0C1/1C1 January 2011 Solutions

1.

(1)(i) $(x-3)(x^2+5) = x(x^2+5) - 3(x^2+5) = x^3 + 5x - 3x^2 - 15$

(1)(ii) $(a-b-2)(a+b-1) = a(a+b-1) - b(a+b-1) - 2(a+b-1)$
$= a^2 + ab - a - ba - b^2 + b - 2a - 2b + 2 = a^2 - b^2 - 3a - b + 2$

(1)(iii) $(1-x)(2-(x+3)) = (1-x)(-1-x) = -1 - x + x + x^2 = x^2 - 1$

(1)(iv) $x(x-1)(1-3x) = x(x - 3x^2 - 1 + 3x) = x(-3x^2 + 4x - 1) = -3x^3 + 4x^2 - x.$

(2) In 1(iv) the term in x^3 is $-3x^3$, the coefficient of x is -1 and the constant term is 0.

(3)

(i) $\dfrac{x^4}{x^7} = x^{4-7} = x^{-3}$ (ii) $x^2\sqrt[3]{x} = x^{2+1/3} = x^{7/3}$ (iii) $(x^4)^{1/6} = x^{4\times1/6} = x^{4/6} = x^{2/3}$

2. (1) $x^2 + 3x - 10 = (x-2)(x+5)$ so $x^2 + 3x - 10 = 0 \iff x = 2, -5$.
Or use the formula to give solutions

$$\frac{-3 \pm \sqrt{3^2 - 4.(-10)}}{2} = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2} = 2, -5.$$

(2)

$$3x^2 - x - 5 = x^2 - 2x - 3 \iff 2x^2 + x - 2 = 0 \iff x = \frac{-1 \pm \sqrt{1^2 - 4.2.(-2)}}{4}$$

$$\iff x = \frac{-1 \pm \sqrt{17}}{4}.$$

(3)

$$\frac{x+1}{x-3} = \frac{x-1}{3} \iff 3(x+1) = (x-3)(x-1) \iff 3x + 3 = x^2 - 4x + 3$$

$$\iff x^2 - 7x = 0 \iff x(x-7) = 0 \iff x = 0, 7.$$

(4)

$$\begin{aligned}\frac{1}{4-x}+\frac{1}{x}=\frac{-2}{x+8} \iff & x(x+8)+(x+8)(4-x) \;=\; -2x(4-x)\\ \iff & x^2+8x-x^2-4x+32 \;=\; 2x^2-8x\\ \iff & 2x^2-12x-32 \;=\; 0\\ \iff & x^2-6x-16 \;=\; 0\\ \iff & (x+2)(x-8)=0\\ \iff & x=-2,8 \quad \text{(or by using the formula)}\end{aligned}$$

(5) Put $y = x^2$, so the equation becomes

$$y^2-3y+2=0 \iff (y-2)(y-1)=0 \iff y=1,2$$

Hence $x^2 = 1, 2$ so $x = 1, -1, \sqrt{2}, -\sqrt{2}$.

3.

(i) $25^x = 5 \iff (5^2)^x = 5 \iff 5^{2x} = 5 \iff 2x = 1 \iff x = 1/2.$

(ii) $\log_5\left(\dfrac{4}{x-1}\right) = -1 \iff \left(\dfrac{4}{x-1}\right) = 5^{-1} \iff \dfrac{4}{x-1} = \dfrac{1}{5}$
$\iff x-1 = 20 \iff x = 21$

(iii) $\log_3(9^{x+2}) = 3x \iff (x+2)\log_3(9) = 3x$
$\iff 2(x+2) = 3x \iff x = 4.$

(iv) $x\log_2(x) = \log_3(x) \iff \dfrac{x\log_3(x)}{\log_3(2)} = \log_3(x) \iff$
$x = \log_3(2)$ using the change of base formula $\log_2(x) = \dfrac{\log_3(x)}{\log_3(2)}$.

<u>or</u> $x\log_2(x) = \log_3(x) \iff x\log_2(x) = \dfrac{\log_2(x)}{\log_2(3)} \iff x = \dfrac{1}{\log_2(3)}$

(v) $\log_x(x^2-5x+9) = 1 \iff x^2-5x+9 = x^1 \iff x^2-6x+9 = 0 \iff$
$(x-3)(x-3) = 0 \iff x = 3.$

4. (1) If $y = mx + c$ passes through both $(-1, 4)$ and $(1, 8)$ then

$$4 = -m + c \quad \text{and} \quad 8 = m + c.$$

Subtracting the first equation from the second gives $4 = 2m$ so $m = 2$ and with the second equation this gives $8 = (2)(1) + c$ so $c = 6$ and the equation of the line $\mathcal{C}$ is $y = 2x + 6$.

(2) Substituting $x = -2$, $y = 2$ into $y = 2x + 6$ gives $2 = (2)(-2) + 6$ which is true so this point lies on the line $\mathcal{C}$.

(3) If $y = 0$ then $0 = 2x + 6$ and $x = -3$ so $\mathcal{C}$ crosses the x-axis at the point $A = (-3, 0)$.

(4) The distance from $(-3, 0)$ to $(1, 8)$ is

$$\sqrt{((-3-1)^2+(0-8)^2} = \sqrt{16+64} = \sqrt{80} = \sqrt{16}\sqrt{5} = 4\sqrt{5}.$$

(5) Considering the right angled triangle formed by the points $A = (-3, 0)$, $(1, 8)$ and $(1, 0)$ (notice that $(-3, 0)$ and $(1, 8)$ lie on the line $\mathcal{C}$ and $(-3, 0)$ and $(1, 0)$ lie on the x-axis) we

see that the cosine of the angle $\mathcal{C}$ makes with the x-axis is the distance from $(-3,0)$ to $(1,0)$, divided by the distance from $(-3,0)$ to $(1,8)$, i.e. $4/4\sqrt{5} = 1/\sqrt{5}$.

(6) If the lines $y = 2x+6$ and $y = 14-2x$ intersect at (x,y) then

$$2x+6 = y = 14-2x$$

so $2x+2x = 14-6$, i.e. $x = 2$ and $y = (2)(2)+6 = 10$. Hence the two lines intersect at the point $(2,10)$.

5.
(1) At the point of intersection we must have $3x+1-x^2 = y = 5-x$. Hence $0 = x^2-4x+4 = (x-2)(x-2)$, so $x = 2$. For this value of x, $y = 5-2 = 3$ so the required point is $(2,3)$.

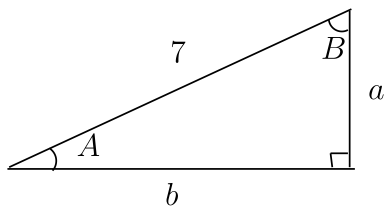
(2) At A the slope of $\mathcal{E}$ is -1 and the slope of $\mathcal{C}$ is $d/dx(3x+1-x^2) = 3-2x$ evaluated at $x = 2$, i.e. -1 again. So $\mathcal{C}$ and $\mathcal{E}$ are parallel at $(2,3)$ and $\mathcal{E}$ must be the tangent to $\mathcal{C}$ at this point.

(3) Let the normal be $y = mx+c$. Since it is normal to $\mathcal{E}$ its slope m must satisfy $m(-1) = -1$ so $m = 1$. Also since this line goes through the point $(2,3)$, $3 = 2m+c$. Hence $c = 1$ and the normal is $y = x+1$.

(4) The normal intersects $\mathcal{C}$ when $3x+1-x^2 = y = x+1$. Hence $0 = x^2-2x = x(x-2)$. The solution $x = 2$ gives the point A so the other point on intersection must be when $x = 0$, and $y = 0+1 = 1$, i.e. the point $(0,1)$.

(5) The slope of $\mathcal{C}$ at $x = 0$ is given by $d/dx(3x+1-x^2) = 3-2x$ evaluated at $x = 0$, i.e. 3. Hence the required tangent $y = mx+c$ to $\mathcal{C}$ at $(0,1)$ satisfies $m = 3$, $1 = m(0)+c$, so the tangent here is $y = 3x+1$.

6. (a)



$\cos(A) = b/7$ so $b = 7\cos(A) = 14/3$.
$\sin^2(A)+\cos^2(A) = 1$ so $\sin^2(A) = 1-4/9 = 5/9$ and $\sin(A) = \sqrt{5/9} = \sqrt{5}/3$.
$a/7 = \sin(A)$ so $a = 7\sin(A) = 7\sqrt{5}/3$.
$\cos(B) = a/7 = \sin(A) = \sqrt{5}/3$
$\sin(B) = b/7 = \cos(A) = 2/3$.

(b) $2\cos(A)\cos(B) = \cos(A+B)+\cos(A-B) \qquad \star$

Substituting $A = B = \pi/8$ in $\star$ gives

$$2\cos^2(\pi/8) = \cos(\pi/4)+\cos(0) = 1/\sqrt{2}+1$$

so

$$\cos(\pi/8) = \sqrt{\left(\frac{1+1/\sqrt{2}}{2}\right)} = \sqrt{\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)}.$$

Substituting $A = \pi/4$, $B = \pi/8$ into $\star$ gives

$$2\cos(\pi/4)\cos(\pi/8) = \cos(3\pi/8) + \cos(\pi/8)$$

so

$$\cos(3\pi/8) = ((2/\sqrt{2}) - 1)\cos(\pi/8) = (\sqrt{2} - 1)\cos(\pi/8).$$

7.(1)

(i) $d/dx(3x^3 - 3) = 9x^2$

(ii) $d/dx(x^{1/3}) = \dfrac{x^{-1+1/3}}{3} = \dfrac{x^{-2/3}}{3}$

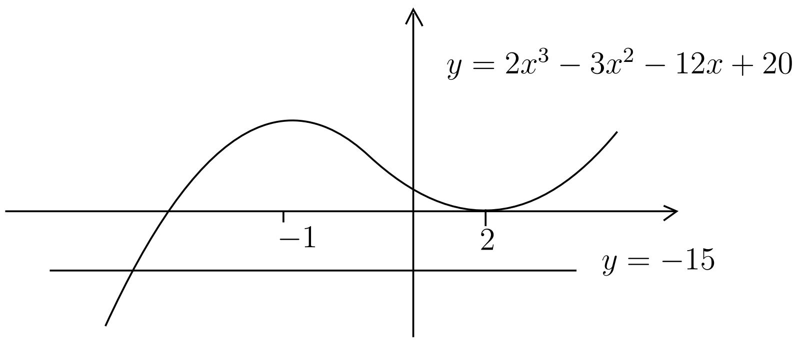
(iii) $d/dx(e^{2x-1}) = 2e^{2x-1}$

(2) For $f(x) = 2x^3 - 3x^2 - 12x + 20$, $df/dx = 6x^2 - 6x - 12$ and $d^2f/dx^2 = 12x - 6$. Hence the stationary points are when

$$df/dx = 3x^2 - 6x - 12 = 6(x - 2)(x + 1) = 0,$$

i.e. $x = 2, -1$. When $x = 2$ $d^2f/dx^2 = 12(2) - 6 = 18 > 0$ so this is a (local) minimum. When $x = -1$, $d^2f/dx^2 = 12(-1) - 6 = -18 < 0$ so this is a (local) maximum point.

The graph of this function looks as below.



In other words it goes up from the left to reach a local maximum at $x = -1$, when in fact $f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 20 = 27$. It then goes down to a local minimum at $x = 2$ (when $f(1) = 2(2)^3 - 3(2)^2 - 12(2) + 20 = 0$) and thereafter moves up to the right.] Since the line $y = -15$ lies below the curve at the local minimum this line only cuts the curve (to the left of $x = -1$) in one place so $2x^3 - 3x^2 - 12x + 20 = -15$ has only one solution.

8.

(1) $$\frac{d}{dx}(x+1)^2 \sin(x) = \sin(x)\frac{d}{dx}(x+1)^2 + (x+1)^2\frac{d}{dx}\sin(x)$$
$$= 2(x+1)\sin(x) + (x+1)^2\cos(x)$$

(2) $$\frac{d}{dx}(\cos^2(x)) = \frac{d}{dx}(\cos(x)) \cdot \cos(x) + \cos(x) \cdot \frac{d}{dx}(\cos(x))$$
$$= (-\sin(x))\cos(x) + \cos(x)(-\sin(x)) = -2\cos(x)\sin(x)$$

(3) $$\frac{d}{dx}\left(\frac{x+1}{x+2}\right) = \frac{(x+2)-(x+1)}{(x+2)^2} = = \frac{1}{(x+2)^2}$$

(4) Putting $u = \sin x$, $y = \ln u$,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (\cos x) = \frac{\cos x}{\sin x} = \cot x$$

(5) Putting $u = e^x + 1$, $y = \sqrt{u}$,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot e^x = \frac{e^x}{2\sqrt{e^x+1}}$$