0C1/1C1 January 2013 Solutions

1. (1)(i) \((x^2 - 3)(x + 4) = x^2(x + 4) - 3(x + 4) = x^3 + 4x^2 - 3x - 12\)
   
   (1)(ii) \((a - b + 1)(a + b - 1) = a(a + b - 1) - b(a + b - 1) + (a + b - 1)\)
   \[= a^2 + ab - a - ba - b^2 + b + a + b - 1 = a^2 - b^2 + 2b - 1\]
   
   (1)(iii) \((2 - x)(1 - (x - 2)) = (2 - x)(1 - x + 2) = (2 - x)(3 - x) = 6 - 5x + x^2\)
   
   (1)(iv) \((1 - 2x)(x - 1)^2 = (1 - 2x)(1 - x)(1 - x) = (1 - 2x)(1 - 2x + x^2) =\)
   \[1 - 2x + x^2 - 2x + 4x^2 - 2x^3 = -2x^3 + 5x^2 - 4x + 1.\]

(2) In 1(iv) the term in \(x^2\) is \(5x^2\), the coefficient of \(x\) is \(-4\) and the constant term is 1.

(3)

(i) \(\frac{x^4}{x^8} = x^{4-8} = x^{-4}\)   (ii) \(x^{-2} \sqrt{x} = x^{-2+1/3} = x^{-5/3}\)   (iii) \((x^4)^{5/6} = x^{4 \times 5/6} = x^{20/6} = x^{10/3}\)

2. (1) \(x^2 - 6x + 9 = (x - 2)(x - 4)\) so \(x^2 - 6x + 8 = 0 \iff x = 2, 4.\)
   
   Or use the formula to give solutions
   \[-(-6) \pm \sqrt{(-6)^2 - 4.8} \over 2 = 6 \pm 2 \over 2 = 2, 4.\]

(2)

\(5x^2 + 4x - 2 = 3x^2 + x - 1 \iff 2x^2 + 3x - 1 = 0 \iff x = \frac{-3 \pm \sqrt{3^2 - 4.2(-1)}}{4}\)
\[\iff x = \frac{-3 \pm \sqrt{17}}{4}.\]

(3)

\(x + 5 \over x - 5 = \frac{x - 2}{2} \iff 2(x + 5) = (x - 5)(x - 2) \iff 2x + 10 = x^2 - 7x + 10\)
\[\iff x^2 - 9x = 0 \iff x(x - 9) = 0 \iff x = 0, 9.\]
(4) 

\[
\frac{2}{x+2} - \frac{1}{x} = \frac{1}{x-4} \iff 2x(x-4) - (x-4)(x+2) = x(x+2) \\
\iff 2x^2 - 8x - x^2 + 2x + 8 = x^2 + 2x \\
\iff -8x + 8 = 0 \\
\iff x = 1
\]

(5) Put \(y = (x+1)^2\), so the equation becomes

\[
y^2 - 5y + 4 = 0 \iff (y-4)(y-1) = 0 \iff y = 1, 4
\]

Hence \((x+1)^2 = y = 1, 4\) so \(x+1 = \pm 1, \pm 2\) and \(x = 0, -2, 1, -3\) are the solutions.

3. 

   (i) \(9^x = 3 \iff (3^2)^x = 3 \iff 3^{2x} = 3^1 \iff 2x = 1 \iff x = 1/2\).

   (ii) \(\log_3 \left( \frac{2}{x+8} \right) = -2 \iff \left( \frac{2}{x+8} \right) = 3^{-2} \iff \frac{2}{x+8} = \frac{1}{9} \iff x + 8 = 18 \iff x = 10\)

   (iii) \(\log_3 (9^{x+1}) = x \iff (x+1) \log_3 (9) = x \iff 2(x+1) = x \iff x = -2\).

   (iv) \(x \log_x (3) = \log_x (2) \iff x = \frac{\log_x (2)}{\log_x (3)} \iff x = \log_3 (2)\)

   (v) \(\log_x (4x-4) = 2 \iff 4x - 4 = x^2 \iff (x-2)^2 = 0 \iff x = 2\)

4. (1) If \(y = mx + c\) passes through both \((-2, 1)\) and \((1, 7)\) then

\[
1 = -2m + c \quad \text{and} \quad 7 = m + c.
\]

Subtracting the first equation from the second gives \(6 = 3m\) so \(m = 2\) and with the second equation this gives \(7 = 2 + c\) so \(c = 5\) and the equation of the line \(C\) is \(y = 2x + 5\).

(2) Substituting \(x = 3, y = 10\) into \(y = 2x + 5\) gives \(10 = (2)(3) + 5\) which is not true so this point does not lie on the line \(C\).
(3) If the lines $y = 2x + 5$ and $y = 1 - 2x$ intersect at $(x, y)$ then

$$2x + 5 = y = 1 - 2x$$

so $2x + 2x = 1 - 5$, i.e. $x = -1$ and $y = (2)(-1) + 5 = 3$. Hence $A = (-1, 3)$.

(4) The gradients of the lines $y = 2x + 5$, $y = 1 - 2x$ are $2$, $-2$ respectively and since $(2)(-2) \neq -1$ they are not perpendicular.

(5) The distance from $(-1, 3)$ to $(0, 5)$ is

$$\sqrt{((-1 - 0)^2 + (3 - 5)^2} = \sqrt{1 + 4} = \sqrt{5}.$$ 

(6) Considering the right angled triangle formed by the points $A = (-1, 3)$, $(0, 5)$ and $(0, 3)$ (notice that $(-1, 3)$ and $(0, 5)$ lie on the line $C$ and the line segment from $(-1, 3)$ and $(0, 3)$ is parallel to the $x$-axis and of length 1) we see that the cosine of the angle $C$ makes with the $x$-axis is the distance from $(-1, 3)$ to $(0, 3)$, divided by the distance from $(-1, 3)$ to $(0, 5)$, i.e. $1/\sqrt{5}$.

5.  

(1) The curves intersect when

$$x^2 - 2 = y = 2x^2 + 7x - 2,$$

equivalently

$$0 = x^2 + 7x = x(x + 7).$$

Thus the two points are when $x = 0$ and $y = 0^2 - 2 = -2$ and when $x = -7$ and $y = (-7)^2 - 2 = 47$, i.e. the points $(0, -2)$ and $(-7, 47)$.

(2) The slopes of these curves at $x$ are $(d/dx)(x^2 - 2) = 2x$ and $(d/dx)(2x^2 + 7x - 2) = 4x + 7$ respectively so these will be equal when $2x = 4x + 7$, i.e. $x = -7/2$.

(3) The point $(-1, -7)$ is on the curve $D$ since substituting in these values into the defining equation gives $-7 = 2(-1)^2 + (-1)7 - 2$. At this value of $x$ the slope is as above $4(-1) + 7 = 3$ so if $y = mx + c$ is to be the tangent it must satisfy that $m = 3$ and $-7 = (3)(-1) + c$ so $c = -4$, i.e. the equation of the tangent is $y = 3x - 4$.

(4) The line $y = 3x - 4$ meets the curve $C$ when $3x - 4 = y = x^2 - 2$, equivalently $x^2 - 3x + 2 = 0 = (x - 2)(x - 1)$, i.e. $x = 1$ and $x = 2$. At $x = 1$
the $y$ coordinate on $C$ is $1^2 - 2 = -1$ and at $x = 2$ the $y$ coordinate on $C$ is $2^2 - 2 = 2$. Hence the two points are $(1, -1)$ and $(2, 2)$.

6. (a) (1) Domain of $f$ is all the reals except 0.

   (2) $f(f(x)) = \frac{2}{f(x)} + 1 = 2/((2/x) + 1) + 1 = 2x/(2 + x) + 1$.

   (3) We must have $x = f(f^{-1}(x)) = \frac{2}{f^{-1}(x)} + 1$. Multiplying both sides by $f^{-1}(x)$ gives
   
   
   $$xf^{-1}(x) = 2 + f^{-1}(x)$$

   so $(x - 1)f^{-1}(x) = 2$ and $f^{-1}(x) = \frac{2}{x - 1}$.

(b)

\[ \text{Diagram} \]

(1) $\cos(A) = b/6$ so $b = 6 \cos(A) = 4$.

(2) $\sin^2(A) + \cos^2(A) = 1$ so $\sin^2(A) = 1 - (2/3)^2 = 5/9$ and $\sin(A) = \sqrt{5/9} = \sqrt{5}/3$.

[From the diagram $\sin(A) \geq 0$ so we must take the positive square root here.]

(3) $\cot(A) = (\cos(A) / \sin(A)) = (2/3)/(\sqrt{5}/3) = 2/\sqrt{5}$

(4) $\cos(A) = 2 \cos^2(A/2) - 1$ so $\cos(A/2) = \sqrt{\cos(A) + 1}/2 = \sqrt{(2/3 + 1)/2 = \sqrt{5}/6}$

[Notice it is the positive root since from the diagram $\cos(A/2)$ is positive.]

(5) $\cos(A - \pi/4) = \cos(A) \cos(\pi/4) + \sin(A) \sin(\pi/4) = (2/3)(1/\sqrt{2}) + (\sqrt{5}/3)(1/\sqrt{2}) = (2 + \sqrt{5})/(3\sqrt{2})$. 

4
7. (1)

(i) \[\frac{d}{dx}(6x^6 - 6) = 36x^5\]

(ii) \[\frac{d}{dx}(x^{-4/3}) = \frac{-4x^{-4/3-1}}{3} = \frac{-4x^{-7/3}}{3}\]

(iii) \[\frac{d}{dx}(e^{1-2x}) = -2e^{1-2x}\]

(2) For \(f(x) = x^3 - 6x^2 + 9x\), \(df/dx = 3x^2 - 12x + 9\) and \(d^2f/dx^2 = 6x - 12\). Hence the stationary points are when

\[df/dx = 3x^2 - 12x + 9 = 3(x - 1)(x - 3) = 0,\]

i.e. \(x = 1, 3\). When \(x = 1\) \(d^2f/dx^2 = 6(1) - 12 = -6 < 0\) so this is a (local) maximum. When \(x = 3\), \(d^2f/dx^2 = 6(3) - 12 = 6 > 0\) so this is a (local) minimum point.

The graph of this function looks as follows:

![Graph of the function](image)

[In other words it goes up from the left to reach a local maximum at \(x = 1\), when in fact \(f(1) = (1)^3 - 6(1)^2 + 9(1) = 4\). It then goes down to a local minimum at \(x = 3\) (when \(f(3) = 0\) and thereafter moves up to the right.) Since the line \(y = 6\) lies above the curve at the local maximum this line only cuts the curve (beyond \(x = 3\)) in one place so \(x^3 - 6x^2 + 9x = 6\) has only one solution.

8.
(1) \[ \frac{d}{dx}(x^2 + 1)^{-3} = (2x)(-3)(x^2 + 1)^{-3-1} = -6x(x^2 + 1)^{-4} \]

(2) \[ \frac{d}{dx}(\sin(x) \cos(2x)) = \frac{d}{dx}(\sin(x)) \cdot \cos(2x) + \sin(x) \cdot \frac{d}{dx}(\cos(2x)) \]
\[ = (\cos(x)) \cos(2x) + \sin(x)(-2 \sin(2x)) = \cos(x) \cos(2x) - 2 \sin(x) \sin(2x) \]

(3) \[ \frac{d}{dx} \left( \frac{1-x}{1+x} \right) = \frac{(-1)(1+x) - (1)(1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2} \]

(4) Putting \( u = 2 + \sin(x), \ y = \ln u, \)
\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (\cos(x)) = \frac{\cos(x)}{2 + \sin(x)} \]

(5) Putting \( u = \sqrt{x}, \ y = 2e^u, \)
\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2e^u \cdot \left( \frac{1}{2\sqrt{u}} \right) = \frac{e\sqrt{x}}{\sqrt{x}} \]