0C1/1C1 January 2010 Solutions

1.

(1)(i) $(x^2-1)(x+5) = x^2(x+5)-(x+5) = x^3+5x^2-x-5$

(1)(ii) $(a-b-1)(a+b-2) = a(a+b-2)-b(a+b-2)-(a+b-2)$
$= a^2+ab-2a-ab-b^2+2b-a-b+2 = a^2-b^2-3a+b+2$

(1)(iii) $(1-a)(b-(a-1)) = (1-a)(b-a+1) = b-a+1-ab+a^2-a$
$= a^2-ab+b-2a+1$

(1)(iv) $x(x+1)(1-3x) = x(x-3x^2+1-3x) = x(-3x^2-2x+1) = -3x^3-2x^2+x.$

(2) In 1(iv) the term in x^2 is $-2x^2$, the coefficient of x is 1 and the constant term is 0.

(3)

(i) $\dfrac{x^3}{x^6} = x^{3-6} = x^{-3}$ (ii) $x\sqrt[3]{x} = x^{1+1/3} = x^{4/3}$ (iii) $(x^6)^{1/3} = x^{6\times 1/3} = x^2$

2. (1) $x^2-6x+8 = (x-2)(x-4)$ so $x^2-6x+8=0 \iff x=2,4.$
Or use the formula to give solutions

$$\frac{-(-6)\pm\sqrt{(-6)^2-4.8}}{2} = \frac{6\pm\sqrt{4}}{2} = 4,2.$$

(2)

$$4x^2+2x-3 = 2x^2-x-2 \iff 2x^2+3x-1=0 \iff x = \frac{-3\pm\sqrt{3^2-4.2.(-1)}}{4}$$

$$\iff x = \frac{-3\pm\sqrt{17}}{4}.$$

(3)

$$\frac{x+2}{x-2} = \frac{x}{3} \iff 3(x+2) = x(x-2) \iff 3x+6 = x^2-2x$$
$$\iff x^2-5x-6=0 \iff (x-6)(x+1)=0 \iff x=-1,6.$$

(4)

$$\begin{aligned}
\frac{1}{3-x}+\frac{1}{x+1} &= \frac{-2}{x+9}\\
\iff (x+9)(x+1)+(x+9)(3-x) &= -2(x+1)(3-x)\\
\iff x^2+10x+9-x^2-6x+27 &= 2x^2-4x-6\\
\iff 2x^2-8x-42 &= 0\\
\iff x^2-4x-21 &= 0\\
\iff (x+3)(x-7) &= 0\\
\iff x = -3,7 &\quad \text{(or by using the formula)}
\end{aligned}$$

(5) Put $y = 2^x$, so $2^{x+1} = 2y$. Then

$$2^{2x} - 2^{x+1} + 1 = 0 \iff y^2 - 2y + 1 = 0$$

$$\iff (y-1)(y-1) = 0 \iff y = 1 \iff 2^x = 1 \iff x = 0.$$

3.(1)

(i) $8^x = 2 \iff (2^3)^x = 2 \iff 2^{3x} = 2 \iff 3x = 1 \iff x = 1/3.$

(ii) $\log_3\left(\frac{27}{x+1}\right) = 2 \iff 3^{\log_3\left(\frac{27}{x+1}\right)} = 3^2 \iff \frac{27}{x+1} = 9$
$\iff x + 1 = 3 \iff x = 2$

(iii) $\log_2(4^{x+1}) = x \iff (x+1)\log_2(4) = x$
$\iff 2(x+1) = x \iff x = -2.$

(iv) $\log_x(x^3 - 2x + 1) = 3 \iff x^{\log_x(x^3-2x+1)} = x^3 \iff x^3 - 2x + 1 = x^3$
$\iff -2x + 1 = 0 \iff x = 1/2.$

(2) When $t = 0$ the volume is $V_0a^0 = V_0$. When $t = 3$ the volume is V_0a^3 and we are told this is twice the initial volume, so $V_0a^3 = 2V_0$. Hence $a^3 = 2$ and $a = \sqrt[3]{2}$.

When the volume is 5 times the initial volume (i.e. $5V_0$) then t must be such that $V_0a^t = 5V_0$, so $5 = a^t = (\sqrt[3]{2})^t$. Taking logs to the base 2 gives $\log_2 5 = t/3$ so $t = 3\log_2 5 = \log_2(125)$.

4. (1) If $y = mx + c$ passes through both $(-3, -3)$ and $(1, 9)$ then

$$-3 = -3m + c \quad \text{and} \quad 9 = m + c.$$

Subtracting the first equation from the second gives $12 = 4m$ so $m = 3$ and with the second equation this gives $9 = (3)(1) + c$ so $c = 6$ and the equation of the line $\mathcal{C}$ is $y = 3x + 6$.

(2) Substituting $x = -4$, $y = -6$ into $y = 3x + 6$ gives $-6 = (3)(-4) + 6$ which is true so this point lies on the line $\mathcal{C}$.

(3) If $y = 0$ then $0 = 3x + 6$ and $x = -2$ so $\mathcal{C}$ crosses the x-axis at the point $A = (-2, 0)$. If $x = 0$ then $y = (3)(0) + 6 = 6$ so $\mathcal{C}$ crosses the y-axis at the point $B = (0, 6)$.

(4) The distance from $(-2, 0)$ to $(0, 6)$ is

$$\sqrt{((0-2)^2 + (6-0)^2} = \sqrt{4 + 36} = \sqrt{40} = \sqrt{4}\sqrt{10} = 2\sqrt{10}.$$

(5) Considering the right angled triangle formed by the points $A = (-2, 0)$, $B = (0, 6)$ and $(0, 0)$ (notice that A and B lie on the line $\mathcal{C}$ and B and $(0, 0)$ lie on the x-axis) we see that the sine of the angle $\mathcal{C}$ makes with the x-axis is the distance from B to $(0, 0)$, divided by the distance from A to B, i.e. $6/2\sqrt{10} = 3/\sqrt{10}$.

(6) If the lines $y = 3x + 6$ and $y = 11x - 10$ intersect at (x, y) then

$$3x + 6 = y = 11x - 10$$

so $11x - 3x = 10 + 6$, i.e. $x = 2$ and $y = (3)(2) + 6 = 12$. Hence the two lines intersect at the point $(2, 12)$.

5. (1) The curves intersect when

$$2x^2 + x - 1 = y = x^2 + 2x + 1,$$

equivalently

$$x^2 - x - 2 = (x+1)(x-2) = 0.$$

Thus the two points are when $x = -1$ and $y = (-1)^2 + 2(-1) + 1 = 0$ and when $x = 2$ and $y = (2)^2 + 2(2) + 1 = 9$, i.e. the points $(-1, 0)$ and $(2, 9)$.

(2) The slopes of these curves at x are $(d/dx)(2x^2 + x - 1) = 4x + 1$ and $(d/dx)(x^2 + 2x + 1) = 2x + 2$ respectively so these will be equal when $4x + 1 = 2x + 2$, i.e. $x = 1/2$.

(3) The point $(-2, 1)$ is on the curve $\mathcal{E}$ since substituting in these values into the defining equation gives the true equation $1 = (-2)^2 + 2(-2) + 1$. At this value of x the slope is as above $2(-2) + 2 = -2$ so if $y = mx + c$ is to be the tangent it must satisfy that $m = -2$ and $1 = (-2)(-2) + c$ so $c = -3$, i.e. the equation of the tangent is $y = -2x - 3$.

(4) For the line $y = mx + c$ to be perpendicular to this tangent we must have $m \times (-2) = -1$ so $m = 1/2$ and for it to pass through the point $(-2, 7)$ we must have $7 = (1/2)(-2) + c$ giving $c = 8$. Thus the required line is $y = 8 + x/2$.

6.(1)

(i) $d/dx(4x^3 + 3) = 12x^2$

(ii) $d/dx(x^{1/4}) = \dfrac{x^{-1+1/4}}{4} = \dfrac{x^{-3/4}}{4}$

(iii) $d/dx(\cos(2x - 1)) = -2\sin(2x - 1)$

(2) For $f(x) = x^3 + 3x^2 - 9x + 5$, $df/dx = 3x^2 + 6x - 9$ and $d^2f/dx^2 = 6x + 6$. Hence the stationary points are when

$$df/dx = 3x^2 + 6x - 9 = 3(x-1)(x+3) = 0,$$

i.e. $x = 1, -3$. When $x = 1$ $d^2f/dx^2 = 6(1) + 6 = 12 > 0$ so this is a (local) minimum. When $x = -3$, $d^2f/dx^2 = 6(-3) + 6 = -12 < 0$ so this is a (local) maximum point.

The graph of this function looks as below.

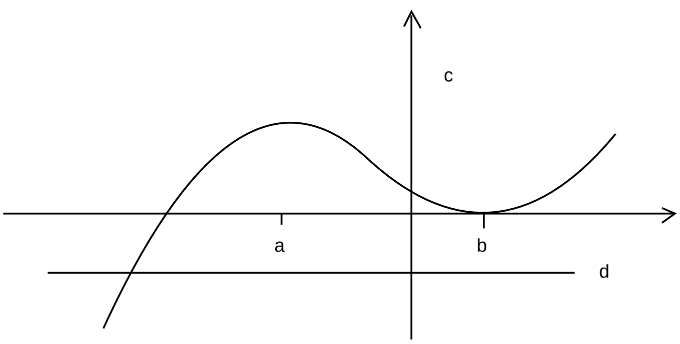
PSfrag replacements

-3

1

$y = x^3 + 3x^2 - 9x + 5$

$y = -1$



In other words it goes up from the left to reach a local maximum at $x = -3$, when in fact $f(1) = (1)^3 - 6(1)^2 + 9(1) = 4$. It then goes down to a local minimum at $x = 1$ (when $f(1) = (1)^3 + 3(1)^2 - 9(1) + 5 = 0$) and thereafter moves up to the right.] Since the line $y = -1$ lies below the curve at the local minimum this line only cuts the curve (to the left of $x = 1$) in one place so $x^3 + 2x^2 - 9x + 6 = 0$ has only one solution.

7.

$$(1)\quad \frac{d}{dx}(x-1)^2e^x = \left(\frac{d}{dx}(x-1)^2\right)e^x + (x-1)^2\frac{d}{dx}(e^x)$$

$$= 2(x-1)e^x + (x-1)^2e^x = (x^2-1)e^x$$

$$(2)\quad \frac{d}{dx}(x(1-\ln x)) = \frac{d}{dx}(x) - \left(\frac{d}{dx}(x)\right)\ln x - x\frac{d}{dx}(\ln x)$$

$$= 1 - \ln x - x/x = -\ln x$$

$$(3)\quad \frac{d}{dx}\left(\frac{x-1}{x+1}\right) = \frac{(x+1)-(x-1)}{(x+1)^2} == \frac{2}{(x+1)^2}$$

(4) Putting $u = \cos x$, $y = \ln u$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (-\sin x) = \frac{-\sin x}{\cos x} = -\tan x$$

(5) Putting $u = \sqrt{x}$, $y = e^u$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \frac{1}{2\sqrt{x}} = (1/2)x^{-1/2}e^{\sqrt{x}}$$

8(1)

$$\begin{aligned} \int (x + \frac{1}{\sqrt{x}})^2 \, dx. &= \int x^2 + 2x^{1/2} + \frac{1}{x} \, dx \\ &= \frac{x^3}{3} + \frac{4x^{3/2}}{3} + \ln x + c \end{aligned}$$

(2)

$$\begin{aligned} \int_0^{\pi/2} e^x + \cos(x) \, dx &= [e^x + \sin(x)]_0^{\pi/2} \\ &= [e^{\pi/2} + \sin(\pi/2)] - [e^0 + \sin(0)] \\ &= e^{\pi/2} + 1 - 1 - 0 \;=\; e^{\pi/2} \end{aligned}$$

(3) Since $x^2 + 1$ is positive for $-1 \le x \le 1$ the area A is given by

$$\begin{aligned} \int_{-1}^{1} x^2 + 1 \, dx &= \left[\frac{x^3}{3} + x\right]_{-1}^{1} \\ &= \left(\frac{1}{3} + 1\right) - \left(\frac{(-1)^3}{3} + (-1)\right) \;=\; 8/3 \end{aligned}$$

PSfrag replacements

A

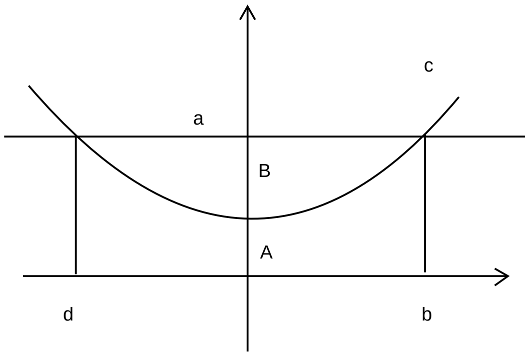
2

$y = x^2 + 1$

1

-1

B



The line $y = x^2 + 1$ intersects the line $y = 2$ when $x^2 + 1 = 2$, that is $x^2 = 1$, equivalently $x = \pm 1$. For $-1 \leq x \leq 1$, $y = x^2 + 1 \leq 2$. Hence the required area B is the area in the rectangle given by the x axis and the lines $y = 2$, $x = 1$, $x = -1$, minus the area A calculated above, i.e. $2 - (8/3) = 4/3$.