)(\alpha_i \wedge \beta_j),$$

for $i = 1, ..., 2^{m-2}$ and

$$ME(K_1 \cup K_2)(\alpha_i \wedge \beta_j) = ME(\overline{K}_1 \cup K_2)(\alpha_i \wedge \beta_j), \quad i = 2^{m-1} + 1, ..., 2^m$$

so the result follows.

■

An obvious consequence obtained by induction is

Corollary 13 *Let $K_1, ..., K_n$ be sets of linear constraints over pairwise disjoint languages $\mathcal{L}_1, ..., \mathcal{L}_n$ respectively and let $\sigma_1, ..., \sigma_n$ be some atoms of $\mathcal{L}_1, ..., \mathcal{L}_n$ respectively. Then*

$$ME(K_1 \cup \; ... \; \cup K_n)(\sigma_1 \wedge \; ... \; \wedge \sigma_n) = ME(K_1)(\sigma_1)...ME(K_n)(\sigma_n).$$

From this we get

Corollary 14 *Let $\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n$ be pairwise disjoint languages and let $\sigma_0, \sigma_1, ..., \sigma_n$ be atoms of $\mathcal{L}_0, \mathcal{L}_1, ..., \mathcal{L}_n$ respectively. Let*

$$K = \{Bel(\sigma_1 \wedge \sigma_0) = ab_1, \; ..., \; Bel(\sigma_n \wedge \sigma_0) = ab_n, Bel(\sigma_0) = a\}$$

$(0 \leq a \leq 1, \; 0 \leq b_i \leq 1)$. *Then*

$$ME(K)(\sigma_1 \wedge \; ... \; \sigma_n \wedge \sigma_0) = ab_1...b_n.$$

Proof. The proof is just like the proof of Corollary 7. Let

$$K' = \{Bel(\sigma_1) = b_1,\ ...,\ Bel(\sigma_n) = b_n,\ Bel(\sigma_0) = a\}.$$

By Corollary 13 and *Irrelevant Information*

$$ME(K')(\sigma_1 \wedge\ ...\ \wedge \sigma_n \wedge \sigma_0) = ab_1...b_n, \tag{23}$$

$$ME(K')(\sigma_1 \wedge \sigma_0) = ab_1,\ ...,\ ME(\sigma_n \wedge \sigma_0) = ab_n.$$

Thus also

$$ME(K')(\sigma_1 \wedge \neg\sigma_0) = b_1 - ab_1,\ ...,\ Bel(\sigma_n \wedge \neg\sigma_0) = b_n - ab_n.$$

Let

$$K'' = \{Bel(\sigma_1 \wedge \neg\sigma_0) = b_1 - ab_1,\ ...,\ Bel(\sigma_n \wedge \neg\sigma_0) = b_n - ab_n,\ Bel(\sigma_0) = a\}.$$

By *Obstinacy* and *Equivalence*

$$ME(K') = ME(K' \cup K \cup K'') = ME(K \cup K'')$$

and by *Relativization* (with $\phi = \sigma_0$),

$$ME(K \cup K'')(\sigma_1 \wedge\ ...\ \wedge \sigma_n \wedge \sigma_0) = ME(K)(\sigma_1 \wedge\ ...\ \wedge \sigma_n \wedge \sigma_0),$$

so the result follows by (23). ■

Let $K(a_1)$ be a set of linear constraints involving propositional variables $Q_1, Q_2, ..., Q_q$, $P_1(a_1), P_2(a_1), ..., P_p(a_1)$. As before, let $\rho_1,\ ...,\ \rho_{2^q}$ denote the atoms of $\{Q_1,\ ...,\ Q_q\}$, and let $\sigma_1, ..., \sigma_{2^p}$ stand for $\bigwedge_{j=1}^p P_j^{\delta_j}$, where

$$P_j^\delta = \begin{cases} P_j & if\ \delta = 1, \\ \neg P_j & if\ \delta = 0. \end{cases}$$

Theorem 15 *Let $K(a_1)$ be as above. There exist $S_1, ..., S_{2^q},\ X_{1,1}, ..., X_{1,2^p}, X_{2,1}, ...$ $..., X_{2^q,2^p}$ such that*

$$\lim_{n\to\infty} ME(\bigcup_{j=1}^n K(a_j))(\bigwedge_{i=1}^r \sigma_{j_i}(a_{n_i})) = \sum_k S_k \prod_{i=1}^r X_{k,j_i}$$

Proof. The proof is similar to that of Theorem 10, so we will only sketch it. Let

$$K^n = \bigcup_{j=1}^n K(a_j) \qquad n \in \mathbb{N}.$$

Let

$$ME(K^n)(\rho_k) = S_k^n \quad 1 \le k \le 2^q.$$

By *Renaming*, $ME(K^n)(\rho_k \wedge \sigma_j(a_m))$, for $1 \le m \le n$, $1 \le k \le 2^q$, $1 \le j \le 2^p$, is independent of m. Let

$$ME(K^n)(\rho_k \wedge \sigma_j(a_1)) =\ ...\ = ME(K^n)(\rho_k \wedge \sigma_j(a_n)) =\ S_k^n X_{k,j}^n.$$

Now let K_1^n be the set of constraints

$$\{Bel(\rho_k) = S_k^n,\ Bel(\rho_k \wedge \sigma_j(a_m)) = S_k^n X_{k,j}^n\ |k = 1, ..., 2^q,\ j = 1, ..., 2^p,\ m = 1, ..., n\}.$$

By *Obstinacy* and *Equivalence* (using the fact that $K^n \cup K_1^n$ is equivalent to K_1^n),

$$ME(K^n) = ME(K^n \cup K_1^n) = ME(K_1^n).$$

Using *Relativization* (with $\phi = \rho_k$) and Corollary 14 we can conclude

$$ME(K_1^n)\left(\rho_k \wedge \bigwedge_{i=1}^{n} \sigma_{j_i}(a_i)\right) = S_k^n \prod_{i=1}^{n} X_{k,j_i}^n.$$

Consequently, *Bel* defined by

$$Bel\left(\rho_k \wedge \bigwedge_{i=1}^{n} \sigma_{j_i}(a_i)\right) = S_k^n \prod_{i=1}^{n} X_{k,j_i}^n$$

($k = 1, ..., 2^q$ and $j_i \in \{1, ..., 2^p\}$), satisfies K^n and since it is the solution of K^n which maximizes the entropy, no other solution of K^n determined analogously by $\vec{x}$, $\vec{s}$ ($\vec{x} = (x_{1,1}, ..., x_{1,2^p}, x_{2,1}, ..., x_{2^q,2^p})$, $\vec{s} = (s_1, ..., s_{2^q})$) has larger entropy. In other words, $(X_{1,1}^n, ..., X_{2^q,2^p})$, $(S_1^n, ..., S_{2^q}^n)$ is the point for which

$$f_n(\vec{x}, \vec{s}) = -\sum_{k,j_1,...,j_n} s_k \prod_{i=1}^{n} x_{k,j_i} \log\left(s_k \prod_{i=1}^{n} x_{k,j_i}\right)$$

is maximal amongst all $\vec{x} \geq 0$, $\vec{s} \geq 0$ for which *Bel* defined by

$$Bel\left(\rho_k \wedge \bigwedge_{i=1}^{n} \sigma_{j_i}(a_i)\right) = s_k \prod_{i=1}^{n} x_{k,j_i} \tag{24}$$

($k = 1, ..., 2^q$ and $j_i \in \{1, ..., 2^p\}$), gives a solution of K^n, i.e.

$$Bel(\rho_k \wedge \sigma_j(a_1))) = s_k x_{k,j}$$

gives a solution of K^1. Now notice that assuming $\sum_{i=1}^{2^p} x_{k,i} = 1$ for each k (which holds when *Bel* as above is a solution of $K(a_1)$) we have

$$\sum_{\substack{r_1,...,r_{2^p} \\ \sum r_i = n}} \binom{n}{r_1, ..., r_{2^p}} x_{k,1}^{r_1} ... x_{k,2^p}^{r_{2^p}} = (x_{k,1} + \ ... \ + x_{k,2^p})^n = 1$$

(this is used to obtain the last equality below), so

$$\begin{aligned}
f_n(\vec{x}, \vec{s}) &= \\
&= -\sum_{k,j_1,...,j_n} s_k \prod_{i=1}^{n} x_{k,j_i} \log\left(s_k \prod x_{k,j_i}\right) \\
&= -\sum_k s_k \sum_{\substack{r_1,...,r_{2^p} \\ \Sigma r_i = n}} \binom{n}{r_1, ..., r_{2^p}} x_{k,1}^{r_1} ... x_{k,2^p}^{r_{2^p}} \log\left(s_k x_{k,1}^{r_1} ... x_{k,2^p}^{r_{2^p}}\right) \\
&= -\sum_k s_k \log s_k \\
&\quad -\sum_k s_k \sum_{\substack{r_1,...,r_{2^p} \\ \Sigma r_i = n}} \binom{n}{r_1, ..., r_{2^p}} x_{k,1}^{r_1} ... x_{k,2^p}^{r_{2^p}} (r_1 \log x_{k,1} + \ ... \ + r_{2^p} \log x_{k,2^p}).
\end{aligned}$$

Since

$$\sum_{\substack{r_2,\ldots,r_2^p \\ \Sigma_{i=2}^{2^p} r_i = n-r_1}} \binom{n-r_1}{r_2,...,r_{2^p}} x_{k,2}^{r_2}...x_{k_2^p}^{r_{2^p}} = (1-x_{k,1})^{n-r_1}$$

and

$$\binom{n}{r_1,...,r_{2^p}} = \binom{n}{r_1}\binom{n-r_1}{r_2,...,r_{2^p}},$$

we have

$$\begin{aligned}\sum_{\substack{r_1,\ldots,r_{2^p} \\ \Sigma r_i = n}} \binom{n}{r_1,...,r_{2^p}} x_{k,1}^{r_1}...x_{k,2^p}^{r_{2^p}}(r_1 \log x_{k,1}) &= \sum_{r_1=0}^{n} \binom{n}{r_1} x_{k,1}^{r_1}(1-x_{k,1})^{n-r_1} r_1 \log x_{k,1} \\ &= n x_{k,1} \log x_{k,1}\end{aligned}$$

in view of the equality

$$\sum_{j=0}^{n} \binom{n}{j} j x^j (1-x)^{n-j} = nx$$

which we have already mentioned. Similarly for $2, ..., 2^p$ in place of 1. Consequently,

$$f_n(\vec{x},\vec{s}) = -\sum_k s_k \log s_k - n \sum_k \sum_j s_k x_{k,j} \log x_{k,j}$$

Now let $y_{k,j} = s_k x_{k,j}$. Substituting $\sum_{j=1}^{2^p} y_{k,j} = s_k$ and $\frac{y_{k.j}}{\Sigma_{j=1}^{2^p} y_{k,j}} = x_{k,j}$ we get just as before

$$\begin{aligned} f_n(\vec{x},\vec{s}) \;=\; g_n(\vec{y}) &= -\sum_k (\sum_j y_{k,j}) \log(\sum_j y_{k,j}) \\ &\quad + n\left(-\sum_{k,j} y_{k,j}\log y_{k,j} + \sum_k (\sum_j y_{k,j}) \log(\sum_j y_{k,j})\right).\end{aligned}$$

Maximizing f_n subject to Bel defined on the language

$$\{Q_1, ..., Q_q, P_1(a_1), P_2(a_1), ..., P_p(a_1), P_1(a_2), ..., P_p(a_n)\}$$

as in (24) being a solution of K^n thus amounts to maximizing g_n subject to Bel defined on the language $\{Q_1, ..., Q_q, P_1(a_1), P_2(a_1), ..., P_p(a_1)\}$ by

$$Bel(\rho_k \wedge \sigma_j(a_1)) = y_{k,j}$$

being a solution of K^1 i.e. of $K(a_1)$. If $\vec{Y^n}$ is the solution of K^1 which maximizes g_n then

$$ME(K^n)\left(\rho_k \wedge \bigwedge_{i=1}^{n} \sigma_{j_i}(a_i)\right) = \begin{cases} \frac{\prod_i Y_{k,j_i}}{(\Sigma_j Y_{k,j})^{n-1}} & \text{if } \Sigma_j Y_{k,j}^n \neq 0, \\ 0 & \text{if } \Sigma_j Y_{k,j} = 0. \end{cases} \tag{25}$$

We will show that the points $\vec{Y^n}$ which maximize g_n subject to K^1 tend to the point which maximizes

$$h(\vec{y}) = -\sum_{k,j} y_{k,j} \log y_{k,j} + \sum_k (\sum_j y_{k,j}) \log(\sum_j y_{k,j})$$

subject to K^1 (and, in case when there are more such points, that point amongst them for which

$$h_0(\vec{y}) = -\sum_k (\sum_j y_{k,j}) \log(\sum_j y_{k,j})$$

is maximal. Let us denote this point $\vec{Y}$. Note that $\vec{Y}$ is uniquely defined since the function h is concave on $\{\vec{y} \mid y_{k,j} \geq 0, k = 1, ...2^q, \; j = 1, ..., 2^p\}$. (This can be seen by induction on p, noting that

$$\begin{aligned}
h(\vec{y}) \quad = \quad & \sum_k \left(-\sum_{j=1}^{2^{p-1}} y_{k,j} \log y_{k,j} + (\sum_{j=1}^{2^{p-1}} y_{k,j}) \log(\sum_{j=1}^{2^{p-1}} y_{k,j}) \right) + \\
& \sum_k \left(-\sum_{j=2^{p-1}+1}^{2^p} y_{k,j} \log y_{k,j} + (\sum_{j=2^{p-1}+1}^{2^p} y_{k,j}) \log(\sum_{j=2^{p-1}+1}^{2^p} y_{k,j}) \right) + \\
& \sum_k \left(-(\sum_{j=1}^{2^{p-1}} y_{k,j}) \log(\sum_{j=1}^{2^{p-1}} y_{k,j}) - (\sum_{j=2^{p-1}+1}^{2^p} y_{k,j}) \log(\sum_{j=2^{p-1}+1}^{2^p} y_{k,j}) + \right. \\
& \left. (\sum_{j=1}^{2^p} y_{k,j}) \log(\sum_{j=1}^{2^p} y_{k,j}) \right)
\end{aligned}$$

and using the fact that the function $f(u, v) = -u \log u - v \log v + (u + v) \log(u + v)$ $(u \geq 0, \; v \geq 0)$ is, as already remarked, concave.). Consequently, the set $Max(K^1, h)$ of solutions of K^1 which maximize h is convex, so the set of $(\sum_j y_{1,j}, ..., \sum_j y_{2^q,j})$ such that $\vec{y} \in Max(K^1, h)$ is convex and thus there is a unique $(\sum_j y_{1,j}, ..., \sum_j y_{2^q,j})$ amongst them which maximizes h_0 (entropy). The convexity of $Max(K^1, h)$ also guarantees that $(\sum_j y_{1,j}, ..., \sum_j y_{2^q,j})$ determines $\vec{y} \in Max(K^1, h)$ uniquely, since adding $\{\sum_j y_{k,j} = c_k : \; k = 1, ..., 2^q\}$ to K^1 produces a set of linear constraints with a convex set of solutions, for which h_0 is constant, and amongst which there is a unique one maximizing entropy of $\vec{y}$, that is $h(\vec{y}) + h_0(\vec{y})$.

The result now follows as in the proof of Theorem 10.

■

The Minimum Distance solution

In this and the succeeding sections we consider the corresponding situation with two other choices of inference process, the minimum distance inference process, MD, and the limiting centre of mass inference process, CM^∞.

To begin with MD, this can be defined analogously to ME but with the alternate measure of 'information content',

$$\sum_{i=1}^{2^n} (Bel(\alpha_i) - 1/2^n)^2 \tag{26}$$

(or, equivalently, $\sum_{i=1}^{2^n} (Bel(\alpha_i))^2$) replacing the Shannon information content

$$n + \sum_{i=1}^{2^n} Bel(\alpha_i) \log(Bel(\alpha_i)).$$

In other words $MD(K)$ is that probability function Bel satisfying K for which the expression (26) is *minimal*. [See [6] for further motivation and properties of this inference process. In particular it is shown there that MD, like ME, satisfies *Language Invariance, Obstinacy* and *Renaming*.]

In this case rather more work is required to prove a result which for ME was rather straightforward, namely:

Theorem 16 *If $K(a_1)$ is such that the Q_j form a complete set of reasons then the limits*

$$\lim_{n\to\infty} MD(\bigcup_{j=1}^{n} K(a_j))(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i}))$$

exist and equal the canonical solution.

As before, we denote $K^n = \bigcup_{j=1}^n K(a_j)$ Before proving the above theorem, we first consider the special case of

$$K(a_1) = \{Bel(P(a_1)) = a\} \quad \text{where} \quad 1 > a > \frac{1}{2}. \tag{27}$$

By *Renaming*, $MD(K^n)\,(\bigwedge P^{\epsilon_i}(a_i))$ depends only on $\sum_{i=1}^n \epsilon_i$. Let

$$MD(K^n)\left(\bigwedge_{i=1}^{2^n} P^{\epsilon_i}(a_i)\right) = Z_m, \quad m = \sum_{i=1}^{n} \epsilon_i.$$

(We should really write Z_m^n etc. to indicate the dependence on n but n will be clear from the context so we omit it, assuming only, where necessary, that n is large.) $\vec{Z} = (Z_0, Z_1, ..., Z_n)$ is that point which minimizes the function

$$f(\vec{z}) = \sum_{m=0}^{n} \binom{n}{m} z_m^2$$

subject to $\vec{z} \geq 0$ and the constraints

$$\sum_{\epsilon_1,\ldots,\epsilon_n} z_{\Sigma\epsilon_i} = \sum_{m=0}^{n} \binom{n}{m} z_m = 1, \tag{28}$$

$$\sum_{m=1}^{n} \binom{n-1}{m-1} z_m = a. \tag{29}$$

Suppose that $Z_m > 0$ just when $m \in S \subseteq \{0, 1, ..., n\}$. By *Obstinacy*,

$$MD(K^n) = MD\left(K^n \cup \left\{Bel\left(\bigwedge P^{\epsilon_i}(a_i)\right) = 0 \mid \sum_{i=1}^{n} \epsilon_i = m \notin S\right\}\right).$$

In other words $\vec{Z}$ is the point for which $Z_m = 0$ if $m \notin S$ and for which $\sum_{m\in S} \binom{n}{m} z_m^2$ is minimal subject to the above constraints *and* $z_m > 0$ whenever $m \in S$. By the Lagrange Multiplier method, for $m \in S$

$$2\binom{n}{m} Z_m - \binom{n}{m}\lambda - \binom{n-1}{m-1}\mu = 0$$

so

$$Z_m = \begin{cases} \frac{1}{2}(\lambda + \frac{m}{n}\mu) & if\ m \in S, \\ 0 & otherwise, \end{cases} \tag{30}$$

where (in view of (28) and (29)) the λ and μ satisfy

$$A\lambda + B\mu = 2, \qquad B\lambda + C\mu = 2a, \tag{31}$$

for

$$A = \sum_{m\in S}\binom{n}{m}, \quad B = \sum_{m\in S}\binom{n}{m}\frac{m}{n}, \quad C = \sum_{m\in S}\binom{n}{m}\left(\frac{m}{n}\right)^2. \tag{32}$$

Notice that by Schwarz Inequality (i.e. $(\sum a_i b_i)^2 \le (\sum a_i^2)(\sum b_i^2)$ with equality only when $a_i = cb_i$ for all i) $AC - B^2 > 0$ unless $|S| = 1$, a possibility which our next lemma will, for large enough n, shortly dismiss. Assuming $AC - B^2 > 0$, from (31) we obtain

$$\lambda = \frac{2(C - aB)}{AC - B^2}, \qquad \mu = \frac{2(aA - B)}{AC - B^2}$$

so

$$Z_m = \begin{cases} \frac{(C-aB)+\frac{m}{n}(aA-B)}{AC-B^2} & if\ m \in S, \\ 0 & otherwise. \end{cases} \tag{33}$$

Lemma 17 *The following situations cannot occur:*
(i) $m < j < k$ and $Z_m > 0$, $Z_j = 0$ and $Z_k > 0$.
(ii) $j < m < k$ and $Z_j = 0$, $Z_m > 0$ and $Z_k = 0$.

Proof. Assume that $Z_j = 0$ (as is the case in (i) and (ii) above). Z_m, Z_j, Z_k must minimize the function

$$F(z_m, z_j, z_k) = \binom{n}{m} z_m^2 + \binom{n}{j} z_j^2 + \binom{n}{k} z_k^2$$

subject to $z_m, z_j, z_k \ge 0$,

$$\binom{n}{m} z_m + \binom{n}{j} z_j + \binom{n}{k} z_k = \binom{n}{m} Z_m + \binom{n}{j} Z_j + \binom{n}{k} Z_k$$

and

$$\binom{n}{m}\frac{m}{n} z_m + \binom{n}{j}\frac{j}{n} z_j + \binom{n}{k}\frac{k}{n} z_k = \binom{n}{m}\frac{m}{n} Z_m + \binom{n}{j}\frac{j}{n} Z_j + \binom{n}{k}\frac{k}{n} Z_k,$$

or equivalently, $z_m, z_j, z_k \ge 0$ and

$$z_m = Z_m - \frac{k-j}{k-m}\frac{\binom{n}{j}}{\binom{n}{m}} z_j, \qquad z_k = Z_k - \frac{m-j}{m-k}\frac{\binom{n}{j}}{\binom{n}{k}} z_j$$

(using the fact that $Z_j = 0$). We have

$$\frac{dz_m}{dz_j} = -\frac{k-j}{k-m}\frac{\binom{n}{j}}{\binom{n}{m}}, \qquad \frac{dz_k}{dz_j} = -\frac{m-j}{m-k}\frac{\binom{n}{j}}{\binom{n}{k}}$$

so substituting for z_m, z_k in F and differentiating with respect to z_j, we obtain

$$\left.\frac{dF}{dz_j}\right|_{z_j=0} = -2\binom{n}{j}\left(\frac{k-j}{k-m}Z_m + \frac{j-m}{k-m}Z_k\right).$$

Note that both in case (i) and (ii), $\frac{dF}{dz_j} < 0$ since $\frac{k-j}{k-m}Z_m > 0$ both in (i) and (ii) whilst $\frac{j-m}{k-m}Z_k$ is greater than 0 in (i) and equal to 0 in (ii). The minimum value of F subject to the constraints cannot be taken at Z_m, Z_j, Z_k as in (i) or (ii) since increasing Z_j by a very small amount would reduce F whilst still preserving the constraints (note that in (i), Z_m, Z_k would remain nonnegative since they start off strictly positive and in (ii) Z_m would remain nonnegative since it starts of strictly positive whilst Z_k would increase with a small increase of Z_j since $\frac{dz_k}{dz_j} > 0$). But this contradicts $\vec{Z} = MD(K^n)$. ■

By the above lemma, the set $S = \{m : \; Z_m > 0\}$ has the form $\{s, s+1, ..., t\}$ for some $0 \leq s \leq t \leq n$, where $s = 0$ or $t = n$. Also $s < t$ otherwise $s = t = 0$ or $s = t = n$ giving $a = 0, 1 \notin (1/2, 1)$. We shall now look more closely at what s and t can be.

Lemma 18 $s \leq [an] < t$.

Proof. By (33) we have

$$Z_j = \begin{cases} \frac{(C-aB)+\frac{j}{n}(aA-B)}{AC-B^2} & if \; j \in S, \\ \\ 0 & otherwise, \end{cases}$$

where A, B, C are as above in (32). Since, as already remarked, $B^2 < AC$,

$$\frac{(C-aB)+\frac{j}{n}(aA-B)}{AC-B^2} \leq 0 \quad \Leftrightarrow \quad (C-aB)+\frac{j}{n}(aA-B) \leq 0.$$

Substituting for A, B, C we can see that this holds just when

$$\sum_{s \leq m \leq t} \binom{n}{m} \left(\frac{m}{n} - a\right) \left(\frac{m}{n} - \frac{j}{n}\right) \leq 0.$$

This does hold for $j = s$ when $t \leq [an]$ (since then $\frac{m}{n} - a \leq 0$ and $\frac{m}{n} \geq \frac{j}{n}$ for $s \leq m \leq t$), and with $j = t$ when $[an] < s$ (since then $\frac{m}{n} - a \geq 0$ and $\frac{m}{n} \leq \frac{j}{n}$ for $s \leq m \leq t$). However, by definition of s and t, $Z_s > 0$ and $Z_t > 0$ so neither $t \leq [an]$ nor $[an] < s$ can hold and the result follows. ■

We will need the following technical result.

Lemma 19 *Let* $0 \leq \gamma < \delta < 1$ *and* $n > \frac{1}{\delta - \gamma}$. *Then*

$$\binom{n}{[\delta n]} \left(\frac{\delta}{1-\delta}\right)^{(\delta-\gamma)n} (n(\delta-\gamma)+1) \geq \sum_{[\gamma n] < m \leq [\delta n]} \binom{n}{m} \geq \binom{n}{[\delta n]} \left(\frac{\gamma}{1-\gamma}\right)^{(\delta-\gamma)n-2}.$$

Proof. For $[\gamma n] < m < [\delta n]$ we have

$$\binom{n}{m} = \frac{[\delta n]}{n-[\delta n]+1} \cdot \frac{[\delta n]-1}{n-([\delta n]-1)+1} \cdot \; ... \cdot \; \frac{m+1}{n-(m+1)+1} \binom{n}{[\delta n]}. \tag{34}$$

Consequently, for $[\gamma n] < m \leq [\delta n]$,

$$\binom{n}{m} \leq \left(\frac{[\delta n]}{n-[\delta n]+1}\right)^{[\delta n]-m} \binom{n}{[\delta n]} \leq \left(\frac{\delta n}{n-[\delta n]}\right)^{(\delta-\gamma)n} \binom{n}{[\delta n]}$$

and the first inequality follows (since there are $[\delta n] - [\gamma n] \leq (\delta - \gamma)n + 1$ of the eligible m's. For the second inequality, using (34) we have

$$\binom{n}{m} \geq \left(\frac{m+1}{n-m}\right)^{[\delta n]-m} \binom{n}{[\delta n]} \geq \left(\frac{\gamma n}{n-\gamma n}\right)^{[\delta n]-m} \binom{n}{[\delta n]}.$$

Consequently (considering just one term in the sum, with $m = [\gamma n] + 1$),

$$\sum_{[\gamma n] < m \leq [\delta n]} \binom{n}{m} \geq \binom{n}{[\delta n]} \left(\frac{\gamma}{1-\gamma} \right)^{(\delta - \gamma)n - 2}$$

as required. ■

Now we can say more about the position of s.

Lemma 20 *Let $\epsilon > 0$, $a - \epsilon > \frac{1}{2}$ Then for sufficiently large n, $s > [n(a-\epsilon)]$.*

Proof. We have $Z_s > 0$ by definition of s and as in Lemma 18 we can see that this means that

$$\sum_{s \leq m \leq t} \binom{n}{m} (m - na)(m - s) > 0,$$

i.e.

$$\sum_{s \leq m \leq na} \binom{n}{m} (na - m)(m - s) < \sum_{na < m \leq t} \binom{n}{m} (m - na)(m - s).$$

Also,

$$\sum_{na < m \leq t} \binom{n}{m} (m - na)(m - s) \leq \sum_{na \leq m \leq n} \binom{n}{m} n^2 \leq \binom{n}{[na]} n^3,$$

whilst if $s \leq [n(a - \epsilon)]$ then

$$\sum_{s \leq m \leq na} \binom{n}{m} (na - m)(m - s) \geq \sum_{[n(a-\epsilon)] < m \leq [n(a - \frac{\epsilon}{2})]} \binom{n}{m}$$

which by Lemma 19 is (for large n)greater or equal to

$$\binom{n}{[n(a - \frac{\epsilon}{2})]} \left(\frac{a - \epsilon}{1 - (a - \epsilon)} \right)^{n\frac{\epsilon}{2} - 2} \geq \binom{n}{[na]} \left(\frac{a - \epsilon}{1 - (a - \epsilon)} \right)^{n\frac{\epsilon}{2} - 2}.$$

Consequently, if $s \leq [n(a - \epsilon)]$ then

$$n^3 > \left(\frac{a - \epsilon}{1 - (a - \epsilon)} \right)^{n\frac{\epsilon}{2} - 2}$$

which is false for large n since $\frac{a-\epsilon}{1-(a-\epsilon)} > 1$. Hence $s > [n(a - \epsilon)]$ (and also $s > n(a - \epsilon)$) for n sufficiently large. ■

Corollary 21 *$t = n$.*

Proof. This follows from Lemma 17 since $s \neq 0$. ■

Proposition 22 *Let $K(a_1)$ be as in (27). Then*

$$\lim_{n \to \infty} MD(K^n)(P(a_1) \wedge P(a_2)) = a^2.$$

Proof. We have

$$MD(K^n)(P(a_1) \wedge P(a_2)) = \sum_{s \leq m \leq n} \binom{n-2}{m-2} Z_m = \sum_{s \leq m \leq n} \binom{n}{m} \frac{m(m-1)}{n(n-1)} Z_m$$

so since $\sum_{s\le m\le n}\binom{n}{m}Z_m=1$ and $Z_m=\frac{1}{2}(\lambda+\frac{m}{n}\mu)$ for $s\le m\le n$,

$$\begin{aligned}|MD(K^n)(P(a_1)\wedge P(a_2))-a^2| &= \left|\sum_{s\le m\le n}\binom{n}{m}\left(\frac{m(m-1)}{n(n-1)}-a^2\right)\frac{1}{2}(\lambda+\frac{m}{n}\mu)\right|\\ &\le \sum_{s\le m\le n}\binom{n}{m}\left|\frac{m(m-1)}{n(n-1)}-a^2\right|\frac{1}{2}(\lambda+\frac{m}{n}\mu).\end{aligned}\tag{35}$$

Now let $\epsilon>0$ be small (so that $a-\epsilon>\frac{1}{2}$) and let n be large (so that $s>n(a-\epsilon)$). Then for $s\le m\le n(a+\epsilon)$ and large n

$$\left|\frac{m(m-1)}{n(n-1)}-a^2\right|\le 4a\epsilon$$

so

$$\sum_{s\le m\le n(a+\epsilon)}\binom{n}{m}\left|\frac{m(m-1)}{n(n-1)}-a^2\right|\frac{1}{2}(\lambda+\frac{m}{n}\mu)\le 4a\epsilon. \tag{36}$$

Also,

$$\sum_{n(a+\epsilon)<m\le n}\binom{n}{m}\left|\frac{m(m-1)}{n(n-1)}-a^2\right|\frac{1}{2}(\lambda+\frac{m}{n}\mu)\le\sum_{n(a+\epsilon)<m\le n}\binom{n}{m}\frac{1}{2}(\lambda+\frac{m}{n}\mu).$$

This can be further estimated from above by

$$\frac{\Sigma_{n(a+\epsilon)<m\le n}\binom{n}{m}\frac{1}{2}(\lambda+\frac{m}{n}\mu)}{\Sigma_{na<m\le n(a+\epsilon)}\binom{n}{m}\frac{1}{2}(\lambda+\frac{m}{n}\mu)} \tag{37}$$

since $\sum_{s\le m\le n}\binom{n}{m}\frac{1}{2}(\lambda+\frac{m}{n}\mu)=1$ and $s\le[na]$. We will show that for any fixed ϵ this tends to 0 as n tends to ∞, which in view of (35) and (36) will complete the proof of our proposition since ϵ can be arbitrarily small. First suppose that $\mu>0$. Then (37) is at most

$$\frac{\Sigma_{n(a+\epsilon)<m\le n}\binom{n}{m}\frac{1}{2}(\lambda+\frac{n}{n}\mu)}{\Sigma_{na<m\le n(a+\epsilon)}\binom{n}{m}\frac{1}{2}(\lambda+\frac{[an]+1}{n}\mu)}. \tag{38}$$

Also since $Z_s=\frac{1}{2}(\lambda+\frac{s}{n}\mu)>0$, i.e. $\lambda>-\frac{s}{n}\mu$ and thus

$$\frac{\lambda+\mu}{\lambda+\frac{[an]+1}{n}\mu}\le\frac{-\frac{s}{n}\mu+\mu}{-\frac{s}{n}\mu+\frac{[an]+1}{n}\mu},$$

(because $g(x)=\frac{x+\mu}{x+\frac{[an]+1}{n}\mu}$ is a decreasing function), the expression (38) is at most

$$\frac{\Sigma_{n(a+\epsilon)<m\le n}\binom{n}{m}(-\frac{s}{n}\mu+\mu)}{\Sigma_{na<m\le n(a+\epsilon)}\binom{n}{m}(-\frac{s}{n}\mu+\frac{[an]+1}{n}\mu)}\le n\,\frac{\Sigma_{n(a+\epsilon)<m\le n}\binom{n}{m}}{\Sigma_{na<m\le n(a+\epsilon)}\binom{n}{m}}$$

(since $\frac{-\frac{s}{n}\mu+\mu}{-\frac{s}{n}\mu+\frac{[an]+1}{n}\mu}=\frac{n-s}{[an]+1-s}\le n$). Also, since $\binom{n}{m}$ decrease with increasing m when $m>\frac{n}{2}$,

$$\sum_{n(a+\epsilon)<m\le n}\binom{n}{m}\le n\binom{n}{[n(a+\epsilon)]}$$

and by Lemma 19

$$\sum_{na<m\leq n(a+\epsilon)} \binom{n}{m} \geq \binom{n}{[n(a+\epsilon)]} \left(\frac{a}{1-a}\right)^{n\epsilon-2}$$

so

$$n\,\frac{\Sigma_{n(a+\epsilon)<m\leq n}\binom{n}{m}}{\Sigma_{na<m\leq n(a+\epsilon)}\binom{n}{m}} \leq n^2\left(\frac{1-a}{a}\right)^{n\epsilon-2} \to 0 \;\; as \;\; n\to\infty.$$

Now suppose $\mu \leq 0$. Then (37) is at most

$$\frac{\Sigma_{n(a+\epsilon)<m\leq n}\binom{n}{m}\lambda}{\Sigma_{na<m\leq n(a+\epsilon)}\binom{n}{m}(\lambda+\frac{[n(a+\epsilon)]}{n}\mu)}. \tag{39}$$

Since $Z_n = \frac{1}{2}(\lambda+\mu) > 0$ and thus $\mu > -\lambda$, we have $\frac{\lambda}{\lambda+\frac{[n(a+\epsilon)]}{n}\mu} \leq \frac{n}{n-[n(a+\epsilon)]} \leq n$ (we assume that $a < 1$ so $n-[n(a+\epsilon)] \geq 1$ for small ϵ and large n) and thus (37) is less or equal to

$$n\,\frac{\Sigma_{n(a+\epsilon)<m\leq n}\binom{n}{m}}{\Sigma_{na<m\leq n(a+\epsilon)}\binom{n}{m}}$$

which is what we had for $\mu > 0$ and which tends to 0 as n tends to ∞. ■

The same approach gives

Proposition 23 *Let $K(a_1) = \{Bel(P(a_1) = a\}$, $\frac{1}{2} < a < 1$. Then*

$$\lim_{n\to\infty} MD(K^n)\left(\bigwedge_{i=1}^{r} P(a_i)\right) = a^r.$$

Proof

$$MD(K^n)\left(\bigwedge_{i=1}^{r} P(a_i)\right) = \sum_{s\leq m\leq n}\binom{n}{m}\frac{m(m-1)...(m-r+1)}{n(n-1)...(n-r+1)}\frac{1}{2}(\lambda+\frac{m}{n}\mu)$$

so $|MD(K^n)\left(\bigwedge_{i=1}^{r} P(a_i)\right) - a^r|$ can be estimated to be small just as $|MD(K^n)(P(a_1)\wedge P(a_2)) - a^2|$ was estimated in Proposition 22.

Corollary 24 *Let $K(a_1) = \{Bel(P(a_1)) = a\}$, $\frac{1}{2} < a < 1$. Then*

$$\lim_{n\to\infty} MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i)\wedge\bigwedge_{i=l+1}^{r}\neg P(a_i)\right) = a^l(1-a)^{r-l}.$$

Proof. This follows by induction on $k = r-l$ (simultaneously for all $r \geq l$). Proposition 23 deals with $k = 0$ (i.e $l = r$, any r). Assume that the result holds for $k-1$, and consider $k = r-l > 0$. We have

$$MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i)\wedge\bigwedge_{i=l+1}^{r}\neg P(a_i)\right) =$$

$$MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i)\wedge\bigwedge_{i=l+1}^{r-1}\neg P(a_i)\right) - MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i)\wedge\bigwedge_{i=l+1}^{r-1}\neg P(a_i)\wedge P(a_r)\right) \tag{40}$$

where $\bigwedge_{i=l+1}^{r-1} \neg P(a_i)$ above is left out when $l = r-1$. By *Renaming*,

$$MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r-1} \neg P(a_i) \wedge P(a_r)\right) = MD(K^n)\left(\bigwedge_{i=1}^{l+1} P(a_i) \wedge \bigwedge_{i=l+2}^{r} \neg P(a_i)\right)$$

so (40) by inductive hypothesis tends to

$$a^l(1-a)^{r-1-l} - a^{l+1}(1-a)^{r-1-l} \;=\; a^l(1-a)^{r-1-l}(1-a) = a^l(1-a)^{r-l}$$

as required. ∎

Corollary 25 *Let* $K(a_1) = \{Bel(P(a_1)) = a\}$, $0 \leq a \leq 1$. *Then*

$$\lim_{n\to\infty} MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r} \neg P(a_i)\right) = a^l(1-a)^{r-l}.$$

Proof. The cases of $a = 0$ and $a = 1$ are trivial. For $a = \frac{1}{2}$, the result follows by *Renaming* since K^n must give all of $\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r} \neg P(a_i)$ the same value $\frac{1}{2^r}$. Thus it remains to show it for $0 < a < \frac{1}{2}$. By *Equivalence*,

$$MD(K^n) = MD(\{Bel(\neg P(a_i)) = 1-a \mid i = 1, ..., n\})$$

so by *Renaming*

$$MD(K^n)\left(\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r} \neg P(a_i)\right) =$$

$$= MD(\{Bel(P(a_i)) = 1-a \mid i = 1, ..., n\})\left(\bigwedge_{i=1}^{l} \neg P(a_i) \wedge \bigwedge_{i=l+1}^{r} P(a_i)\right)$$

$$= MD(\{Bel(P(a_i)) = 1-a \mid i = 1, ..., n\})\left(\bigwedge_{i=1}^{r-l} P(a_i) \wedge \bigwedge_{i=r-l+1}^{r} \neg P(a_i)\right)$$

which tends to $(1-a)^{r-l}a^l$ by Proposition 23, as required. ∎

Corollary 26 *Let* $K(a_1) = \{Bel(P(a_1)) = a\}$, $0 \leq a \leq 1$. *Then*

$$\lim_{n\to\infty} MD(K^n)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})\right) = a^l(1-a)^{r-l}$$

where $l = \sum_{i=1}^{r} \epsilon_i$

Proof. This follows from the previous Corollary again by *Renaming.* ∎

To conclude the proof of Theorem 16 we need some further properties of MD, a strong version of the *Relativization Principle*[5] and a weak version of the *Irrelevant Information Principle.*

[5]It was incorrectly asserted in [6] that MD, and CM^∞, fail to satisfy this principle.

Theorem 27 *Let* $0 < c < 1$, $K_1, K_2, K \in C\mathcal{L}$ $(\mathcal{L} = \{p_1, ..., p_n\})$ *and*

$$K_1 = \left\{ Bel(\phi) = c,\ \sum_{j=1}^{r} a_{ji} Bel(\theta_j \wedge \phi) = b_i Bel(\phi) \mid i = 1, ..., m \right\},$$

$$K_2 = \left\{ \sum_{j=1}^{s} c_{ji} Bel(\psi_j \wedge \neg\phi) = d_i Bel(\neg\phi) \mid i = 1, ..., k \right\},$$

$$K = \left\{ Bel(\phi) = 1,\ \sum_{j=1}^{r} a_{ji} Bel(\theta_j) = b_i \mid i = 1, ..., m \right\}. \tag{41}$$

Then $MD(K_1 \cup K_2)(\chi \wedge \phi) = cMD(K)(\chi)$ *for any* $\chi \in S\mathcal{L}$.

Proof. Without loss of generality assume that the atoms of $\mathcal{L}$, $\alpha_1, ..., \alpha_{2^n}$, are ordered so that ϕ is equivalent to $\bigvee_{i=1}^{t} \alpha_i$. Let $\vec{\tau} = MD(K)$ and $\vec{\delta} = MD(K_1 \cup K_2)$. Then $\tau_{t+1} = \ ...\ = \tau_{2^n} \ = 0$ and Bel_1 defined by

$$Bel_1(\alpha_i) = \begin{cases} c\tau_i & for\ i = 1, ..., t, \\ \delta_i & for\ i = t+1, ...2^n \end{cases}$$

is a solution to $K_1 \cup K_2$ whilst Bel_2 defined by

$$Bel_2(\alpha_i) = \begin{cases} \frac{\delta_i}{c} & for\ i = 1, ..., t, \\ 0 & for\ i = t+1, ...2^n \end{cases}$$

is a solution of K. By the choice of $\vec{\delta}$ and $\vec{\tau}$,

$$\sum_{i=1}^{t} \left(\frac{\delta_i}{c}\right)^2 \leq \sum_{i=1}^{t} \tau_i^2$$

and

$$\sum_{i=1}^{t} (c\tau_i)^2 + \sum_{i=t+1}^{2^n} \delta_i^2 \leq \sum_{i=1}^{2^n} \delta_i^2$$

so

$$\sum_{i=1}^{t} \tau_i^2 \ \leq \ \sum_{i=1}^{t} \left(\frac{\delta_i}{c}\right)^2$$

By the strict convexity of the function $e(x_1, ..., x_t) = \sum_{i=1}^{t} x_i^2$ on the set of solutions of K we have $\tau_i = \frac{\delta_i}{c}$ for $i = 1, ..., t$ and the result follows. ∎

In what follows we shall refer to this principle as the *Strong Relativization Principle.*

Theorem 28 *Let* $K \in C\mathcal{L}$ *and let* β *be an atom of the language* $\mathcal{L}'$ *where* $\mathcal{L} \cap \mathcal{L}' = \emptyset$. *Then for* $\theta \in S\mathcal{L}$

$$MD(K)(\theta) = MD(K \cup \{Bel(\beta) = 1\})(\theta \wedge \beta).$$

Proof. Let $\alpha_1, ..., \alpha_{2^n}$ be the atoms of $\mathcal{L}$, $\beta_1, ..., \beta_{2^{n'}}$ the atoms of $\mathcal{L}'$ and assume (without loss of generality) that $\beta = \beta_1$. Let

$$MD(K)(\alpha_i) = \delta_i, \qquad MD(K \cup \{Bel(\beta) = 1\})(\alpha_i \wedge \beta_j) = \tau_{ij},$$

for $i = 1, ..., 2^n,\ j = 1, ..., 2^{n'}$. Clearly $\tau_{ij} = 0$ for $j = 2, ..., 2^{n'}$. Let Bel_1 be defined on $S\mathcal{L}$ by

$$Bel_1(\alpha_i) = \tau_{i1}, \quad i = 1, ..., 2^n,$$

and let Bel_2 be defined on $S(\mathcal{L} \cup \mathcal{L}')$ by

$$Bel_2(\alpha_i \wedge \beta_j) = \begin{cases} \delta_i & if\ j = 1 \\ 0 & otherwise. \end{cases}$$

Then Bel_1 is a solution to K and Bel_2 is a solution to $K \cup \{Bel(\beta) = 1\}$. Consequently, by the choice of $\vec{\delta}$ and $\vec{\tau}$

$$\sum_{i=1}^{2^n} \tau_{i1}^2 \leq \sum_{i=1}^{2^n} \delta_i^2 \leq \sum_{i=1}^{2^n} \tau_{i1}^2$$

which implies that $\tau_{i1} = \delta_i$ for $i = 1, ..., 2^n$ by the strict convexity of the function $e(x_1, ..., x_{2^n}) = \sum_{i=1}^{2^n} x_i^2$ on the convex set of solutions of K. The result follows. ■

Using Corollary 26 and Theorems 27 and 28 Theorem 16 follows by direct a translation of the proof of Theorem 9.

We would conjecture that an analogous result to Theorem 10 also holds for MD (and even gives the same answer as ME). This has already been proved in a number of cases although confirming it in full generality remains a topic for future investigation.

The CM^∞ solution

Turning now to CM^∞, its motivation (which was first explained in [9], see also [6]) is rather different from that of MD or ME. Briefly, given a set of constraints K as above (on a probability function Bel on sentences of the language with propositional variables $p_1, p_2, ..., p_n$) an initially perhaps rather obvious choice of a particular 'assigning' probability function satisfying K might be the 'most average' solution to K, or more formally the centre of mass of the polytope of solutions of K (assuming uniform density). Attractive as this choice might appear, based as it is on some idea of indifference, it actually has a serious flaw. Namely language invariance fails. That is, if we instead had considered K as a set of constraints on a probability function Bel defined on the sentences of some other overlying language, for example the larger language with $n+1$ propositional variable $p_1, p_2, ..., p_n, p_{n+1}$, then this centre of mass solution may well not agree with the centre of mass solution for the smaller language on arguments common to both of them. In other words assigned beliefs depend on the chosen overlying language, despite the fact that in the real world we apparently do not consider this to be relevant. This clearly calls into question the intuition behind selecting the centre of mass point in the first place. However some reconciliation is possible by noticing that if we continue to enlarge our language then these centre of mass probability functions do settle down i.e. converge, on their common arguments. The inference process selecting these limiting probability functions, which *does* satisfy language invariance, is denoted CM^∞ (the 'centre of mass as the language size tends to infinity').

Fortunately this inference process has, as shown in [9], an alternative characterization much more akin to those of ME and MD. Namely, $CM^\infty(K)$ is that solution Bel to K for which the sum

$$\sum_{i \notin I} \log(Bel(\alpha_i))$$

is maximal, where $I = \{\ i \mid \text{for all } Bel \text{ satisfying } K,\ Bel(\alpha_i) = 0\}$. [It is easy to show, see for example [6] page 74, that this maximum is not $-\infty$, that is that there is a solution Bel to K such that $Bel(\alpha_i) > 0$ for all $i \notin I$, and that CM^∞ satisfies the principles of *Obstinacy* and *Renaming*.]

For this inference process we can prove the following result:-

Theorem 29 *If $K(a_1)$ is such that the Q_j form a complete set of reasons (i.e. $K(a_1)$ is as in (1)) then the limits*

$$\mathrm{Lim}_{n\to\infty} CM^\infty(\bigcup_{j=1}^{n} K(a_j))(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i}))$$

exist and equal the canonical solution corresponding to a complete set of 3 reasons, Q_1', Q_2', Q_3', with $\beta_1' = 1, \beta_2' = 1/2, \beta_3' = 0$.

In other words a complete set of reasons again emerges, only in this case, unlike ME and MD, it is not necessarily the complete set from $K(a_1)$!
As before, we denote $K^n = \bigcup_{j=1}^{n} K(a_j)$, and we consider a simpler situation first.

Proposition 30 *Let*

$$K(a_1) = \{Bel(P(a_1)) = a\} \tag{42}$$

where $0 < a < 1/2$. Then

$$\lim_{n\to\infty} CM^\infty(K^n)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_i)\right) = \begin{cases} \frac{a}{2^{r-1}} & \text{if } l = \Sigma\epsilon_i > 0, \\ \frac{a}{2^{r-1}} + 1 - 2a & \text{if } l = \Sigma\epsilon_i = 0. \end{cases}$$

Proof. Note that K^n has a solution which is non-zero for each $\bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)$ (e.g the ME solution), so by *Open-Mindedness*, $CM^\infty(K^n)\,(\bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)) \neq 0$. As in the case of MD, by *Renaming*, $CM^\infty(K^n)\,(\bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i))$ depends only on $\sum_{i=1}^{n} \epsilon_i$. Let

$$CM^\infty(K^n)\left(\bigwedge_{i=1}^{n} P^{\epsilon_i}(a_i)\right) = Z_m, \quad m = \sum_{i=1}^{n} \epsilon_i.$$

(Again, we should really write Z_m^n and λ_n, μ_n (see below) etc. to indicate the dependence on n but n will be clear from the context so we omit it, assuming only, where necessary, that n is large.) $\vec{Z} = (Z_0, Z_1, ..., Z_n)$ is that point which maximizes the function

$$f(\vec{z}) = \sum_{m=0}^{n} \binom{n}{m} \log z_m$$

subject to $\vec{z} > 0$ and the constraints

$$\sum_{\epsilon_1,\ldots,\epsilon_n} z_{\Sigma\epsilon_i} = \sum_{m=0}^{n} \binom{n}{m} z_m = 1 \tag{43}$$

$$\sum_{m=1}^{n} \binom{n-1}{m-1} z_m = a. \tag{44}$$

By the Lagrange Multiplier method,

$$Z_m = \frac{1}{\lambda + \frac{m}{n}\mu},$$

where the λ and μ satisfy

$$\sum_{m=0}^{n} \binom{n}{m} \frac{1}{\lambda + \frac{m}{n}\mu} = 1 \tag{45}$$

and

$$\sum_{m=1}^{n} \binom{n-1}{m-1} \frac{1}{\lambda + \frac{m}{n}\mu} = \sum_{m=0}^{n} \binom{n}{m} \frac{\frac{m}{n}}{\lambda + \frac{m}{n}\mu} = a. \tag{46}$$

Notice that multiplying (46) by μ and (45) by λ and adding them together we obtain that

$$\lambda + \mu a = \sum_{m=0}^{n} \binom{n}{m} = 2^n.$$

Notice also that $\lambda \geq 1$ since $1 \geq Z_0 = \frac{1}{\lambda} > 0$ and $\lambda + \mu > 0$ since $Z_n = \frac{1}{\lambda+\mu} > 0$. We will need the following lemma:

Lemma 31 *Let $\epsilon > 0$ and $n > \frac{4}{\epsilon}$. Then*

$$\frac{\binom{n}{m}}{\binom{n}{\left[\frac{n}{2}\right]}} < \left((1-\epsilon)^{\frac{\epsilon}{4}}\right)^n.$$

for all $m < n(\frac{1}{2} - \epsilon)$ and all $m > n(\frac{1}{2} + \epsilon)$.

Proof. For $m < n(\frac{1}{2} - \epsilon)$,

$$\begin{aligned}
\frac{\binom{n}{m}}{\binom{n}{\left[\frac{n}{2}\right]}} &= \frac{n!}{m!(n-m)!} \frac{\left(\left[\frac{n}{2}\right]\right)!\left(n-\left[\frac{n}{2}\right]\right)!}{n!} \\
&= \left(\frac{\left[\frac{n}{2}\right]}{n-\left[\frac{n}{2}\right]+1}\right)\left(\frac{\left[\frac{n}{2}\right]-1}{n-\left[\frac{n}{2}\right]+2}\right) \cdots \left(\frac{m+2}{n-(m+1)}\right)\left(\frac{m+1}{n-m}\right)
\end{aligned}$$

In this product all the terms are less than 1 and the final $\left[\frac{n\epsilon}{2}\right]$ are at most

$$\frac{m+\left[\frac{n\epsilon}{2}\right]}{n-\left[\frac{n\epsilon}{2}\right]-m} \leq \frac{\frac{n}{2}-\frac{n\epsilon}{2}}{\frac{n}{2}+\frac{n\epsilon}{2}}.$$

Hence

$$\frac{\binom{n}{m}}{\binom{n}{\left[\frac{n}{2}\right]}} \leq \left(\frac{1-\epsilon}{1+\epsilon}\right)^{\left[\frac{n\epsilon}{2}\right]} < \left((1-\epsilon)^{\frac{\epsilon}{4}}\right)^n$$

since $1-\epsilon < 1$ and $\frac{n\epsilon}{4} < \left[\frac{n\epsilon}{2}\right]$ for $n > \frac{4}{\epsilon}$. For $m > n(\frac{1}{2}+\epsilon)$ the result follows since $\binom{n}{m} = \binom{n}{n-m}$. ■

Returning now to the main proof, let us set $\delta(\epsilon) = (1-\epsilon)^{\frac{\epsilon}{4}}$ (note that $\delta(\epsilon) < 1$). Given $\epsilon > 0$ and $n > \frac{4}{\epsilon}$, we have for all $m < n(\frac{1}{2}-\epsilon)$ and all $m > n(\frac{1}{2}+\epsilon)$

$$\frac{\binom{n}{m}\frac{1}{\lambda+\frac{m}{n}\mu}}{\binom{n}{\left[\frac{n}{2}\right]}\frac{1}{\lambda+\left[\frac{n}{2}\right]\frac{\mu}{n}}} < (\delta(\epsilon))^n \frac{\lambda+\left[\frac{n}{2}\right]\frac{\mu}{n}}{\lambda+\frac{m}{n}\mu}. \tag{47}$$

We now consider two cases. First let $\mu \geq 0$. Then from (47) we have for $m > 0$ and $m < n(\frac{1}{2}-\epsilon)$ or $m > n(\frac{1}{2}+\epsilon)$

$$\frac{\binom{n}{m}\frac{1}{\lambda+\frac{m}{n}\mu}}{\binom{n}{\left[\frac{n}{2}\right]}\frac{1}{\lambda+\left[\frac{n}{2}\right]\frac{\mu}{n}}} < (\delta(\epsilon))^n \frac{\lambda+\mu}{\lambda+\frac{\mu}{n}} \leq n(\delta(\epsilon))^n.$$

Hence it follows from (45) and (46) that

$$\left|\frac{1}{\lambda} + \sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{1}{\lambda + \frac{m}{n}\mu} - 1\right| \le n^2(\delta(\epsilon))^n. \tag{48}$$

$$\left|\sum_{n\left(\frac{n}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{\frac{m}{n}}{\lambda + \frac{m}{n}\mu} - a\right| \le n^2(\delta(\epsilon))^n. \tag{49}$$

Also,

$$\left|\sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{1}{\lambda + \frac{m}{n}\mu} - \sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \left(\frac{1}{\lambda + \frac{\mu}{2}}\right)\right| =$$

$$= \left|\sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{\mu\left(\frac{1}{2} - \frac{m}{n}\right)}{\left(\lambda + \frac{\mu}{2}\right)\left(\lambda + \frac{m}{n}\mu\right)}\right| \ \le\ 2\epsilon$$

(using (45), $\frac{\mu}{\lambda+\frac{\mu}{2}} \le 2$ and $\left|\frac{1}{2} - \frac{m}{n}\right| < \epsilon$). Similarly

$$\left|\sum_{\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{\frac{m}{n}}{\lambda + \frac{m}{n}\mu} - \sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{\frac{1}{2}}{\lambda + \frac{\mu}{2}}\right| =$$

$$= \left|\sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{\left(\frac{m}{n} - \frac{1}{2}\right)\lambda}{\left(\lambda + \frac{m}{n}\mu\right)\left(\lambda + \frac{\mu}{2}\right)}\right| \ \le\ \epsilon.$$

Combining these with (48) and (49) gives

$$\left|\frac{1}{\lambda} + \sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{1}{\lambda + \frac{\mu}{2}} - 1\right| \le n^2(\delta(\epsilon))^n + 2\epsilon, \tag{50}$$

$$\left|\sum_{n\left(\frac{1}{2}-\epsilon\right)\le m\le n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{1}{\lambda + \frac{\mu}{2}} - 2a\right| \le 2n^2(\delta(\epsilon))^n + 2\epsilon. \tag{51}$$

Note that it follows that

$$\left|\frac{1}{\lambda} - (1 - 2a)\right| \le 3n^2(\delta(\epsilon))^n + 4\epsilon. \tag{52}$$

Now recall that the above estimates were derived for an arbitrary ϵ assuming that $n > \frac{4}{\epsilon}$ and that μ ($= \mu_n$, i.e. dependent on n) is greater or equal to 0. Consider what happens if $\mu < 0$. In this case let $\lambda' = \lambda + \mu (\ge 0)$, $\mu' = -\mu$. The constraints (45) and (46) become

$$\sum_{m=0}^{n} \binom{n}{m} \frac{1}{\lambda' + \frac{n-m}{n}\mu'} = 1 \tag{53}$$

and

$$\sum_{m=0}^{n} \binom{n}{m} \frac{\frac{n-m}{n}}{\lambda' + \frac{n-m}{n}\mu'} = 1 - a. \tag{54}$$

But since $\binom{n}{m} = \binom{n}{n-m}$, these are (up to transposing a and $1-a$) exactly the same constraints as before. Arguing as in the previous case we obtain that

$$\left| \frac{1}{\lambda'} - (1 - 2(1-a)) \right| \leq 3n^2(\delta(\epsilon))^n + 4\epsilon$$

but since $1 - 2(1-a) = 2a - 1 < 0$ and $\frac{1}{\lambda'} = \frac{1}{\lambda+\mu} = Z_n > 0$ this cannot occur when $3n^2(\delta(\epsilon))^n + 3\epsilon < |2a-1|$. Looking at a small ϵ, we can see that this means that eventually, for large n, μ must be greater or equal to 0 and since we are interested in the limit as n tends to ∞, we can ignore the possibility of $\mu < 0$ altogether.

By an exactly similar method as that used to obtain (48) and (49) we can show that

$$CM^{\infty}(K^n)(P(a_1) \wedge P(a_2)) = \sum_{m=0}^{n} \binom{n}{m} \frac{\frac{m(m-1)}{n(n-1)}}{\lambda + \frac{m}{n}\mu}$$

can be approximated analogously by

$$\sum_{n\left(\frac{1}{2}-\epsilon\right) \leq m \leq n\left(\frac{1}{2}+\epsilon\right)} \binom{n}{m} \frac{\frac{1}{4}}{\lambda + \frac{\mu}{2}}$$

and thus in view of (51),

$$\lim_{n\to\infty} CM^{\infty}(K^n)(P(a_1) \wedge P(a_2)) = \frac{a}{2}.$$

Similarly in general

$$\lim_{n\to\infty} CM^{\infty}(K^n)\left(\bigwedge_{i=1}^{r} P(a_i)\right) = \lim_{n\to\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{\frac{m(m-1)\ldots(m-r+1)}{n(n-1)\ldots(n-r+1)}}{\lambda + \frac{m}{n}\mu} = \frac{a}{2^{r-1}}.$$

The required answer now follows (using *Renaming*) by induction on $r - l$ (simultaneously for all $r \geq l$) since the above deals with $r - l = 0$ (any r) and for $l > 0$, $r - l > 0$

$$CM^{\infty}(K^n)\left(\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r} \neg P(a_i)\right) =$$

$$CM^{\infty}(K^n)\left(\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r-1} \neg P(a_i)\right) - CM^{\infty}(K^n)\left(\bigwedge_{i=1}^{l} P(a_i) \wedge \bigwedge_{i=l+1}^{r-1} \neg P(a_i) \wedge P(a_r)\right)$$

$$\to \frac{a}{2^{r-1-1}} - \frac{a}{2^{r-1}} = \frac{a}{2^{r-1}}.$$

Consequently, for $l = 0$,

$$CM^{\infty}(K^n)\left(\bigwedge_{i=1}^{r} \neg P(a_i)\right) \to 1 - (2^r - 1)\frac{a}{2^{r-1}} = 1 - 2a + \frac{a}{2^{r-1}}.$$

■

By *Renaming* we now have

Corollary 32 *Let* $K(a_1) = \{Bel(P(a_1) = a\}$ *where* $1 > a > \frac{1}{2}$. *Then*

$$\lim_{n\to\infty} CM^\infty(K^n)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_i)\right) = \begin{cases} \frac{1-a}{2^{r-1}} & if\ l = \Sigma\epsilon_i < r, \\ \frac{1-a}{2^{r-1}} + 2a - 1 & if\ l = \Sigma\epsilon_i = r. \end{cases}$$

Finally, since for

$$K(a_1) = \{Bel(P(a_1) = \frac{1}{2}\}$$

we have

$$\lim_{n\to\infty} CM^\infty(K^n)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_i)\right) = \frac{1}{2^r},$$

we can summarize these cases as follows:

Corollary 33 *Let* $K(a_1) = \{Bel(P(a_1) = a\}$ *where* $0 \leq a \leq 1$. *Then*

$$\lim_{n\to\infty} CM^\infty(K^n)\left(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_i)\right) = \begin{cases} \frac{a}{2^{r-1}} & if\ a < \frac{1}{2}\ and\ l = \Sigma\epsilon_i > 0, \\ \frac{a}{2^{r-1}} + 1 - 2a & if\ a < \frac{1}{2}\ and\ l = \Sigma\epsilon_i = 0, \\ \frac{1-a}{2^{r-1}} & if\ a > \frac{1}{2}\ and\ l = \Sigma\epsilon_i < r, \\ \frac{1-a}{2^{r-1}} + 2a - 1 & if\ a > \frac{1}{2}\ and\ l = \Sigma\epsilon_i = r, \\ \frac{1}{2^r} & if\ a = \frac{1}{2} \end{cases}$$

The rest of the proof of Theorem 29 follows just like that for MD via showing that CM^∞ satisfies the same strong version of the *Relativization Principle* and the same weak version of the *Irrelevant Information Principle.*

Theorem 34 *Let* $0 < c < 1$, $K_1, K_2, K \in C\mathcal{L}$ $(\mathcal{L} = \{p_1, ..., p_n\})$ *and let*

$$\begin{aligned} K_1 &= \left\{ Bel(\phi) = c,\ \sum_{j=1}^{r} a_{ji} Bel(\theta_j \wedge \phi) = b_i Bel(\phi) \mid i = 1, ..., m \right\}, \\ K_2 &= \left\{ \sum_{j=1}^{s} c_{ji} Bel(\psi_j \wedge \neg\phi) = d_i Bel(\neg\phi) \mid i = 1, ..., k \right\}, \\ K &= \left\{ Bel(\phi) = 1,\ \sum_{j=1}^{r} a_{ji} Bel(\theta_j) = b_i \mid i = 1, ..., m \right\}. \end{aligned} \tag{55}$$

Then $CM^\infty(K_1 \cup K_2)(\chi \wedge \phi) = cCM^\infty(K)(\chi)$ *for any* $\chi \in S\mathcal{L}$.

Proof. Without loss of generality assume that the atoms of $\mathcal{L}$, $\alpha_1, ..., \alpha_{2^n}$ are ordered so that ϕ is equivalent to $\bigvee_{i=1}^{t} \alpha_i$. Let $\vec{\tau} = CM^\infty(K)$ and $\vec{\delta} = CM^\infty(K_1 \cup K_2)$. Then $\tau_{t+1} = \ ... \ = \tau_{2^n} = 0$ and Bel_1 defined by

$$Bel_1(\alpha_i) = \begin{cases} c\tau_i & for\ i = 1, ..., t, \\ \delta_i & for\ i = t+1, ...2^n \end{cases}$$

is a solution to $K_1 \cup K_2$ and Bel_2 defined by

$$Bel_2(\alpha_i) = \begin{cases} \frac{\delta_i}{c} & for\ i = 1, ..., t, \\ 0 & for\ i = t+1, ...2^n \end{cases}$$

is a solution of K. By *Open Mindedness* (satisfied by CM^∞), if $\delta_i = 0$ for some $1 \leq i \leq t$ then this must be forced by the constraints $K_1 \cup K_2$, and hence by K_1 since the constraints in K_1 and K_2 refer to completely disjoint set of atoms. Hence $c\tau_i = 0$, too. Conversely, if $\tau_i = 0$ then this must be forced by K, equivalently $c\tau_i$ must be forced to be zero by K_1, so $\delta_i = 0$. Thus for $1 \leq i \leq t$, $\delta_i = 0 \Leftrightarrow \tau_i = 0$. By the choice of $\vec{\delta}$ and $\vec{\tau}$,

$$\begin{aligned}
\sum_{\substack{1\leq i\leq 2^n\\ \delta_i\neq 0}} \log \delta_i &= \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log(\delta_i/c) + \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log c + \sum_{\substack{t+1\leq i\leq 2^n\\ \delta_i\neq 0}} \log \delta_i \\
&\leq \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log \tau_i + \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log c + \sum_{\substack{t+1\leq i\leq 2^n\\ \delta_i\neq 0}} \log \delta_i \\
&= \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log c\tau_i + \sum_{\substack{t+1\leq i\leq 2^n\\ \delta_i\neq 0}} \log \delta_i \\
&\leq \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log \delta_i + \sum_{\substack{t+1\leq i\leq 2^n\\ \delta_i\neq 0}} \log \delta_i \\
&= \sum_{\substack{1\leq i\leq 2^n\\ \delta_i\neq 0}} \log \delta_i,
\end{aligned}$$

so

$$\sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log c\tau_i = \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log \delta_i.$$

By the strict concavity of the function $e(x_1, ..., x_t) = \sum_{\substack{1\leq i\leq t\\ \delta_i\neq 0}} \log x_i$ on the convex set of solutions of K we have $c\tau_i = \delta_i$ for $i = 1, ..., t$ and the result follows. ■

Theorem 35 *Let $K \in C\mathcal{L}$ and let β be an atom of the language $\mathcal{L}'$ where $\mathcal{L} \cap \mathcal{L}' = \emptyset$. Then for $\theta \in S\mathcal{L}$*

$$CM^\infty(K)(\theta) = CM^\infty(K \cup \{Bel(\beta) = 1\})(\theta \wedge \beta).$$

Proof. Let $\alpha_1, ..., \alpha_{2^n}$ be the atoms of $\mathcal{L}$, $\beta_1, ..., \beta_{2^{n'}}$ the atoms of $\mathcal{L}'$ and (without loss of generality) $\beta = \beta_1$. Let

$$CM^\infty(K)(\alpha_i) = \delta_i, \quad CM^\infty(K \cup \{Bel(\beta) = 1\})(\alpha_i \wedge \beta_j) = \tau_{ij}$$

for $i = 1, ..., 2^n,\ j = 1, ..., 2^{n'}$. Clearly $\tau_{ij} = 0$ for $j = 2, ..., 2^{n'}$. Let Bel_1 be defined on $S\mathcal{L}$ by

$$Bel_1(\alpha_i) = \tau_{i1} \quad (i = 1, ..., 2^n)$$

and let Bel_2 be defined on $S(\mathcal{L} \cup \mathcal{L}')$ by

$$Bel_2(\alpha_i \wedge \beta_j) = \begin{cases} \delta_i & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then Bel_1 is a solution to K and Bel_2 is a solution to $K \cup \{Bel(\beta) = 1\}$. Also, by the *Open Mindedness* of CM^∞, $\delta_i = 0$ just when this is forced by the constraints K. Hence if $\delta_i = 0$ then $\tau_{i1} = 0$ (since Bel_1 satisfies K), and similarly if $\tau_{i1} = 0$ then $\delta_i = 0$ (since Bel_2 satisfies $K \cup \{Bel(\beta) = 1\}$). Consequently, by the choice of $\vec{\delta}$ and $\vec{\tau}$

$$\sum_{\delta_i\neq 0} \log \tau_{i1} \leq \sum_{\delta_i\neq 0} \log \delta_i \leq \sum_{\delta_i\neq 0} \log \tau_{i1}$$

which implies that $\tau_{i1} = \delta_i$ for $i = 1, ..., 2^n$ by the strict concavity of the function $e(x_1, ..., x_{2^n}) = \sum_{\delta_i\neq 0} \log x_i$ on the convex set of solutions of K The result follows.

■

Analogously to the proof of Theorem 9 Theorem 29 now follows directly from Corollary 33 and Theorems 34 and 35.

We would conjecture that an analogous result to Theorem 10 also holds for CM^∞ (again with a limiting set of 3 complete reasons as in Theorem 16), a result we have already proved in a number of cases though not yet in complete generality. Clearly such an answer seems open to criticism in the car/skid scenario since it would imply that once examples of both skidding and non-skidding cars had been observed the conditional probability of any particular future car skidding should be 1/2, independent of whatever other patterns or propensities had been observed.

De Finetti's Theorem

The fact that each of the inference processes ME, MD, CM^∞ gives limit probabilities corresponding to canonical solutions of complete sets of reasons (at least for the cases so far proven) is intriguing. Is there some common reason for the emergence of 'reasons' like this? It is certainly not the case that this behavior is exhibited by all inference processes. For example, for

$$\alpha = \bigwedge_{i=1}^{n} p_i^{\epsilon_i}$$

let

$$\sigma(\alpha) = \frac{1}{(n+1)\binom{n}{t}}$$

where $t = \sum \epsilon_i$, and define the (language invariant) inference process D_2 to select from the solutions to K that probability function Bel which minimizes the 'cross-entropy'

$$\sum_{i=1}^{2^n} (Bel(\alpha_i) \log(Bel(\alpha_i)/\sigma(\alpha_i)).$$

In this case then, for $K(a_1)$ the empty set of constraints,

$$\mathrm{Lim}_{n\to\infty} D_2(\bigcup_{j=1}^{n} K(a_j))(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})) = \frac{1}{(r+1)\binom{r}{m}}\ , \tag{56}$$

where $m = \sum \epsilon_i$, and these values do not correspond to the canonical solution of any complete set of reasons (as will be apparent shortly).

There is another way of saying that the limit solutions for ME, MD, CM^∞ correspond to canonical solutions of complete sets of reasons. According de Finetti's Theorem, [3], (see also [4]), if B is an *exchangeable* probability function on the sentences of the language with propositional variables $P(a_1), P(a_2), \ldots$, that is

$$B(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i}))$$

depends only on r and $\sum \epsilon_i$, then

$$B(\bigwedge_{i=1}^{r} P^{\epsilon_i}(a_{n_i})) = \int_0^1 x^{\Sigma\epsilon_i}(1-x)^{r-\Sigma\epsilon_i} dF(x)$$

for some normalized measure F on $[0,1]$. Furthermore the values of B will correspond to the canonical solution of a complete set of reasons just if F is finite discrete, that is if all the measure in F (the λ's) is concentrated on a finite number of discrete points (the β's).

Now it is easy to check that in the cases of ME, MD, CM^{∞} discussed in this paper the limiting probability functions $\mathrm{Lim}_{n\to\infty} ME(\bigcup_{j=1}^{n} K(a_j))$ are exchangeable (essentially because these inferences processes satisfy *Renaming*, see [6]) so to say that the solution agrees with a canonical solution of a complete set of reasons is equivalently saying that the corresponding de Finetti measure is finite discrete. [This explains the above example. The values given in (56) correspond to the standard (uniform) Lebesgue measure which, of course, is not discrete.]

Conclusion

In this paper we have provided a framework and methodology for inductive reasoning as a limiting case of probabilistic propositional uncertain reasoning. We have shown that in a number of important cases the limit is well defined and furthermore corresponds to the canonical solution based on a complete set of reasons. The problem of fully explaining this phenomenon and showing its persistence for still more general knowledge bases will be the subject of a forthcoming paper by the first two authors.

The emergence of 'reasons' in this fashion is particularly intriguing in the case of the maximum entropy inference process which, we have previously argued, for example in [7], corresponds to the idealization of common sense, and so one might argue, should be normative for intelligent agents like ourselves. Such a conclusion for induction would stand squarely opposed to the conventional Carnapian approach (see for example [5], [1], [2]) based on considerations of symmetry etc. which yield continuous de Finetti measures.

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