December 4, 2013

0C1/1C1, MATH19541/19821 Course Notes

Introduction

Lecturer: Prof Jeff Paris, room 1.119, School of Mathematics, Alan Turing Building (opposite the Citroën Dealer on Upper Brook Street), tel 0161-275-5880, email jeff.paris@manchester.ac.uk. Office hour 1.30-2.30 Fridays.
Best website `www.maths.manchester.ac.uk/~jeff/`

Students: Are from

1. Optometry, 1C1
2. Foundation Year, 0C1

(and maybe elsewhere?)

Lecture times: Monday at 1.00 in Renold J17, Wednesday at 9.00 in Renold E7.

Tutorial/Computer sessions: All tutorials are in the Sackville G.11 Cluster, for 1C1 Wednesdays at 10.00, for 0C1 Tuesdays and Fridays at 10.00 EXCEPT Friday 25th October when the session will be at 4.00. Note that these tutorials start in Week 1.

Reading Week: Week 6 is a reading week and there will be no lectures or tutorials then for this course.

In Class Tests: Week 5 (Wednesday October 23rd) and Week 10 (Wednesday November 27th).

Web site: All the arrangements, times and places, syllabus, past exam papers, further example sheets, details about the web exercises etc. for the course can be accessed through Blackboard and at the School of Mathematics service courses web pages at
http://www.maths.manchester.ac.uk/study/undergraduate/service-teaching/ (click on 'Foundation Studies'/'Optometry' then 'MATH19821'/'MATH19541'). You should spend a little time familiarizing yourself with the pages which relate to this course and check them (quite) regularly for new announcements.

Complete course notes are available to download from these web pages, *and also on my page* `www.maths.manchester.ac.uk/~jeff/`. However it is important for to keep up with the course by attending the lectures and making notes of your own. There are at least four good reason's for doing this. Firstly it is the best way to understand and retain the material, secondly you need to be ready to answer the tests as they come up, thirdly I will be giving additional examples (and insight) in the lectures, and finally it's just so much more enjoyable and satisfying going to the lectures when you concentrate and have a reasonable idea what it's all about.

Tutorials/Computer sessions: In addition to the lectures you will have a weekly tutorial session in one of the computing clusters. There you can work on problems from the course and get help and feed back if necessary. These sessions are based around the examples sets which are available on-line. Despite the fact that these sets can be accessed remotely *you are required to attend one session per week and an attendance list will be taken.* Information on how to register to use this software is available on the course web pages.

Important point: Doing the problem sets is the main way to learn maths, not watching the lecturer. In the tutorials there is the chance to get help with questions on the examples sets and feed back on your attempts to these and the two in-class tests. Also, if there are things in the lectures that you don't understand then you can ask about them in the tutorials.

The problem sets are accessed via the web. A session introducing the system and showing how to access and answer the problem sets will take place in the tutorial session of Week 1. There will be ten such sets. You attempt them by entering your answers into the computer interface; the system will tell you whether your answers are right or wrong, and will make some general comment if they are wrong. The system will also tell you how well you did on each set. You can attempt each set as many times as you like, but each must be attempted by a due date: Set 1 must be attempted by the end of week 3, set 2 by the end of week 4 and so on. 10% of the marks for the course are available from the computerized problem sets. You get the credit for making a serious attempt at the sets, 'serious attempt' currently being equated with scoring at least 60% (for example 5 out of 8). You may attempt the sets as often as you like in order to achieve this threshold and even if you get a question wrong the first time you may still be given the opportunity to try it again for half a mark. Once you reach the threshold you can still try the test again for further practice *and you will not lose the original credit you received for reaching the threshold.* Note that it is not the case that you get one mark for each test on which you score at least 60%. If you achieve that for all 10 tests then you will get 10 marks but if only for 9 then the mark you receive will go down to 7 and if only 8 then it will be reduced still further, to 4, and so on.

Calculators are allowed for these tests but will not be allowed (nor required, assuming you know your multiplication tables up to 12 times 12) in the examination and In Class Tests.

You will need to become familiar with the notation for inputting your answers. The following examples provide some useful pointers:

Formula	Input	Formula	Input
$2xy$	`2x*y`	x^3	`x^3`
$\frac{2}{3}$	`(2/3)`	$\frac{x+1}{x-1}$	`(x+1)/(x-1)`
$\log x$	`log(x)`	$\frac{2\ln x}{3}$	`(2/3)ln(x)`
$\cos x$	`cos(x)`	$\sqrt{2}$	`sqrt(2)`
π	`pi`	$\frac{3\pi}{4}$	`(3/4)*pi`

Formulae Tables: Generic formulae tables, covering many courses, not just this one, are available at
http://www.maths.manchester.ac.uk/service/images/formtabsV2.pdf
You will not be provided with, nor will you be allowed to use, formulae tables in the examination.

HELM As part of the HELM project the University has copies of .pdf files on many mathematical topics, some of which you may find helpful (and interesting even). They are accessible via
`www.maths.manchester.ac.uk/study/undergraduate/`
`information-for-current-students/service-teaching/resources/`

Manchester Mathematics Resource Centre: This provides a drop in help facility, for details see:
http://www.maths.manchester.ac.uk/service/resource-centre/

Assessment: Apart from the online tests (worth 10% of the final mark) there will be two in-class tests (worth, respectively 8% and 7% of the final mark) and an exam in January (worth 75% of the final mark).

The in-class tests will take place in the usual lecture room on Wednesday 23rd October and Wednesday 27th November and each will last 35 minutes. You may consult the Course Notes in these tests but the use of any electronic devices, including mobiles, is prohibited. If you are caught using any such device you will receive zero marks. It is important that you turn up punctually for these tests equipped with a pen or pencil. Do not cheat in these tests, if you are caught you will lose all the marks for the test and face disciplinary action. Sample in-class tests, with solutions, for you to practice on are available at the end of these notes.

The exam will take place in January and lasts 2 hours. Previous exam papers, with solutions and in some cases feedback, are available on the course and my web pages, again for you to practice on.

The rubric for the exam will read:

Answer SIX of the EIGHT questions
If more than SIX questions are attempted then credit
will be given for the FIRST SIX answers.

Electronic calculators are not permitted

The 'FIRST SIX' means the first six appearing in your answer book. You will not be given credit for any answers beyond this point. If you decide that you do not want a solution to be marked you should put a line through it. If you subsequently decide that you do want it to be marked you should write 'STET' beside it. If you then change your mind again . . . !!

Marking of the exam and in-class tests The exam will be marked as follows: If a part of a question is worth just 1 mark then you'll get 1 mark for a correct answer (obviously) and no marks for an incorrect answer. So showing your working will not get you any marks. If a question is worth more than 1 mark and you have given the correct answer (showing working or not) then you will get the full marks. However if your answer is incorrect and you have shown some working then you may still get some marks for that.

The exam will be worth 75% of the marks for the course and there will be a choice of 6 questions out of 8. Each question will be worth 10 marks altogether and the final total will be scaled by 5/4 to bring it up to the required 75%. *Formulae tables will not be provided, nor allowed, in the examination. Similarly calculators will not be permitted.*

In the case of the in-class tests each part is worth one mark and that's what you get for a correct answer, with zero otherwise. You will not receive any marks for your working but in the case of an incorrect answer you may receive some useful feedback on it.

A short overview of the course

This course is intended for people who have GCSE in Maths (or some broadly equivalent background), but not A-level. There may be people who do in fact have A-level, but the assumed background is GCSE. Naturally this will mean that for some people the material will be familiar, and for others it will not be.

Purpose of course: To cover some of the mathematics that you will need for your degree course.

Apart from forming part of your degree this course will be of lasting value to you for the following reasons (in increasing order of importance – in my opinion):

- The techniques you will learn will enable you to solve real world problems such as you might meet in your future career.
- Many of the concepts and nomenclatures that you will meet in your chosen discipline will necessarily be expressed in mathematical jargon and you will feel comfortable with it.
- By learning mathematics you acquire transferrable skills of analytic thinking and the ability to precisely abstract, formulate and solve problems not just in mathematics but over a wide range of real world contexts.

This course starts at a basic level: this is because we assume that you did no maths at A level, and therefore that you have been away from it for two years. So the introduction is gentle. The first semester has two main parts: a revision of algebra, followed by an introduction to the differential calculus. [You will meet the second part of the calculus, the integral calculus, in the follow on course 0C2/1C2.] Some of you may have met this before, but many of you (I hope most) will not have.

Calculus is the key topic in this first semester. This is a major, even *the* tool for solving problems in 'continuous mathematics' especially those involving the dynamics of moving objects, fluids, money, commodities, time, etc..

A short syllabus for the first semester is:

- Revision of numbers, arithmetic operations, brackets, algebraic manipulation [3].
- Powers, roots, the exponential and logarithmic functions [3].
- Solving simple equations, including quadratics [2].
- The standard functions, circular measure and the trigonometric functions [3].
- Useful identities involving trigonometric functions [2]
- Simple coordinate geometry, straight lines, circles, points of intersection, slopes [4].
- Differentiation, derivatives of the standard functions, products, quotients, and 'function of a function' [3].
- Maxima and minima, simple graph sketching [2].

There is no set book for the course. Any A-level book will cover the material of the first semester. The HELM notes on the webpage are probably as good as any. Note that the material in this course is covered at A-level in the 'Pure Core' modules C1–C4 (though in fact there is very little here from C4).

A final point. Maths has the reputation of being a hard subject. However it's actually very easy, once you've put in a little effort to understand a topic it seems so self evident that you can't forget it. It's like learning to swim or ride a bicycle, just show a little determination at first at the rest is easy, pleasurable even!

Algebra – a reminder

Types of number

The *Natural or Whole Numbers*, sometimes referred to as the 'God given' numbers,

$$1, 2, 3, 4, \ldots$$

denoted by N or $\mathbb{N}$.[1]

Extending $\mathbb{N}$ by adding zero and negative whole numbers gives the *Integers*

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

denoted by Z or $\mathbb{Z}$.

The *Fractions* or Rational Numbers such as

$$1/3, 13/6, -10/141, 7/21, \ldots$$

that is numbers of the form p/q where the *numerator* p is an integer and the *denominator* q is a natural number. Since we identify 3/1 with 3, $-10/1$ with -10 we see that the rationals in turn extend the integers. Note that we, for example, identify

$$1/2, 2/4, 3/6, 4/8, \ldots,$$

in other words cancelling out a common factor from the numerator and the denominator in a fraction does not change the number it names.

A rational number p/q is in its *simplest form* if there is no common factor in p and q which can be canceled out. So, for example, 2/5 is in it's simplest form but 6/15 is not because 3 can be canceled from both the numerator and denominator.

Next come the decimal or *Real* numbers such as 1.25, 0.375 etc. These extend the rationals since we can form the decimal version of a rational number p/q by 'dividing out'. So in particular $5/4 = 1.25$. Doing this with 1/3 yields $0.3333\ldots$ so in this case the decimal expansion goes on for ever repeating 3's. Similarly

$$-19/7 = -2.714285714285714285\ldots.$$

Indeed we could do the same thing with 5/4 by repeatedly dividing even when we reach remainder 0 to give the expansion

$$5/4 = 1.250000\ldots.$$

What we might notice about all these decimals of rational numbers is that they eventually keep just repeating blocks of digits (714285 in the case of -19/7). In fact it is a theorem (i.e. proven mathematical fact) that the rational numbers are *exactly* the ones which eventually repeat in this way.

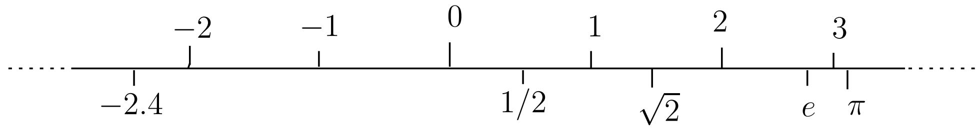
But clearly there are decimal numbers which do not repeat in this way, for example

$$0.101001000100001000001 00\ldots.$$

Such numbers being non-rational are called *Irrational Numbers*. Well known examples of irrational numbers are $\sqrt{2}$ and π.

Given that $\sqrt{2}$ and π are irrational can you easily write down any other irrational numbers?

A useful way of visualizing the real numbers is as a line or (one dimensional) continuum.



– sometimes called the *Real Line*. The decimal numbers are also called the *Real Numbers*, $\mathbb{R}$.

Arithmetic Operations

We can combine numbers to get new numbers by using the *Arithmetic Operations*

Addition/plus $\quad a + b$

Multiplication/times $\quad a \times b$ or $a \cdot b$ or ab.
Which we use depends on the context and improving readability. For example we would write 0.4×1.9 rather than $0.4 \cdot 1.9$ or $0.4\ 1.9$ and $x \cdot y$ or xy rather than $x \times y$.

Subtraction/minus $\quad a - b$

Division $\quad a \div b$ or a/b or $\dfrac{a}{b}$ $\quad$ (not defined if $b = 0$)

Useful facts

- Addition and multiplication are commutative, meaning that

$$a + b = b + a, \qquad ab = ba,$$

  whereas subtraction and division are not, for example

$$5 - 2 = 3 \neq -3 = 2 - 5 \qquad 5/7 \neq 7/5$$

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$$0 - b = -b \qquad a - b = a + (-b) \qquad -(-a) = a$$
$$a - (-b) = a + b \qquad (-a) + (-b) = (-a) - b = -(a + b)$$
$$-a = (-1)a \qquad (-a)b = a(-b) = -(ab) \qquad (-a)(-b) = ab$$
$$(-a)/b = a/(-b) = -(a/b) \qquad (-a)/(-b) = a/b$$

- For adding, multiplying etc. fractions;[2]

$$\frac{p}{q} + \frac{n}{m} = \frac{pm + qn}{qm} \quad \frac{p}{q} - \frac{n}{m} = \frac{pm - qn}{qm} \quad \frac{p}{q} \times \frac{n}{m} = \frac{pn}{qm} \quad \frac{p}{q} \div \frac{n}{m} = \frac{pm}{qn}$$

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[1] In some areas of mathematics 0 counts also as a natural number, in other areas it does not. For this course $\mathbb{N}$ will not include 0.

[2] Some students on this course are weak at adding and multiplying fractions. If you think you may be one of them check out that you get the same answers (in simplest form) as me in the following examples: $\frac{3}{4} + \frac{5}{6} = \frac{19}{12}$, $\frac{6}{5} + \frac{2}{7} = \frac{52}{35}$, $\frac{3}{4} - \frac{5}{6} = \frac{-1}{12}$, $\frac{3}{4} \times \frac{5}{6} = \frac{5}{8}$

- Notice that if p is an integer and q a natural number then $p \div q$, alternatively p/q or $\dfrac{p}{q}$, is the same number as the fraction, or rational, $\dfrac{p}{q}$. So the apparent double meaning of this notation is not a problem.

Arithmetic Expressions and Algebraic Expressions

An *arithmetic expression* is a string of numbers with operations that specify a calculation.

E.g.: $2 + 10, \quad 100 \times 4/7, \quad 25 - 8 \times 4 + 6 \div 3$

(Not all such strings are valid: e.g. $2 \times \div 3$, $1 + 14 -$ are not valid.)

What does the expression $7 + 2 \times 4$ mean?
Should we:

1. Add first, then multiply? $7 + 2 \times 4 = 9 \times 4 = 36$
2. Multiply first, then add? $7 + 2 \times 4 = 7 + 8 = 15$

BDMAS (Brackets, Division, Multiplication, Addition, Subtraction) is a convention for resolving this ambiguity. It consists of two parts

Brackets

When brackets are included in an expression it means that the operations inside the brackets are to be performed first, starting with the innermost brackets and working out.

E.g. $7 + (2 \times 4)$ means multiply first: $7 + (2 \times 4) = 15$
$(7 + 2) \times 4$ means add first: $(7 + 2) \times 4 = 36$

We can use more than one set of brackets:

$$\begin{array}{ccc} (2 + \frac{1}{3}) & \times & (10 + 5) \\ \downarrow & & \downarrow \\ \frac{7}{3} & \times & 15 = 35 \end{array}$$

We can put brackets inside brackets, e.g. in $5 + ((3 + \frac{4}{7}) \times 2)$. When brackets are *nested* in this way, we work out the inside bracket first.

$$\begin{array}{lcl} 5+ & ((3 + \frac{4}{7}) & \times 2) \\ & \downarrow & \\ 5+ & (\frac{25}{7} & \times 2) \\ & \downarrow & \\ 5+ & \frac{50}{7} & = \frac{85}{7} \end{array}$$

Note: Another use of brackets in arithmetic expressions is to separate the minus sign in front of a negative number from the arithmetic operations: e.g. $5 - (-3)$. This use is to increase readability (as already used above).

Precedence

The other way to make $7 + 2 \times 4$ clear is to have a convention for *precedence* of operations: i.e. an agreement about which operations to perform first.

We use the convention that $\times$ *and* $\div$ *are done before* $+$ *and* $-$. So then $7 + 2 \times 4 = 7 + 8 = 15$.

Example:

$$\begin{aligned} & 25 - 8 \times 4 + 6 \div 3 \\ = \quad & 25 - 32 + 2 = -5 \end{aligned}$$

$+$ and $-$ have the same precedence. We do these in order working from the left. E.g.

$$3 - 2 + 1 = 1 + 1 = 2$$

Note this is different to $3 - (2 + 1) = 3 - 3 = 0$.

BDMAS also asserts this same situation for $\times$ and $\div$. However we shall not adopt this convention since it is frequently in conflict with common usage. For example in this course an expression like uv/w stands for $(u \times v)/w$ and u/vw stands for $u/(v \times w)$.

Note: Brackets overrule operator precedence: $(7 + 2) \times 4$ means add first.

An expression in which symbols/variables stand for some (or all) of the numbers is an *algebraic expression*.

Brackets and precedence can be used in algebraic expressions and the same BDMAS conventions apply.

E.g. if a, b and c are variables standing for numbers:

$a(b + c)$	means first find $b + c$ then multiply the result by a.
$ab + c$	means first find $a \times b$ then add c to the result.

Multiplying out brackets

To remove brackets for an algebraic expression we use the *Distributive Law*:

$$x(y + z) = xy + xz = yx + zx = (y + z)x$$

and similarly

$$x(y + z + t) = xy + xz + xt \qquad \text{etc.etc.}$$

To eliminate brackets from

$$(a + b)(c + d)$$

set $x = c + d$, to get

$$\begin{aligned} (a + b)(c + d) &= (a + b)x \\ &= ax + bx \quad \text{by Distributive Law} \\ &= a(c + d) + b(c + d) \quad \text{replacing } x \\ &= ac + ad + bc + bd \quad \text{using the Distributive Law again} \end{aligned}$$

So we get

$$(a+b)(c+d) = ac + ad + bc + bd$$

Similarly using the Distributive Law;

$$x(y-z) = x(y+(-z)) = xy + x(-z) = xy - xz$$

$$x-(y+z) = x + (-1)(y+z) = x + (-1)y + (-1)z = x - y - z$$

$$x - 3(y-z) = x + (-3)(y + (-1)z) = x + (-3)y + (-3)(-1)z = x - 3y + 3z$$

etc., etc..

Some more examples;

$$\begin{aligned}(2x+3)(x-1) &= (2x+3)(x+(-1)) \\ &= 2x \cdot x + 2x(-1) + 3x + 3(-1) \\ &= 2x^2 - 2x + 3x - 3 \\ &= 2x^2 + x - 3 \quad \text{by } \textit{collecting terms}\end{aligned}$$

$$\begin{aligned}(3a-1)(b-3) &= 3ab - 9a - b + (-1)(-3) \\ &= 3ab - 9a - b + 3\end{aligned}$$

In an algebraic expression we can identify a particular term by the letters involved: e.g. in the expression just above

$3ab$ is 'the term in ab'

$-9a$ is 'the term in a'

$-b$ is 'the term in b'

3 is 'the *constant* term'

The number in front of the letters in a term is called the *coefficient*. E.g. in the term $-9a$ the coefficient (of a) is -9. If there is no number written explicitly, the coefficient is 1 or -1 according to the sign of the term. E.g. the coefficient of b above is -1. If no constant term is mentioned explicitly then it is 0. E.g. the constant term in $x^2 - 2x$ is 0.

(An aside: for any that don't know it:

$a > b$ means 'a is greater than b' (i.e. a is strictly to the right of b on the real line)

$a < b$ means 'a is less than b'

$a \geq b$ means 'a is greater than or equal to b'

$a \leq b$ means 'a is less than or equal to b')

Powers and Roots

Definition: If a is a real number, and n is a natural number, we write a^n for

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ times}}$$

In a^n, a is called the *base* and n is called the *power* or *index* or *exponent.*

E.g.

1. $2^4 = 2 \times 2 \times 2 \times 2 = 16$
2. $(1.3)^3 = (1.3) \times (1.3) \times (1.3) = 2.197$

Note: We usually read a^2 as 'a squared' rather than 'a to the power 2', and a^3 as 'a cubed' rather than 'a to the power 3'.

Note: In arithmetic or algebraic expressions, powers take precedence over $+$, $-$, $\times$, $\div$. E.g. in $3^2/12$ we compute 3^2 first: $3^2/12 = 9/12 = 3/4$.

Rules for Powers

We can simplify an expression like $a^n a^m$ because

$$\begin{aligned} a^n \times a^m &= \underbrace{a \times a \times \ldots \times a}_{n \text{ times}} \times \underbrace{a \times \ldots \times a}_{m \text{ times}} \\ &= a^{n+m} \end{aligned}$$

Also, for natural numbers n, m, $n > m$,

$$\begin{aligned} a^n/a^m &= \frac{\overbrace{a \times a \times \ldots \times a}^{n \text{ times}}}{\underbrace{a \times \ldots \times a}_{m \text{ times}}} \\ &= \underbrace{a \times \ldots \times a}_{n-m \text{ times}} \\ &= a^{n-m} \end{aligned}$$

$$\begin{aligned} (a^n)^m &= \underbrace{a^n \times a^n \ldots \times a^n}_{m \text{ times}} \\ &= \underbrace{a \times \ldots \times a}_{mn \text{ times}} \\ &= a^{nm} \end{aligned}$$

$$
\begin{aligned}
(ab)^n &= \underbrace{ab \times ab \ldots \times ab}_{n \text{ times}} \\
&= \underbrace{a \times \ldots \times a}_{n \text{ times}} \times \underbrace{b \times \ldots \times b}_{n \text{ times}} \\
&= a^n b^n
\end{aligned}
$$

So we have the following

Rules for powers:

P1 $a^1 = a$

P2 $a^n a^m = a^{n+m}$

P3 $a^n / a^m = a^{n-m}$

P4 $(a^n)^m = a^{nm}$

P5 $(ab)^n = a^n b^n$

where a is a real number, n, m are natural numbers and $n > m$ in P3.
Examples:

1. $3^2 3^3 = 3^5$ (i.e. $9 \times 27 = 243$)
2. $2^5/2^3 = 2^2$ (i.e. $32/8 = 4$)
3. $(5^2)^3 = 5^6 = (5^3)^2$ (i.e. $25^3 = 15,625$)

Integer and Rational Exponents

We extend the notation a^n to integer n (so now possibly zero or negative whole numbers) by setting:

$$a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}.$$

E.g. $2^{-4} = 1/2^4 = 1/16$
$3^4 \times 3^{-5} = 3^{4-5} = 3^{-1} = 1/3$
$(3^{-2})^2 = 3^{-4} = 1/3^4 = 1/81$

Note that 0^{-n} is not defined when $n > 0$.

For n a natural number we set $a^{1/n}$ to be that number b (if it exists) such that $b^n = a$.

So $a^{1/3}$ is that number which when multiplied by itself 3 times gives a usually called the *cube root* of a and alternatively denoted $\sqrt[3]{a}$.

E.g. Since $4^3 = 64$, $(64)^{1/3} = 4$.

More generally for a a real number and n a *natural number* $a^{1/n}$, alternatively written $\sqrt[n]{a}$ (or just $\sqrt{a}$ if $n = 2$) and called an n'th root of a, or a *square root* of a if $n = 2$.

Example:

1. $8^{1/3} = \sqrt[3]{8} = 2$ since $2^3 = 8$; (2 is the third, or cube, root of 8).
2. $243^{1/5} = \sqrt[5]{243} = 3$ since $3^5 = 243$
3. $(9.261)^{1/3} = \sqrt[3]{9.261} = 2.1$ since $2.1^3 = 9.261$

Note:

1. If n is odd there is one n-th root. E.g. $\sqrt[3]{-27} = -3$. So in this case the notation $a^{1/n}$ is unambiguous.
2. If n is even there are two possibilities:
   (a) If a is positive there are two n-th roots of a (in the reals), one the negative of the other. In this case $a^{1/n}$, alternatively $\sqrt[n]{a}$, stands for the positive root; the other is $-a^{1/n} = -\sqrt[n]{a}$.
   E.g. 625 has two fourth roots: $\sqrt[4]{625} = 5$ and $-\sqrt[4]{625} = -5$.
   (b) If a is negative there are no n-th roots (in the reals).
   E.g. -4 has no square root. (Both 2^2 and $(-2)^2$ give 4.)

For a rational number m/n (so n is a natural number, m and integer), we set

$$a^{m/n} = (a^{1/n})^m.$$

Examples:

1. $125^{2/3} = (125^{1/3})^2 = (\sqrt[3]{125})^2 = 5^2 = 25$
2. $9.261^{2/3} = (2.1)^2 = 4.41$
3. $32^{-2/5} = (32^{1/5})^{-2} = 2^{-2} = 1/2^2 = 1/4$

Subject to the above provisos P1-5 hold for a, b a real numbers and r, s rational numbers (and even for real numbers when $a, b \geq 0$).

P1 $a^1 = a, \quad a^0 = 1$

P2 $a^r a^s = a^{r+s}$

P3 $a^r/a^s = a^{r-s}, \quad a^{-s} = 1/a^s$

P4 $(a^r)^s = a^{rs}$

P5 $(ab)^r = a^r b^r, \quad (a/b)^r = a^r/b^r.$

Examples:

1. $\sqrt{2}\sqrt{3} = 2^{1/2} \times 3^{1/2} = (2 \times 3)^{1/2} = \sqrt{6}$
2. $2^6 = 64$ so $2 = 64^{1/6} = \sqrt[6]{64}$ and $64^{5/6} = (64^{1/6})^5 = 2^5 = 32$.
3. Express $(x^{1/2}y)^3$ in the form $x^m y^n$:

$$(x^{1/2}y)^3 = (x^{1/2})^3(y)^3 = x^{3/2}y^3$$

4. Express $\frac{(ab^2)^{1/2}a^3}{b^{1/2}}$ in the form $a^m b^n$:

$$\frac{(ab^2)^{1/2}a^3}{b^{1/2}} = \frac{a^{1/2}ba^3}{b^{1/2}} = \frac{ba^{1/2}a^3}{b^{1/2}} = b^{1-1/2}a^{3+1/2} = b^{1/2}a^{7/2}$$

Precedence for taking powers goes in order *brackets before powers before multiplication and division.* So for example $(2+3)^2 = 5^2$ and NOT $2^2 + 3^2 = 14$, ab^2 is $a \times (b^2)$ NOT $(ab)^2$, a^2/b is $(a^2)/b$ NOT $(a/b)^2$.

Logarithms

Fact Given real numbers $a, b > 0$, $a \neq 1$, there is a real number x such that $a^x = b$. We call x the logarithm (log for short) of b to the base a, written $x = \log_a b$. So[3]

[L1] $a^{\log_a b} = b$. In other words

$$a^x = b \iff x = \log_a b$$

($\iff$ is shorthand for 'if and only if')

We now derive a number of other properties of logs from [L1].

Since $a^1 = a$ and $a^0 = 1$,

[L2] $\log_a a = 1$ and $\log_a 1 = 0$.

Since $a^x = a^x$,

[3] If any confusion might arise as to what $\log_a$ applies to we insert brackets, e.g. write $\log_a(x+y)$ rather than $\log_a x + y$.

[L3] $\log_a(a^x) = x$.

Since $a^{\log_a bc} = bc = b \times c = a^{\log_a b} \times a^{\log_a c} = a^{\log_a b + \log_a c}$,

[L4] $\log_a bc = \log_a b + \log_a c$

Since $a^{\log_a (b/c)} = b/c = a^{\log_a b}/a^{\log_a c} = a^{\log_a b - \log_a c}$,

[L5] $\log_a(b/c) = \log_a b - \log_a c$.

Also $b^y = (a^{\log_a b})^y = a^{y \log_a b}$ so $\log_a b^y = y \log_a b$,

[L6] $\log_a b^y = y \log_a b$.

Notice that in each of these laws the logarithms must be to the same base.

Examples:

1. $\log_2(2\sqrt{2}) = \log_2 2 + \log_2 \sqrt{2} = 1 + 1/2 = 3/2$
2. $\log_6 2 + \log_6 3 = \log_6(2 \times 3) = \log_6 6 = 1$
3. Express $\log_a uv^3\sqrt{w}$ in terms of $\log_a u$, $\log_a v$ and $\log_a w$:

$$\begin{aligned} \log_a uv^3\sqrt{w} &= \log_a u + \log_a v^3 + \log_a \sqrt{w} \\ &= \log_a u + 3\log_a v + \frac{1}{2}\log_a w \end{aligned}$$

4. Express $\log_a \sqrt{x/y}$ in terms of $\log_a x$ and $\log_a y$:

$$\log_a \sqrt{x/y} = \log_a(x/y)^{1/2} = (1/2)\log_a(x/y) = \frac{1}{2}\log_a x - \frac{1}{2}\log_a y$$

One use of logs that you may have met is in scaling graphs (for example of acidity, i.e. pH). To take an example a single greenfly produces 9 offspring per day. so if on day 1 we had 1 greenfly by day 2 we'd have 10, on day 3 100 ($= 10^2$), on day 4 1000 ($= 10^3$) and so on. A graph of the number N of greenfly against the number of days would be pretty useless since it would hover microscopically close to the x-axis for a while and then suddenly shoot straight up off the page. On the other hand if we plot instead $\log_{10} N$ against days we get a nice straight line graph since $\log_{10} 1 = 0$, $\log_{10} 10 = 1$, $\log_{10}(100) = 2$, $\log_{10}(1000) = 3 \ldots$.

Change of Base

If my calculator only produces logs to the base 10 how can I find $\log_3(5.2)$?

Suppose $x = \log_b c$. Then $b^x = c$ so $x \log_a b = \log_a b^x = \log_a c$. Hence $x = \dfrac{\log_a c}{\log_a b}$, giving

[L7] $\log_b c = \dfrac{\log_a c}{\log_a b}$

So

$$\log_3 5.2 = \frac{\log_{10} 5.2}{\log_{10} 3} = \frac{0.7160\ldots}{0.4771\ldots} = 1.501 \quad \text{to 3 d.p.}$$

Examples:

1. $\log_8 64 = 2$ and $\log_8 2 = 1/3$ so

$$\log_2 64 = \frac{\log_8 64}{\log_8 2} = \frac{2}{(1/3)} = 6$$

Although we have talked about using any number $a > 0, a \neq 1$, for the base of logarithms, the bases that are actually used in practice are restricted to a few numbers. Generally

base 2 is used by computer scientists

base 10 is used by engineers and technologists. If *you* see log without the base given explicitly assume the base is 10.

base e is used by mathematicians ($e = 2.71828\ldots$ is an irrational number). We sometimes write ln for $\log_e$ (called 'natural logarithms').

<u>Quiz Question Example</u> The number $N(t)$ of bacteria in a colony at time t is related to the number N_0 at time 0 by the equation $N(t) = N_0 a^t$ for some $a > 1$. If the colony size trebles every 5 hours what must a be? In that case how long is it before the colony has increased tenfold?

<u>Answer</u> When $t = 5$ $N(t) = 3N_0$ so

$$3N_0 = N(5) = N_0 a^5.$$

$$\therefore \quad 3 = a^5, \quad \text{so} \quad a = 3^{1/5}.$$

[$\therefore$ means 'therefore'.] If the colony has increased tenfold by time t then

$$10N_0 = N(t) = N_0 a^t = N_0 (3^{1/5})^t = N_0 3^{t/5}.$$

Cancelling the N_0 we obtain $10 = 3^{t/5}$. Taking logs to base 10 yields

$$1 = \log_{10}(3^{t/5}) = \frac{t}{5}\log_{10}(3), \quad \text{so} \quad t = \frac{5}{\log_{10}(3)}.$$

Equations and their Solution

Most scientific 'Laws' are expressed as equations involving mathematical expressions, often simply algebraic expressions, which related quantities such as temperature, length, money, mass, time.
Examples:

$$\begin{aligned} a^2 &= b^2 + c^2 && \text{Pythagoras's Theorem} \\ A &= 4\pi r^2 && \text{Surface area of a sphere} \\ PV &= nRT && \text{Ideal Gas Equation} \end{aligned}$$

We can use such laws to *solve* for unknown quantities in terms of quantities that we do know. For example knowing the lengths b, c of the other two sides of a right angle triangle we can solve for the length a of the hypotenuse (i.e. the longest side) using Pythagoras's Theorem, i.e.

$$a = \sqrt{b^2 + c^2}.$$

Here we shall look at solving simple equations in general, not just ones derived from 'Laws'. Usually we use x for the unknown quantity we wish to solve for. The equation is then called an *equation in x* (or 'for x').

E.g.

$$\begin{array}{lrcl} \text{(i)} & 2x + 3 &=& 7 \\ \text{(ii)} & x^2 &=& 2x + 3 \\ \text{(iii)} & 2^x &=& \sqrt{x} \end{array}$$

Finding the values of x which make the equation true/balance is *solving the equation* (for x).

Equations may have one solution (e.g. (i)), no solutions (e.g.(iii)), or several solutions ((ii) has two).

Simple equations (in x say) can often be solved by *Rearranging*, that is applying the same operations (adding something, squaring etc.) to both sides of the equation until x is isolated by itself on one side.

E.g.

$$\begin{array}{rrcll} \text{If} & 2x + 3 &=& 7 & \\ \text{then} & 2x + 3 - 3 &=& 7 - 3 & \text{(subtracting 3)} \\ \text{so} & 2x &=& 4 & \\ \text{and} & x &=& 2 & \text{dividing by 2} \end{array}$$

E.g. Solve $\sqrt{7x^2 - 4} = 2x$. We have

$$
\begin{aligned}
\sqrt{7x^2-4} &= 2x \\
\therefore \quad 7x^2-4 &= (2x)^2 = 4x^2 && \text{squaring both sides} \\
\therefore \quad 7x^2-4x^2 &= 4 && \text{add } 4-4x^2 \text{ to each side} \\
\therefore \quad 3x^2 &= 4 \\
\therefore \quad x^2 &= \tfrac{4}{3} \\
\therefore \quad x &= \sqrt{4/3} \quad \text{or} \quad x = -\sqrt{4/3}
\end{aligned}
$$

Note that at this point we have shown the *if* x is a solution then $x = \sqrt{4/3}$ or $x = -\sqrt{4/3}$. However we have not shown that these actually are solutions. In fact $x = -\sqrt{4/3}$ is not a solution because by our convention that $\sqrt{a}$ means the positive square root,

$$\sqrt{7\left(-\sqrt{(4/3)}\right)^2 - 4} > 0 > 2\left(-\sqrt{(4/3)}\right).$$

It is easy to check that $x = \sqrt{(4/3)}$ is a solution.

When solving equations then one should always check that the answers one has derived actually are solutions to the original equation (and in any case this is a good way of checking that you haven't make a mistake in that derivation). Usually this is clear because the manipulations one does on the equation can all be reversed. E.g. in the example earlier where we went from $2x + 3 = 7$ to $x = 2$ by subtracting 3 from both sides and then dividing both sides by 2 we can equally go back from $x = 2$ to $2x + 3 = 7$ by multiplying both sides by 2 and then adding 3 to both sides.

In the example immediately above however we made a step which does not reverse, we went from

$$\sqrt{7x^2-4} = 2x$$

to

$$7x^2 - 4 = 4x^2$$

by squaring both sides. We cannot go back the other way because the most we can conclude from this last equation is that

$$\text{Either } \sqrt{7x^2-4} = 2x \text{ or } \sqrt{7x^2-4} = -2x.$$

Polynomial Equations

A polynomial equation (in x, of degree n) is one of the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

(where a_n, …, a_0 are real number constants, $a_n \neq 0$). E.g.

$$2x^4 + 5x^3 + 3x^2 + x + 7 = 0$$

Note that we normally omit the terms in a polynomial with coefficient zero, omit the coefficient if it is 1, and replace a negative coefficient as in $+(-a_n)x^n$ by $-a_nx^n$ e.g. write

$$x^5 - 4x^4 + 3x^2$$

rather than

$$1x^5 + (-4)x^4 + 0x^3 + 3x^2 + 0x + 0.$$

A polynomial equation is called *linear* if $n = 1$, *quadratic* if $n = 2$ and *cubic* if $n = 3$. Linear equations are immediately solved by rearrangement:[4]

$$ax + b = 0 \iff ax = -b \iff x = -\frac{b}{a}$$

The solutions to a polynomial equation are called the *roots* of the polynomial.

Sometimes a polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ can be expressed ("factorized") as the product of linear polynomials, called *factors*, and (maybe) a constant.

Examples:

$$\begin{aligned} 2x^2 - 3x - 2 &= (2x+1)(x-2) \\ 2x^3 - 6x^2 - 2x + 6 &= 2(x+1)(x-1)(x-3) \end{aligned}$$

So $x - 2$ is a factor of $2x^2 - 3x - 2$ etc..

Factorizing enables us to solve a polynomial equation since the polynomial can be zero just if one of the linear factors is zero, and being linear we can easily solve these. E.g.

$$\begin{aligned} 2x^3 - 6x^2 - 2x + 6 = 0 &\iff 2(x+1)(x-1)(x-3) = 0 \\ &\iff x + 1 = 0 \text{ or } x - 1 = 0 \text{ or } x - 3 = 0 \\ &\iff x = -1 \text{ or } x = 1 \text{ or } x = 3 \end{aligned}$$

so the solutions to

$$2x^3 - 6x^2 - 2x + 6 = 0$$

(equivalently roots of $2x^3 - 6x^2 - 2x + 6$) are $-1, 1, 3$.

Factorizing quadratics (when possible) is easy: For $a \neq 0$ and $b^2 - 4ac \geq 0$ (so $\sqrt{b^2 - 4ac}$ is a real number)

$$ax^2 + bx + c = a\left(x + \left(\frac{b - \sqrt{b^2 - 4ac}}{2a}\right)\right)\left(x + \left(\frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\right)$$

[4]The symbol $\iff$ stands for 'equivalently' or 'just if' or 'if and only if (iff)'.

since

$$\begin{aligned}
\text{RHS} &= ax^2 + ax\left(\frac{b-\sqrt{b^2-4ac}}{2a}\right) + ax\left(\frac{b+\sqrt{b^2-4ac}}{2a}\right) \\
&\quad + a\left(\frac{b-\sqrt{b^2-4ac}}{2a}\right)\left(\frac{b+\sqrt{b^2-4ac}}{2a}\right) \\
&= ax^2 + 2ax(b/2a) + a\left(\frac{b^2+b\sqrt{b^2-4ac}-b\sqrt{b^2-4ac}-(\sqrt{b^2-4ac})^2}{4a^2}\right) \\
&= ax^2 + bx + \frac{b^2-(b^2-4ac)}{4a} \\
&= ax^2 + bx + c.
\end{aligned}$$

I.e. the factors of ax^2+bx+c are

$$x+\left(\frac{b-\sqrt{b^2-4ac}}{2a}\right) \quad \text{and} \quad x+\left(\frac{b+\sqrt{b^2-4ac}}{2a}\right)$$

and the solutions of

$$ax^2+bx+c=0$$

are

$$-\left(\frac{b-\sqrt{b^2-4ac}}{2a}\right) \quad \text{and} \quad -\left(\frac{b+\sqrt{b^2-4ac}}{2a}\right),$$

i.e.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \frac{-b-\sqrt{b^2-4ac}}{2a}, \quad \text{or} \quad \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \text{for short.}$$

Example. E.g. Solve $x^2+x-6=0$

Here $a=1$, $b=1$, $c=-6$, so $b^2-4ac=1^2-4(1)(-6)=25\geq 0$ and solutions are

$$x=\frac{-1\pm\sqrt{1^2-4(1)(-6)}}{2(1)}=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm\sqrt{25}}{2}=\frac{-1\pm 5}{2}$$

So $x=\frac{-1+5}{2}=2$ or $x=\frac{-1-5}{2}=-3$.

E.g. Solve $2x^2-7x-4=0$. $a=2$, $b=-7$, $c=-4$ so $b^2-4ac=(-7)^2-4(2)(-4)=81$ and

$$x=\frac{7\pm\sqrt{81}}{2(2)}=\frac{7\pm 9}{4}$$

so $x=\frac{7+9}{4}=4$ or $x=\frac{7-9}{4}=-1/2$.

E.g. Solve $x^2-3x+1=0$. Here $b^2-4ac=(-3)^2-4(1)(1)=5$ and

$$x=\frac{3\pm\sqrt{5}}{2(1)}$$

i.e. $x = \frac{3+\sqrt{5}}{2}$ or $x = \frac{3-\sqrt{5}}{2}$
(Since in this course we are mathematicians you should leave it like this with the $\sqrt{}$ signs, that is the correct answer. If you get out your calculators you'll end up with an answer which is only correct to a certain number of decimal places, though that could be good enough for engineers who only need a fair approximation to work with in practice.)

The above method of solving quadratics is foolproof. However it is sometimes quicker to make a guess and check it works. Notice that if

$$x^2 + x - 6 = (x + a)(x + b) = x^2 + x(a + b) + ab$$

then a, b must be such that $a + b = 1$ and $ab = -6$. If I can guess such a, b I will have my factorization. Clearly I can so guess, take $a = 3$, $b = -2$, so

$$x^2 + x - 6 = (x + 3)(x - 2)$$

and the roots of $x^2 + x - 6$ are -3 and 2.

Sometimes an equation can be reduced to quadratic form:

E.g. to solve $\dfrac{1}{x} + \dfrac{1}{x+1} = \dfrac{1}{x+2}$ notice that

$$\frac{1}{x} + \frac{1}{x+1} = \frac{1}{x+2} \iff x(x+1)(x+2)\left(\frac{1}{x} + \frac{1}{x+1}\right) = x(x+1)(x+2)\left(\frac{1}{x+2}\right)$$

(multiplying each side by $x(x+1)(x+2)$)

$$\iff \frac{x(x+1)(x+2)}{x} + \frac{x(x+1)(x+2)}{x+1} = \frac{x(x+1)(x+2)}{x+2}$$

$$\iff (x+1)(x+2) + x(x+2) = x(x+1)$$

(canceling out top and bottom)

$$\iff x^2 + x + 2x + 2 + x^2 + 2x = x^2 + x$$

(multiplying out brackets)

$$\iff x^2 + 4x + 2 = 0$$

$$\iff x = \frac{-4 \pm \sqrt{8}}{2}$$

(Using the quadratic formula)

$$\iff x = -2 \pm \sqrt{2}$$

since $\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$.

[Note that to be a solution $x, x+1, x+2$ cannot be zero so $x(x+1)(x+2) \neq 0$ and it is OK to multiply both sides on line 1 by $x(x+1)(x+2)$.]

E.g. Solve $4^x + 12 = 7(2^x)$.

Note that $4 = 2^2$ so that $4^x = (2^2)^x = 2^{2x} = (2^x)^2$. Put $y = 2^x$, so

$$\begin{aligned}(2^x)^2 + 12 = 7(2^x) &\iff y^2 + 12 = 7y \\ &\iff y^2 - 7y + 12 = 0 \\ &\iff (y-4)(y-3) = 0 \text{ (by the quadratic formula, or `guessing')}\end{aligned}$$

so $y = 4$ or $y = 3$, i.e. $2^x = 4$ or $2^x = 3$, equivalently $x = \log_2 4 = 2$ or $x = \log_2 3$.

Functions

Definition: A *function* f (on the real numbers) is a rule or process (or recipe) which on input (or argument) a number x produces another number (the output or value), denoted $f(x)$ (and read 'f of x').

E.g. f might be the rule which says 'square the input x'. So in this case on input 5.6 f outputs $(5.6)^2$, i.e. $f(5.6) = (5.6)^2 = 31.36$ and in general $f(x) = x^2$.

f might be the rule 'take the $\log_{10}$ of the input and add 3', so $f(100) = \log_{10}(100)+3 = 2 + 3 = 5$ and generally $f(x) = \log_{10}(x) + 3$.

Functions need not be defined by formulas (as above) For e.g. the 'rounding' function is specified by

$$f(x) = x \text{ rounded down to the nearest integer}$$

so that $f(27.41) = 27$, $f(-105.82) = -106$ etc.

The important thing is that the *process* that produces the new number is well defined.

We commonly use $x, t, u, v, \ldots$ to stand for the input and $f, g, h \ldots$ for functions. We may also give the output number a symbol, often y, writing $y = f(x)$.

Domain of a function

For some functions not every number is allowed as an input. E.g. For the function $f(x) = \frac{1}{x}$ the input $x = 0$ is not allowed (it gives 1/0). Any other number is allowed.

For the function $f(x) = \log_{10}(x) + 3$ only strictly positive values of x are allowed.

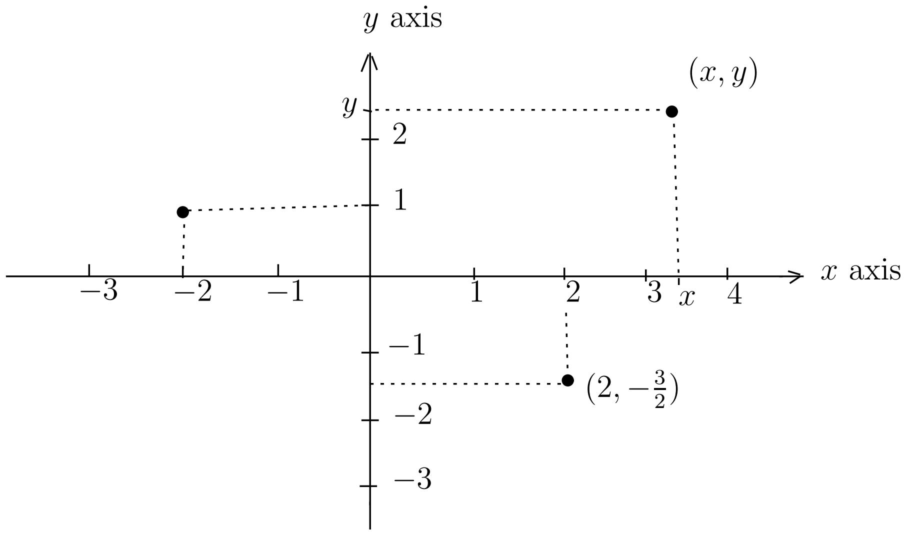
Definition: The collection of numbers that are allowed as input to a function f is called the *domain* of the function f.

So the domain of $f(x) = \frac{1}{x}$ is all the real numbers except 0.

E.g. The domain of the function $f(x) = \sqrt{x}$ is all the numbers greater or equal 0.

Graphs of functions

Recall that given a coordinate system

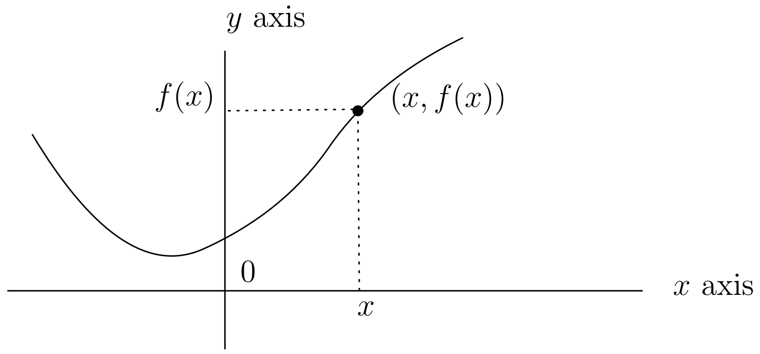


every point P on the plane can be assigned a pair of numbers (x, y) called the *coordinates* of P. [The point $(0, 0)$ is referred to as the *origin*.]

Given a function f the collection of pairs $(x, f(x))$, where x is in the domain of f, is called the *graph* of f.

Plotting the graph of f means marking these points in a coordinate system as above.

Knowing the graph of a function f allows us to reconstruct the rule/function f, or at least an equivalent rule in that it always gives the same output. Namely, given input x output that $y(= f(x))$ such that the point (x, y) is on the graph.

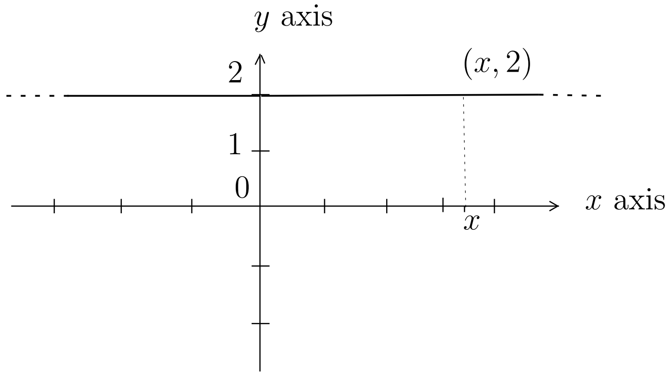


Examples

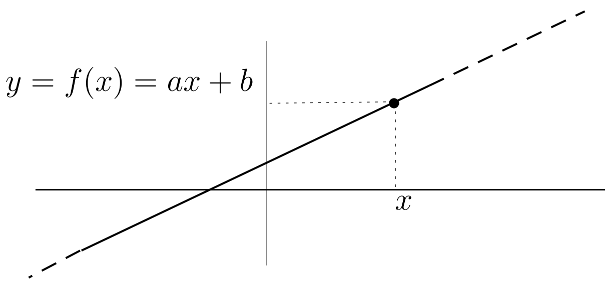
1. **Constant Functions**: $f(x) = b$, where b is a constant.

   Clearly the domain of a constant function is all the reals (and similarly for the linear, quadratic and polynomial functions below).

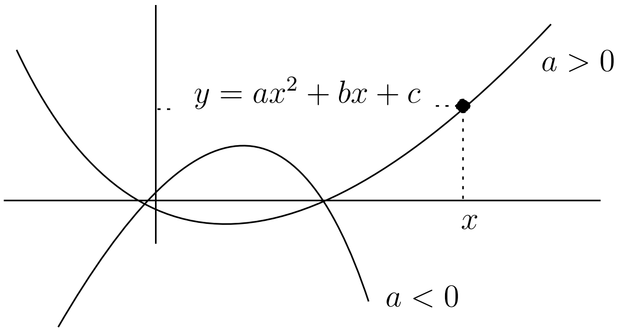
   E.g. $f(x) = 2$.



2. **Linear Functions**: $f(x) = ax + b$ (a and b are constants, $a \neq 0$).
   E.g. $f(x) = 7x - 2$



3. **Quadratic Functions**: $f(x) = ax^2 + bx + c$ (a, b and c are constants, $a \neq 0$).
   E.g. $f(x) = 2x^2 + 3x + 2$



4. **Polynomial Functions**: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ (a_n, a_{n-1} etc. are constants, $a_n \neq 0$ or $n = 0$.)

   Recall that constant a_j in front of x^j is called the *coefficient* of x^j and $a_j x^j$ is the *term* in x^j. (Note, some or all of the a_j's can be 0.) The number n (the highest power of x) is called the *degree* of the polynomial. A special case of this is a constant polynomial a_0 which has degree 0.

   Polynomial functions can be added, subtracted and multiplied to give new polynomial functions in the obvious way.

   E.g. Say $P(x) = x^2 + 4x - 1$ and $Q(x) = 2x^2 - 3$. Then

$$P(x) + Q(x) = (x^2 + 4x - 1) + (2x^2 - 3) = 3x^2 + 4x - 4$$

   and

$$\begin{aligned} P(x)Q(x) &= (x^2 + 4x - 1)(2x^2 - 3) = 2x^4 - 3x^2 + 8x^3 - 12x - 2x^2 + 3 \\ &= 2x^4 + 8x^3 - 5x^2 - 12x + 3 \end{aligned}$$

   Notice that the degree of $P(x) \times Q(x)$ is the *sum* of the degrees of $P(x)$ and $Q(x)$.

5. **Rational Functions**:

Rational functions are functions f of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

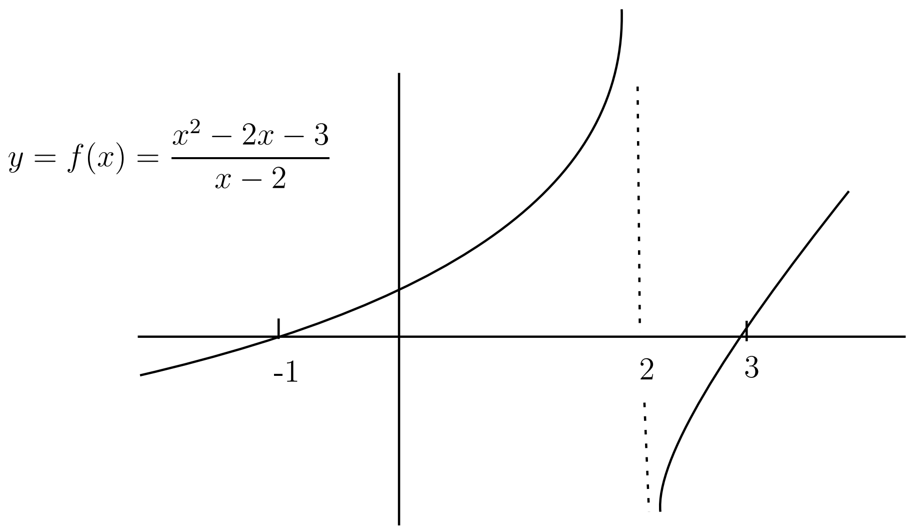
where $P(x), Q(x)$ are polynomials. The domain of such a function is the collection of numbers x such that $Q(x) \neq 0$.

E.g.

$$f(x) = \frac{x^2 - 2x - 3}{x - 2}$$

is a rational function with domain all numbers x except $x = 2$.

(Note, polynomials are the special case of rational functions when the denominator is 1.)



The domain of f here is all the real numbers except 2 ($f(2)$ is not defined).

We can add (and subtract) rational functions by the usual procedure of adding fractions. If $R_1(x) = \frac{P_1(x)}{Q_1(x)}$ and $R_2(x) = \frac{P_2(x)}{Q_2(x)}$ (where $P_1(x)$, $Q_1(x)$, $P_2(x)$ and $Q_2(x)$ are polynomials) then

$$\begin{aligned} R_1(x) + R_2(x) &= \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} \\ &= \frac{P_1(x)Q_2(x)}{Q_1(x)Q_2(x)} + \frac{P_2(x)Q_1(x)}{Q_1(x)Q_2(x)} \\ &= \frac{P_1(x)Q_2(x) + P_2(x)Q_1(x)}{Q_1(x)Q_2(x)} \end{aligned}$$

E.g.

$$\begin{aligned}\frac{x+1}{x^2+1}+\frac{3x}{x-2} &= \frac{(x+1)(x-2)+(3x)(x^2+1)}{(x^2+1)(x-2)}\\ &= \frac{(x^2-x-2)+(3x^3+3x)}{x^3-2x^2+x-2}\\ &= \frac{3x^3+x^2+2x-2}{x^3-2x^2+x-2}\end{aligned}$$

We can multiply and divide rational functions, again by the usual procedures for fractions. For $R_1(x)$ and $R_2(x)$ as above

$$R_1(x)R_2(x)=\frac{P_1(x)}{Q_1(x)}\cdot\frac{P_2(x)}{Q_2(x)}=\frac{P_1(x)P_2(x)}{Q_1(x)Q_2(x)}$$

and

$$R_1(x)/R_2(x)=\frac{P_1(x)}{Q_1(x)}\Big/\frac{P_2(x)}{Q_2(x)}=\frac{P_1(x)}{Q_1(x)}\cdot\frac{Q_2(x)}{P_2(x)}=\frac{P_1(x)Q_2(x)}{Q_1(x)P_2(x)}$$

E.g.

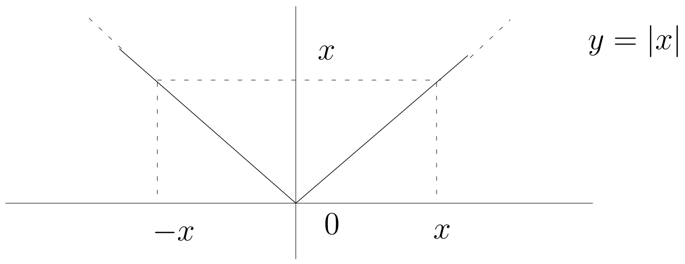
$$\left(\frac{x+1}{x^2+1}\right)\left(\frac{3x}{x-2}\right)=\frac{3x(x+1)}{(x^2+1)(x-2)}=\frac{3x^2+3x}{x^3-2x^2+x-2}$$

and

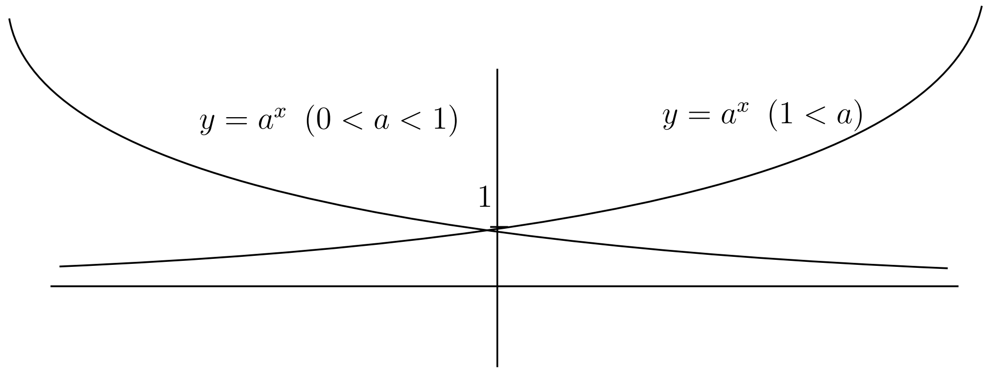
$$\left(\frac{x+1}{x^2+1}\right)\Big/\left(\frac{3x}{x-2}\right)=\frac{(x+1)(x-2)}{3x(x^2+1)}=\frac{x^2-x-2}{3x^3+3x}$$

6. **The Modulus Function,** written $|x|$ (= the 'size' of x) is defined by

$$|x| = \text{'size' of } x = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \qquad \left(= \sqrt{x^2} \text{ in fact}\right)$$



So $|3| = 3$, $|-3| = 3$.



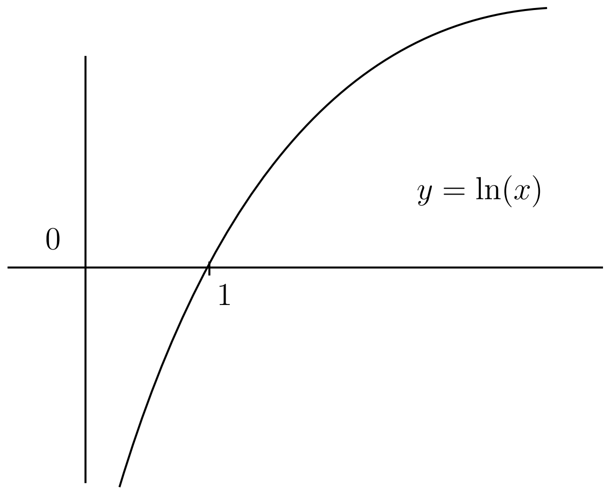
7. **The Exponential Functions:** $f(x) = a^x$, where a is a strictly positive constant.

   E.g. $f(x) = 2^x$, $f(x) = (1/2)^x$.

   Notice that $f(x) = 10^{3x}$ is also an exponential function since $10^{3x} = (10^3)^x = 1000^x$.

   The output a^x is always a positive number, even if x is negative: e.g. if $f(x) = 2^x$ then $f(-1) = 2^{-1} = 1/2$.

8. **Logarithmic Functions**: $f(x) = \log_a x$, where a is a positive constant, $a \neq 1$. [Recall when for base e we may write $\ln(x)$ in place of $\log_e(x)$.]



   The domain of the function $\log_a$ is all the strictly positive real numbers.

Composition of functions

If f, g are functions then $f(g(x))$ (said f *composed with* g) is the result of inputting x to g, to get output $g(x)$ and then inputting this value $g(x)$ into f to get $f(g(x))$.

E.g. if $f(x) = 2 + \sqrt{x}$ and $g(x) = x - 3$ then

$$f(g(x)) = f(x-3) = 2 + \sqrt{x-3}.$$

Given a function f, f^{-1} (called the *inverse function to* f or f *to the minus* 1) is that function (if it exists) such that $f(f^{-1}(x)) = x$. (In most case you will meet it also happens that $f^{-1}(f(x)) = x$ i.e. it takes you back from the value $f(x)$ to the original input x.)

Usually we can find f^{-1} by solving the equation $f(f^{-1}(x)) = x$.

For example if $f(x) = \dfrac{1}{x+3} - 2$ then

$$x = f(f^{-1}(x)) = \frac{1}{f^{-1}(x) + 3} - 2$$

so multiplying both sides by $f^{-1}(x) + 3$ gives

$$\begin{aligned} x(f^{-1}(x) + 3) &= 1 - 2(f^{-1}(x) + 3) \\ \therefore \quad xf^{-1}(x) + 3x &= 1 - 2f^{-1}(x) - 6 \\ \text{so} \qquad f^{-1}(x)(x+2) &= -3x - 5 \\ \text{and} \qquad f^{-1}(x) &= \frac{-3x-5}{x+2}. \end{aligned}$$

Trigonometric Functions

Circular Measure

Clearly in the diagram below angle A is less than angle B:

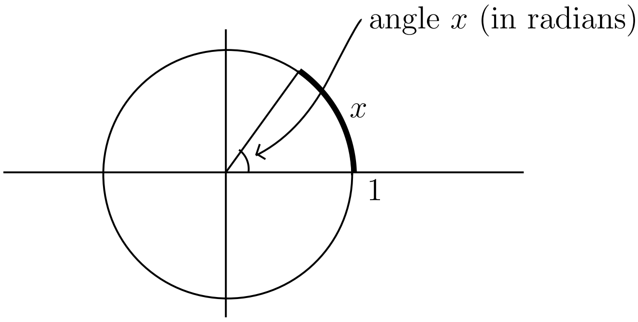
But how do we *measure angles*?

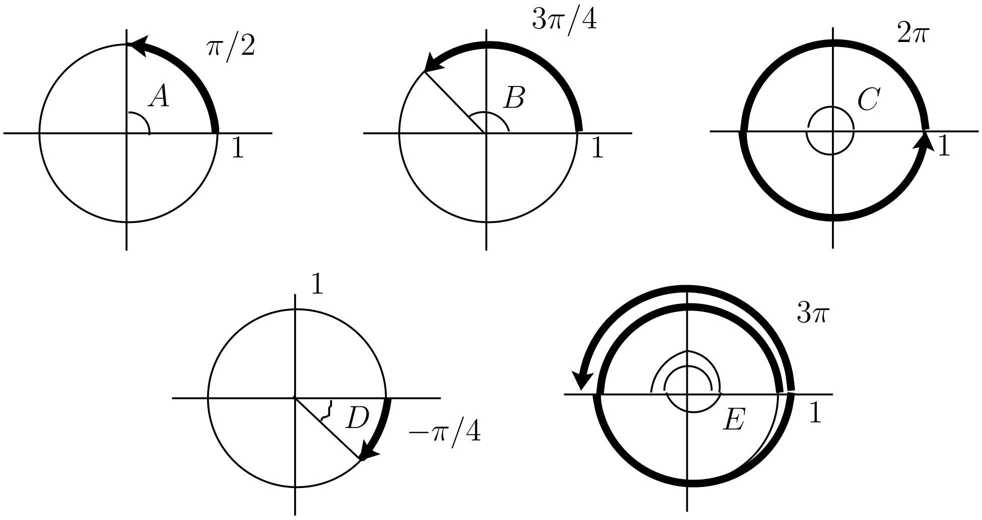
We do this by the length of the anticlockwise arc on the unit circle that the angle determines[5]. E.g.

So in the diagrams below angle A is $\pi/2$, angle B is $3\pi/4$, angle C is 2π, angle D is $-\pi/4$, angle E is 3π.

[5]Just like we measure temperature by the length the mercury goes up the column.

So,





We sometimes refer to these units of angles as ‘radians’. In non-maths circles a more common unit to use is *degrees* where

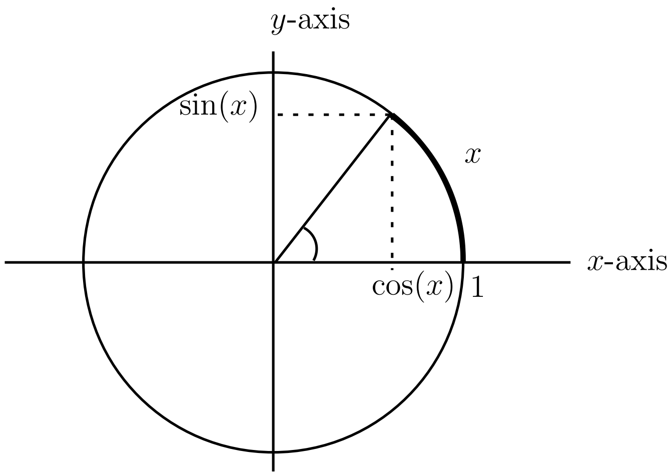
$$2\pi \text{ radians} \quad = \quad 360 \text{ degrees} \quad = \quad 360^\circ$$

E.g. π radians$= 180^\circ$, $\pi/2$ radians $= \ 90^\circ$, and in general

$$D \text{ degrees} \ = \left(\frac{\pi}{180}\right) \times D \ \text{ radians} \qquad R \text{ radians} \ = \left(\frac{180}{\pi}\right) \times R \text{ degrees.}$$

The *Trigonometric Functions cosine*, *sine* of angle x, denoted $\cos(x), \sin(x)$, are the x and y coordinates, respectively, of the point where the line at angle x (in radians) from the x-axis meets the unit circle.

So,



radians	degrees	cos	sin
0	0°	1	0
2π	360°	1	0
$\pi/2$	90°	0	1
π	180°	-1	0
$3\pi/2$	270°	0	-1
$\pi/4$	45°	$1/\sqrt{2}$	$1/\sqrt{2}$
$3\pi/4$	135°	$-1/\sqrt{2}$	$1/\sqrt{2}$
$-\pi/4$	-45°	$1/\sqrt{2}$	$-1/\sqrt{2}$
$7\pi/4$	315°	$1/\sqrt{2}$	$-1/\sqrt{2}$
3π	540°	-1	0

Apart from these most values of sin and cos need are not easily expressed in terms of numbers we are already familiar with and to find them we need to use tables/calculators. Two exceptions however are:

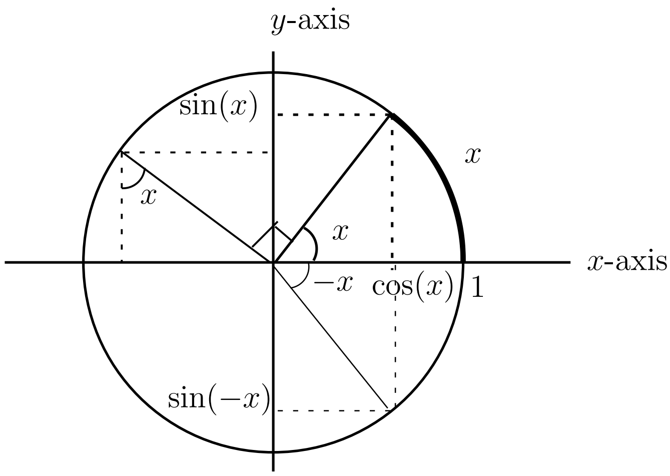
$$\sin(\pi/6) = \sin(30^\circ) = 1/2 = \cos(60^\circ) = \cos(\pi/3)$$
$$\sin(\pi/3) = \sin(60^\circ) = \sqrt{3}/2 = \cos(30^\circ) = \cos(\pi/6)$$

From the diagram below we see that

$$\cos(-x) = \cos(x), \qquad \sin(-x) = -\sin(x)$$
$$\sin(\pi/2 + x) = \cos(x), \qquad \cos(\pi/2 + x) = -\sin(x)$$

so replacing x here by $-x$ gives

$$\begin{aligned} \sin(\pi/2 - x) &= \cos(x) \quad \text{for } x \text{ in radians} \\ \cos(\pi/2 - x) &= \sin(x) \quad \text{for } x \text{ in radians} \\ \sin(90^\circ - x) &= \cos(x) \quad \text{for } x \text{ in degrees} \\ \cos(90^\circ - x) &= \sin(x) \quad \text{for } x \text{ in degrees} \qquad (1) \end{aligned}$$



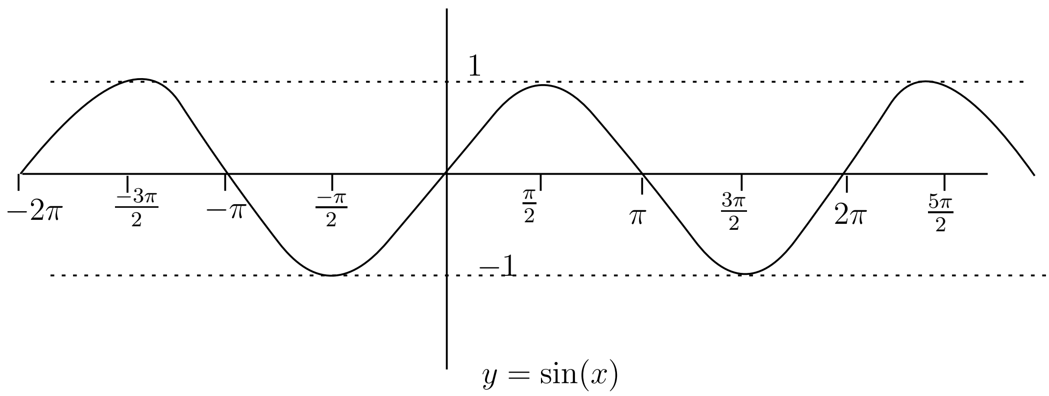
Again from the diagram and Pythagoras Theorem we have

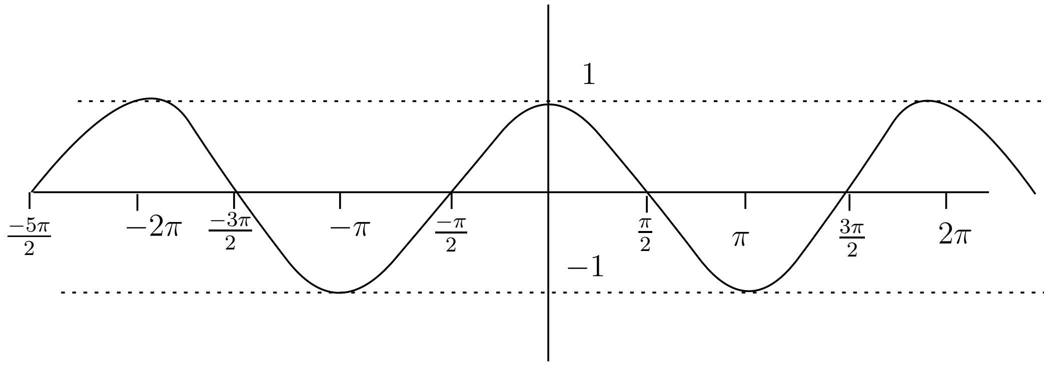
$$(\sin(x))^2 + (\cos(x))^2 = 1$$

Confusingly we usually write $\sin^2(x)$ for $(\sin(x))^2$ and $\cos^2(x)$ for $(\cos(x))^2$ so the above identity becomes

$$\sin^2(x) + \cos^2(x) = 1 \tag{2}$$

Clearly from their definition the functions sin and cos are periodic of period 2π. Their graphs look like:





$y = \cos(x)$

Further Trigonometric Functions

1. tangent function:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

2. secant function:

$$\sec(x) = \frac{1}{\cos x}$$

3. cosecant function:

$$\operatorname{cosec}(x) = \frac{1}{\sin x}$$

4. cotangent function:

$$\cot(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

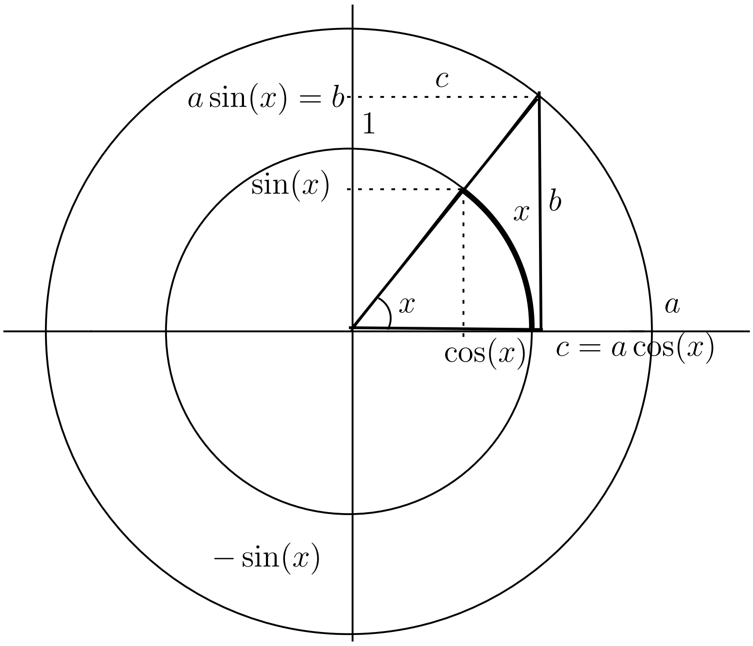
Dividing (2) by $\cos^2(x)$ we obtain

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

which simplifies to

$$\tan^2(x) + 1 = \sec^2(x)$$

Consider the right angled triangles in the diagram below.



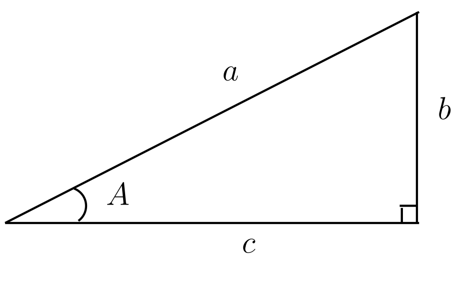
By scaling up (or down) the unit circle to one of radius a we see that $b = a\sin(x)$ and $c = a\cos(x)$. So

$$\sin(x) = \frac{b}{a} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$
$$\cos(x) = \frac{c}{a} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$
$$\tan(x) = \frac{b}{c} = \frac{\text{Opposite}}{\text{Adjacent}}$$

-mnemonic SOH-CAH-TOA.

Examples

In the right angle triangle



if we know the angle A and side a we can find side b since $b = a\sin(A)$. For example if $a = 7.3$ and $A = 33°$ then my calculator tells me $\sin(33°) = 0.545$ (approx) so

$b = 7.3 \times 0.545$ (approx), i.e. 3.976 (approx). Using sin, cos and tan given an angle (not the $\pi/2$!) and a side we can find the other two sides.

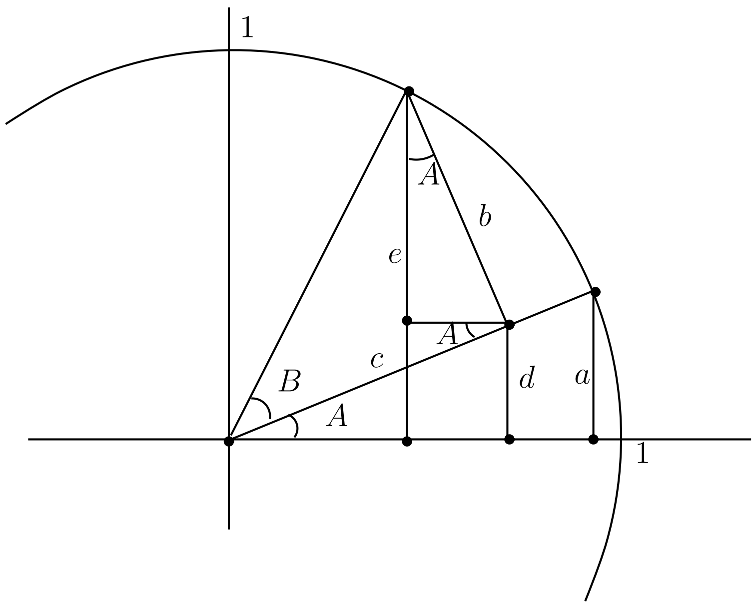
Similarly given two sides we can find the angles. For example suppose that $b = 4.3$, $c = 5.5$. Then $\tan(A) = 4.3 \div 5.5 = 0.782$ (approx) so we're now looking for an angle A whose tangent is 0.782. Fortunately calculators (usually) have a function, $\tan^{-1}$, which we can apply to 0.782 to get such an A between $-\pi/2$ and $\pi/2$ in radians or between $-90°$ and $90°$, depending on which we ask for. In this case

$$A = \tan^{-1}(0.782) = 38°.$$

Similarly calculators have functions $\sin^{-1}$ ($\cos^{-1}$) which take us from the value of $\sin(A)$ ($\cos(A)$) to A between $-\pi/2$ and $\pi/2$ (resp. 0 and π) if we ask for radians and between $-90°$ and $90°$ (resp. $0°$ and $180°$) if we ask for the answer in degrees.

Some Useful Trigonometric Identities

Consider the following diagram:



From this diagram we get:

$$\begin{aligned}
&b = \sin(B)\\
&e = b\cos(A) = \cos(A)\sin(B)\\
&c = \cos(B)\\
&d = c\sin(A) = \sin(A)\cos(B)\\
\therefore\quad &\sin(A+B) = d + e = \sin(A)\cos(B) + \cos(A)\sin(B) \qquad (3)
\end{aligned}$$

A similar diagram gives that

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \qquad (4)$$

Replacing $+B$ by $-B$ and using our earlier observations that $\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$ we obtain

$$\begin{aligned}\sin(A-B) = \sin(A+(-B)) &= \sin(A)\cos(-B) + \cos(A)\sin(-B) \\ &= \sin(A)\cos(B) - \cos(A)\sin(B) \qquad (5)\end{aligned}$$

$$\begin{aligned}\cos(A-B) = \cos(A+(-B)) &= \cos(A)\cos(-B) - \sin(A)\sin(-B) \\ &= \cos(A)\cos(B) + \sin(A)\sin(B) \qquad (6)\end{aligned}$$

The identities (1)-(6) find many uses and you should make an effort to remember them. For example

$$\begin{aligned}\cos(2A) &= \cos(A)\cos(A) - \sin(A)\sin(A) \quad \text{by (4)} \\ &= \cos^2(A) - \sin^2(A) \\ &= \cos^2(A) - (1-\cos^2(A)) \quad \text{by (2)} \\ &= 2\cos^2(A) - 1,\end{aligned}$$

so

$$\frac{\sqrt{3}}{2} = \cos(\pi/6) = 2\cos^2(\pi/12) - 1$$

which gives

$$\cos(\pi/12) = \sqrt{\frac{1}{2}\left(\frac{\sqrt{3}}{2}+1\right)} = \frac{\sqrt{\sqrt{3}+2}}{2}.$$

Similarly

$$\sin(2A) = 2\sin(A)\cos(A)$$

Using (3),(4) we get[6]

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin(A)\cos(B)+\cos(A)\sin(B)}{\cos(A)\cos(B)-\sin(A)\sin(B)}$$

Dividing top and bottom of this right hand side quotient by $\cos(A)\cos(B)$ and using the fact that

$$\tan(A) = \frac{\sin(A)}{\cos(A)}, \qquad \tan(B) = \frac{\sin(B)}{\cos(B)}$$

we obtain

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

and similarly

$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

[6] For the January 2011 exam you will not need to know these identities since they were not covered in the course.

Here's a formula which some of you may need later this year:

$$a\cos(x) + b\sin(x) = \beta\cos(x-\gamma) \tag{7}$$

$$\text{where } \beta = \sqrt{a^2+b^2}, \quad \sin(\gamma) = \frac{b}{\sqrt{a^2+b^2}} \quad \text{and } \cos(\gamma) = \frac{a}{\sqrt{a^2+b^2}},$$

(there always is such a γ between $-180°$ and $180°$, equivalently between $-\pi$ and π radians).

We can check the correctness of (7) by using (4). We get

$$\begin{aligned}
\beta\cos(x-\gamma) &= \sqrt{a^2+b^2}(\cos(x)\cos(\gamma)+\sin(x)\sin(\gamma)) \\
&= \sqrt{a^2+b^2}\left(\left(\frac{a}{\sqrt{a^2+b^2}}\cos(x)\right) + \left(\frac{b}{\sqrt{a^2+b^2}}\sin(x)\right)\right) \\
&= a\cos(x) + b\sin(x).
\end{aligned}$$

Finally in this section you should be aware of the following identities though you will not need them for the exam:

$$\begin{aligned}
\sin(A)+\sin(B) &= \sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)+\sin\left(\frac{A+B}{2}-\frac{A-B}{2}\right) \\
&= \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \\
&+ \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \\
&= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).
\end{aligned}$$

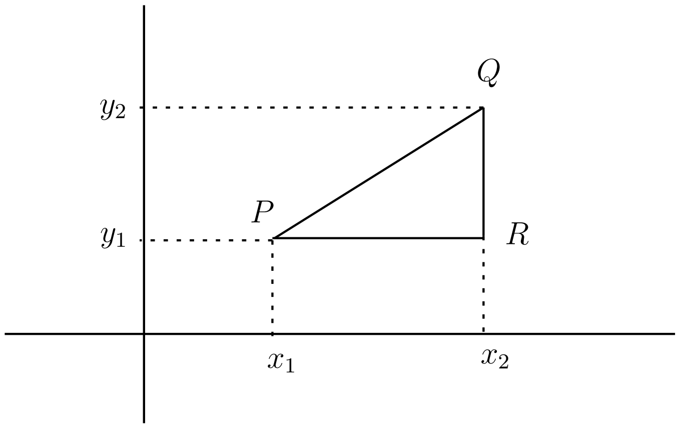
Similar arguments give that:

$$\begin{aligned}
\sin(A)+\sin(B) &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\sin(A)-\sin(B) &= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \\
\cos(A)+\cos(B) &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\
\cos(A)-\cos(B) &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)
\end{aligned}$$

Coordinate Geometry

Distance between two points

Suppose P and Q are points with coordinates (x_1, y_1) and (x_2, y_2) respectively. Let R be the point (x_2, y_1). Then the triangle PRQ is right angled and by Pythagoras's Theorem



$$(\text{length } PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

so

$$\text{length } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

E.g. What is the distance between $(1, 2)$ and $(3, -3)$?
Here $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (3, -3)$ so

$$\text{distance } = \sqrt{(3-1)^2 + (-3-2)^2} = \sqrt{2^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$$

The Equation of a Circle

A circle whose centre is at the point (a, b) and whose radius has length r, is just the collection of points whose distance from (a, b) is r. Thus it is the collection of points (x, y) such that

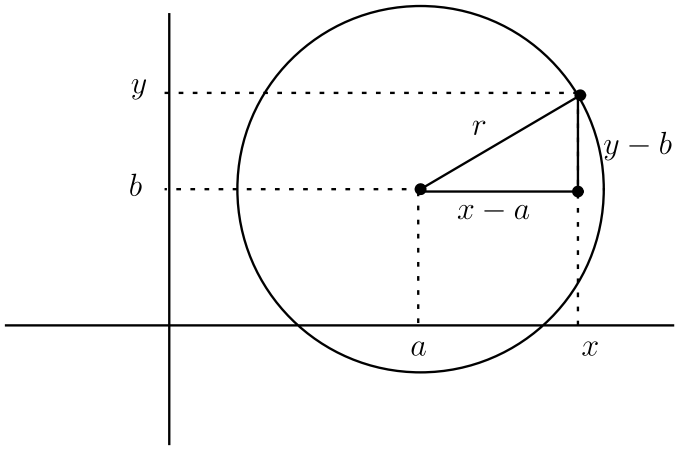
$$r = \sqrt{(x-a)^2 + (y-b)^2}$$

or more simply, by squaring both sides,

$$(x-a)^2 + (y-b)^2 = r^2$$

E.g. the unit circle (radius 1, centre $(0, 0)$) is the set of points (x, y) given by

$$x^2 + y^2 = 1.$$



Straight lines

For a, b, d constants, a, b not both zero, the points (x, y) satisfying the equation

$$ay + bx + d = 0 \tag{8}$$

form a straight line (and conversely). If $a = 0$ this is the vertical line $x = -(d/b)$ going through the point $(-d/b, 0)$. Otherwise we can rearrange equation (8) into the 'standard form'

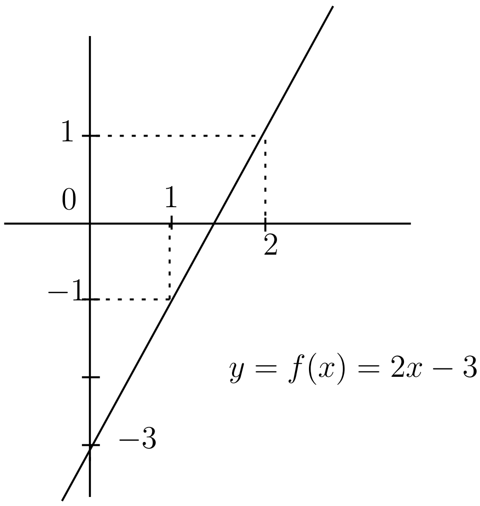
$$y = mx + c \quad \text{where } m = -b/a,\ c = -d/a.$$

In this case we can think of the line given by (8) as the graph of the linear function

$$f(x) = mx + c.$$

E.g. the graph of $f(x) = 2x - 3$ (or $y = 2x - 3$)

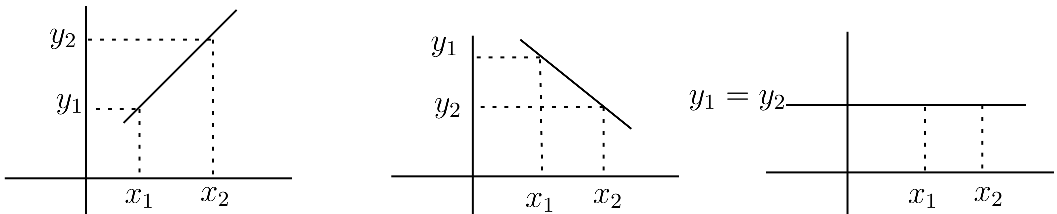
x	$f(x) = 2x - 3$
-1	$2(-1) - 3 = -5$
0	$2(0) - 3 = -3$
1	$2(1) - 3 = -1$
2	$2(2) - 3 = 1$



The point (x_1, y_1) is *on* the line $y = mx + c$ (or the line $y = mx + c$ goes through the point (x_1, y_1)) if $y_1 = mx_1 + c$. E.g. $(2, -5)$ is on the line $y = -3x + 1$ since $-5 = (-3)(2) + 1$.

m is the *gradient* or *slope* of the line $y = mx + c$ since it is the amount y increases when x increases by 1. m is positive if the line slopes up, negative if it slopes down and 0 if it is flat.

E.g.



$$m = \frac{y_2 - y_1}{x_2 - x_1} > 0 \qquad m = \frac{y_2 - y_1}{x_2 - x_1} < 0 \qquad m = \frac{y_2 - y_1}{x_2 - x_1} = 0$$

How do we find the line through the (distinct) points (x_1, y_1), (x_2, y_2)?

If $x_1 = x_2$ then it is the vertical line through $(x_1, 0)$ with equation $x = x_1$. Otherwise we may suppose the line has the form $y = mx + c$. Then since $(x_1, x_2), (y_1, y_2)$ lie on this line:

$$y_1 = mx_1 + c, \tag{9}$$
$$y_2 = mx_2 + c. \tag{10}$$

Subtracting (9) from (10) gives

$$(y_1 - y_2) = m(x_1 - x_2),$$

so

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Substituting this back into (9) gives

$$y_1 = mx_1 + c = \frac{y_1 - y_2}{x_1 - x_2}x_1 + c$$

so

$$c = y_1 - mx_1 = y_1 - \frac{y_1 - y_2}{x_1 - x_2}x_1.$$

In other words the equation of the line which goes through the two points $(x_1, y_1), (x_2, y_2)$ $(x_1 \neq x_2)$ *is*

$$y = \frac{y_1 - y_2}{x_1 - x_2}x + y_1 - \frac{y_1 - y_2}{x_1 - x_2}x_1$$

equivalently $y = mx + c$ *where*

$$m = \frac{y_1 - y_2}{x_1 - x_2} \quad \textit{and} \quad c = y_1 - mx_1.$$

E.g. What line goes through $(1, 2)$ and $(3, 7)$?

Here (x_1, y_1) is $(1, 2)$, and (x_2, y_2) is $(3, 7)$, so

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{2-7}{1-3} = \frac{-5}{-2} = \frac{5}{2}$$

$$c = y_1 - mx_1 = 2 - \frac{5}{2} \times 1 = -\frac{1}{2}.$$

So equation of the line is

$$y = \frac{5}{2}x - \frac{1}{2}.$$

Note that in practice it is often quicker to solve (9) and (10) directly. For example in the above case we have

$$2 = m \times 1 + c, \qquad (11)$$

$$7 = m \times 3 + c. \qquad (12)$$

Subtracting (11) from (12) gives

$$5 = 2m \quad \text{so} \quad m = 5/2$$

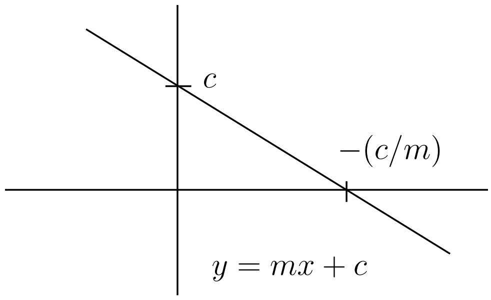
and substituting this value of m in (11) gives

$$2 = (5/2) \times 1 + c$$

so $c = -(1/2)$ and the equation of the line is $y = (5/2)x - 1/2$.

Intersection of a straight line with the axes

Given a (straight) line $y = mx + c$, when $x = 0$, $y = c$. So the point $(0, c)$ lies on the line and this is where the line crosses/intersects the y-axis. When $y = 0$ and $m \neq 0$, $0 = mx + c$ so $x = -(c/m)$. So the point $(-(c/m), 0)$ is on the line and this is where the line intersects the x-axis. [When $m = 0$ the line $y = mx + c$ is horizontal, so parallel to the x-axis and either never crosses it or is identical to it.]



Intersection of straight lines

Two straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are *parallel* if they have the same gradient, i.e. $m_1 = m_2$.

If they are not parallel then they meet/intersect/cross at some point (x, y). This point must be on both lines so:

$$y = m_1x + c_1 \quad \underline{\text{and}} \quad y = m_2x + c_2$$

These are called *simultaneous equations.* We can solve them by noting that they give

$$m_1x + c_1 = m_2x + c_2$$

(said *eliminating* y), which yields

$$x(m_1 - m_2) = c_2 - c_1$$

and hence

$$x = \frac{c_2 - c_1}{m_1 - m_2}.$$

In turn we can now find y since

$$y = m_1x + c_1 = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1.$$

In summary then, the point at which the two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ meet is

$$\left(\frac{c_2 - c_1}{m_1 - m_2}, m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1\right).$$

Example: Where do the lines $y = 3x - 2$ and $y = x + 4$ intersect?

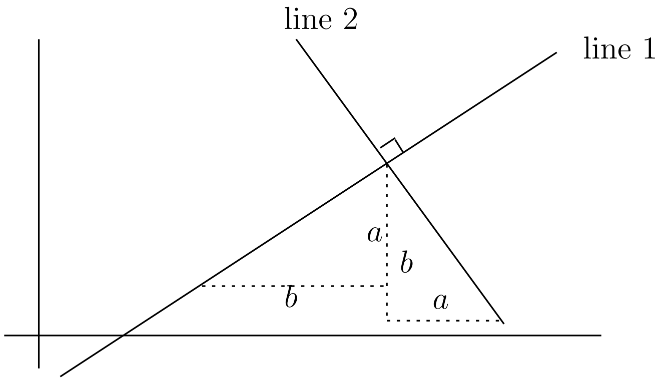
Eliminating y gives

$$\begin{aligned} 3x - 2 &= x + 4 \\ \Rightarrow \quad 3x - x &= 4 + 2 = 6 \\ \Rightarrow \quad 2x &= 6 \quad \Rightarrow \quad x = 3 \end{aligned}$$

Then $y = 3(3) - 2 = 7$, so the point of intersection is $(3, 7)$.
(Check: $7 = 3 + 4$, $7 = 3(3) - 2$ so the point lies on both lines.)
[Again I find it easier to simply solve the two simultaneous equations each time rather than remember the above formula.]

Perpendicular lines

Two lines are *perpendicular*, or *normal to each other*, if they intersect at right angles. The relationship between the gradients m_1, m_2 can be found from a diagram:



Conversely if such triangles can be found then the lines are perpendicular.

$$\begin{aligned} m_1 &= \text{gradient of line 1} = \frac{a}{b} \\ m_2 &= \text{gradient of line 2} = \frac{-b}{a} \end{aligned}$$

So $m_1 m_2 = (a/b) \times (-b/a) = -1$. Hence:

Two lines are perpendicular just if their gradients have product -1.

Example: Show $2x - y + 4 = 0$ and $7 - 5x = 10y$ are perpendicular.

In standard form $2x - y + 4 = 0$ is $y = 2x + 4$ so $m_1 = 2$. Also, $7 - 5x = 10y$ is $y = \frac{-5}{10}x + \frac{7}{10} = -\frac{1}{2}x + \frac{7}{10}$ so $m_2 = -1/2$.
Hence $m_1 m_2 = 2 \times (-1/2) = -1$, so the lines are perpendicular.

Example: Find the equation of the line through the point $(3, -2)$ normal to the line $y = 3x - 11$.

Let the line have equation $y = mx + c$. Then to be normal to $y = 3x - 10$ we must have $3 \times m = -1$ so $m = -1/3$. Since this line must also go through the point $(3, -2)$ we must have

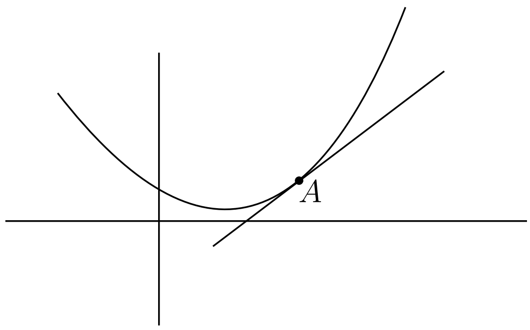
$$-2 = 3m + c = 3 \times (-1/3) + c = -1 + c$$

so $c = -1$ and the required line is $y = -\dfrac{x}{3} - 1$.

Tangents

Given a graph of a function (or more generally a curve, such as a circle – we will use the word 'curve' informally in this course) it clearly appears to have a gradient everywhere that it is 'smooth'. (E.g. we talk of a hill having a gradient.) But how can we measure this gradient at a point?

For a straight line $y = mx + c$ it is just m – and the same at all points on the line. But for a point A on a general curve when it clearly varies depending on A ?



Idea! Take a point B close to A on the curve, then the gradient of the line AB thru' A and B (called the *chord* AB) is close to the gradient of the curve at A.
As B gets closer to A, the line AB gets closer to a straight line through A which has the same gradient as the curve at A. We call this straight line the *tangent* (to the curve) at A and write

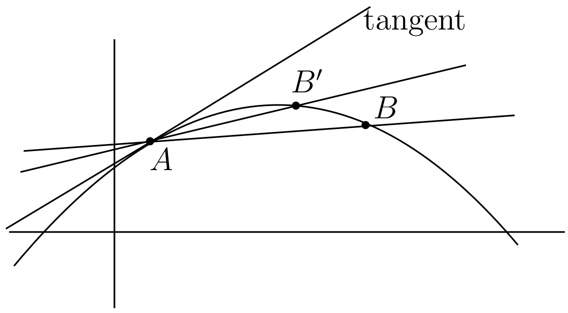
$$\begin{aligned} \text{as } B &\to A \\ \text{line } AB &\to \text{tangent at } A \end{aligned}$$

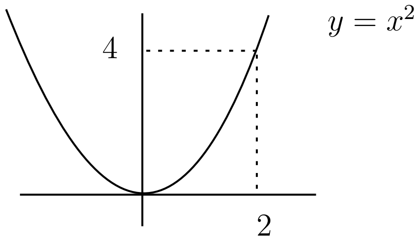
and

$$\text{gradient of } AB \;\to\; \text{grad. of tangent at } A = \text{grad. of curve at } A,$$

where the arrow here is read as 'gets closer to' or 'approaches'. This is also written

$$\lim_{B \to A} (\text{grad. of } AB) = \text{grad. of curve at } A.$$





Example: Consider the curve $y = x^2$.
The point $(2, 4)$ lies on this curve (since $4 = 2^2$). What is the gradient of the curve at this point? Let us take the points on the curve where $x = 2.2$, 2.1, 2.05, 2.01, 2.001, which are approaching $x = 2$, and find the gradients of the lines through these points and $(2, 4)$.

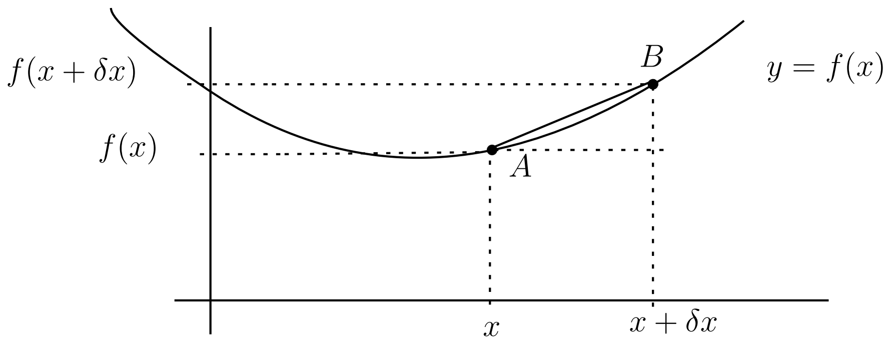
x	2.2	2.1	2.05	2.01	2.001
y	4.84	4.41	4.2025	4.0401	4.004001
change in y	0.84	0.41	0.2025	0.0401	0.004001
change in x	0.2	0.1	0.05	0.01	0.001
gradient	4.2	4.1	4.05	4.01	4.001

As (x, y) approaches $(2, 4)$ the gradient of the lines approaches 4. So the gradient of the curve $y = x^2$ at $(2, 4)$ appears to be 4.

This method is laborious, not general and requires us to say ‘appears to be’, which isn’t very satisfactory. However we can generalize our idea to something wonderful!

Finding the gradient of a graph at a general point

We want to find the gradient at a general point $A = (x, f(x))$ on the graph of the function f.
Let $B = (x + \delta x, f(x + \delta x))$ be a point on the graph close to A, here δx represents a small change in x (it is not δ times x !!).
According to the above 'recipe' to find the gradient at $A = (x, f(x))$ take the limit of the gradient of the line joining $(x, f(x))$ and $B = (x + \delta x, f(x + \delta x))$ as the δx tends to zero.



That is,

$$\text{Gradient of } f \text{ at } A = \text{ limit as } \delta x \to 0 \text{ of } \frac{f(x+\delta x) - f(x)}{x + \delta x - x} \quad \left(= \frac{f(x+\delta x) - f(x)}{\delta x} \right)$$

written

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x},$$

variously denoted

$$f'(x) \text{ or } \frac{df}{dx} \text{ or } \frac{d(f(x))}{dx} \text{ or } \frac{dy}{dx} \text{ when } y = f(x)$$

and called the *derivative* of f, or, when f is implicit, the *derived function.* (As with δx, df,dx are just notation to remind us of its origin as a limit of a ratio, df is *not* d times f etc!!) The process of obtaining $f'(x)$ from $f(x)$ is called *differentiation.*

The theory and application of differentiation is called The Differential Calculus and along with it's sister the Integral Calculus is absolutely central in developing the mathematics of continuous quantities such as time, velocity, waves, money, pressure, electricity, etc.

Example: Let $f(x) = x^2$, as earlier.

Then applying the above, for the point $(x, f(x))$ (i.e. (x, x^2)) on this graph,

$$
\begin{aligned}
\text{gradient of } f \text{ at } (x, x^2) &= \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(\frac{(x+\delta x)^2 - x^2}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(\frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(\frac{2x\delta x + (\delta x)^2}{\delta x} \right) \\
&= \lim_{\delta x \to 0} (2x + \delta x) \; = \; 2x
\end{aligned}
$$

So, $\dfrac{d(f(x))}{dx} = \dfrac{d(x^2)}{dx} = f'(x) = 2x$ and in particular the gradient of f at $x = 2$ is $f'(2) = 2(2) = 4$, as suggested above. Similarly the gradient of $f(x) = x^2$ at $x = 1$ is $f'(1) = 2(1) = 2$.

Another example:

1. Let $y = f(x) = mx + c$. Then

$$
\begin{aligned}
\frac{dy}{dx} = f'(x) &= \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(\frac{m(x+\delta x) + c - (mx + c)}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(\frac{m\delta x}{\delta x} \right) \; = \; \lim_{\delta x \to 0} m \; = \; m
\end{aligned}
$$

   I.e. the gradient of the line $y = mx + c$ is m (which, of course, is what we already thought it was!). Also, if $m = 0$, i.e. $f(x) = c$, then $f'(x) = 0$.

2. Another example $y = f(x) = x^3$. Here

$$
\begin{aligned}
f(x+\delta x) &= (x+\delta x)^3 \\
&= (x+\delta x)(x+\delta x)(x+\delta x) \\
&= x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3
\end{aligned}
$$

   so

$$
\begin{aligned}
\frac{d(x^3)}{dx} = \lim_{\delta x \to 0} \left(\frac{(x+\delta x)^3 - x^3}{\delta x} \right) &= \lim_{\delta x \to 0} \left(\frac{x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - x^3}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(\frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3}{\delta x} \right) \\
&= \lim_{\delta x \to 0} \left(3x^2 + 3x\delta x + (\delta x)^2 \right) \; = \; 3x^2
\end{aligned}
$$

   So the gradient of the graph $f(x) = x^3$ when $x = 4$, i.e. at the point $(4, 64)$, is $3(4)^2 = 48$.

In principle we can differentiate any function using this technique, but in practice the method is laborious. Fortunately, we can derive some rules which make differentiation of the commonly encountered functions much simpler and more routine. Some of these rules are stated below, though we will not prove them: (they are proved by making arguments similar to the ones above).

Derivatives of some common functions

	Function , $y = f(x)$	Derivative , dy/dx	
(a)	c	0	c any constant
(b)	x^r	rx^{r-1}	r any real number $\neq 0$
(c)	e^x	e^x	
(d)	$\ln(x)$	$1/x$	
(e)	$\sin(x)$	$\cos(x)$	x in radians
(f)	$\cos(x)$	$-\sin(x)$	x in radians
(g)	$\tan(x)$	$\sec^2(x)$	x in radians

Rules for differentiation

	Function , $f(x)$	Derivative , $f'(x)$	
(h)	$cg(x)$	$cg'(x)$	constant $\times$ function
(i)	$g(x) + h(x)$	$g'(x) + h'(x)$	sum of functions
(j)	$g(x)h(x)$	$g'(x)h(x) + g(x)h'(x)$	product of functions
(k)	$\dfrac{g(x)}{h(x)}$	$\dfrac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$	quotient of functions

Examples

1. If $y = x^4$ then $\dfrac{dy}{dx} = 4x^{4-1} = 4x^3$ by (b).

2. If $y = x = x^1$ then $\dfrac{dy}{dx} = 1.x^{1-1} = x^0 = 1$ by (b).

3. $\dfrac{d(\sqrt{x})}{dx} = \dfrac{d(x^{1/2})}{dx} = \dfrac{1}{2} \times x^{\frac{1}{2}-1}$ (by (b)) $= \dfrac{x^{-1/2}}{2} = \dfrac{1}{2\sqrt{x}}$.

4. $\dfrac{d}{dx}\left(\dfrac{4}{x}\right) = 4\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$ (by (h)) $= 4\dfrac{d(x^{-1})}{dx} = 4(-1)(x^{-1-1})$ by (b) $= -4x^{-2} = -\dfrac{4}{x^2}$

5. $\dfrac{d(3e^x + 5\ln(x))}{dx} = \dfrac{d(3e^x)}{dx} + \dfrac{d(5\ln(x))}{dx}$ by (i), here $g(x) = 3e^x$, $h(x) = 5\ln(x)$

$= 3\dfrac{d(e^x)}{dx} + 5\dfrac{d(\ln(x))}{dx}$ by (h) $= 3 \cdot e^x + 5 \cdot \dfrac{1}{x}$ by (c), (d) $= 3e^x + \dfrac{5}{x}$.

6. $\dfrac{d}{dx}(\sin(x) + \cos(x)) = \dfrac{d}{dx}\sin(x) + \dfrac{d}{dx}\cos(x)$ by (j) $= \cos(x) - \sin(x)$ by (e), (f) .
Here $g(x) = \sin(x)$, $h(x) = \cos(x)$.

7. Letting $g(x) = x$, $h(x) = \ln(x)$ we get $\dfrac{d}{dx}(x\ln(x)) = \dfrac{d}{dx}(x) \cdot \ln(x) + x \cdot \dfrac{d}{dx}(\ln(x))$ by (j)
$= 1 \cdot \ln(x) + x \cdot \dfrac{1}{x}$ by (d) $= \ln(x) + 1$.

8. Putting $g(x) = e^x$, $h(x) = \tan(x)$ gives $\dfrac{d}{dx}(e^x\tan(x)) = \dfrac{d(e^x)}{dx}\tan(x) + e^x\dfrac{d(\tan(x)}{dx}$ by (j)
$= e^x\tan(x) + e^x\sec^2(x)$ by (c), (g).

9. Setting $g(x) = x^{1/3}$, $h(x) = (x + 1)$ gives

$$\frac{d}{dx}\left(\frac{x^{1/3}}{x+1}\right) = \frac{(x+1)(d/dx)(x^{1/3}) - x^{1/3}(d/dx)(x+1)}{(x+1)^2} \text{ by (k)}$$

$$= \frac{(x+1)(x^{-2/3}/3) - x^{1/3}}{(x+1)^2} = \frac{(x^{1/3})/3 + (x^{-2/3})/3 - x^{1/3}}{(x+1)^2} = \frac{x^{-2/3} - 2x^{1/3}}{3(x+1)^2}$$

since

$$\frac{d}{dx}(x+1) = \frac{d}{dx}(x) + \frac{d}{dx}(1) = 1 + 0 = 1$$

10. $\dfrac{d}{dx}(\cot(x)) = \dfrac{d}{dx}\left(\dfrac{\cos(x)}{\sin(x)}\right)$

$$= \frac{(d/dx)(\cos(x)) \cdot \sin(x) - \cos(x) \cdot (d/dx)(\sin(x))}{\sin^2(x)} \text{ by (k)}$$

$$= \frac{-\sin(x) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)} \text{ by (e),(f)}$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\text{cosec}^2(x)$$

since $\sin^2(x) + \cos^2(x) = 1$.

We can similarly show that

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x) \qquad \frac{d}{dx}(\text{cosec}(x)) = -\text{cosec}(x)\cot(x).$$

Example Find the points where the line $y = 2x - 2$ intersects/crosses the curve $y = x^2 - 5x + 10$ and find the equation of the tangent to this curve at these points.

If (x, y) is on both the line and the curve then

$$2x - 2 = y = x^2 - 5x + 10 \tag{13}$$

so

$$x^2 - 7x + 12 = 0.$$

By inspection (or the formula)

$$x^2 - 7x + 12 = (x - 4)(x - 3)$$

so the solutions to (13) are $x = 3, 4$ and the required points are $(3, 2(3) - 2) = (3, 4)$ and $(4, 2(4) - 2) = (4, 6)$. Since

$$\frac{d}{dx}(x^2 - 5x + 10) = 2x - 5$$

the gradient of the tangent to this curve at $x = 3$ is $2(3) - 5 = 1$. Therefore the tangent $y = mx + c$ at $(3, 4)$ is the line thru' this point with gradient 1, i.e. $y = x + 1$ since $m = 1$ and $4 = 1(3) + c$. Rest is an exercise. *(Answer is $y = 3x + 6$.)*

Function of a function

How can we differentiate the function $y = \sqrt{3x - 1}$?

This is the composition of two functions, $f(x) = \sqrt{x}, g(x) = 3x + 1$, that we can differentiate. Here is a wonderful 'rule' for tackling such cases:

The Chain Rule If y is a function of u, $y = f(u)$, where u is a function of x, $u = g(x)$ (so $y = f(u) = f(g(x))$), the rule for differentiation is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

equivalently,

$$(f(g(x))' = f'(g(x))g'(x)$$

So to differentiate $y = \sqrt{3x - 1}$ let $u = 3x - 1$; then $y = \sqrt{u} = u^{1/2}$. Since

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2u^{1/2}} \quad \text{and} \quad \frac{du}{dx} = 3$$

the Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2u^{1/2}} \times 3 = \frac{3}{2(3x-1)^{1/2}} = \frac{3}{2\sqrt{3x-1}}.$$

Similarly for a function $f(x)$, constants a, b and $u = ax + b$,

$$\frac{d(f(ax+b))}{dx} = \frac{d(ax+b)}{dx} \cdot \frac{d(f(u))}{du} = af'(u) = af'(ax+b).$$

This enables us to generalize out list of derivatives above to the following:

For real $r \neq 0$ and constants a, b,

$$\begin{aligned}
\frac{d}{dx}(b) &= 0 \\
\frac{d}{dx}((ax+b)^r) &= ar(ax+b)^{r-1} \\
\frac{d}{dx}(e^{ax+b}) &= ae^{ax+b} \\
\frac{d}{dx}(\ln(ax+b)) &= \frac{a}{ax+b} \\
\frac{d}{dx}(\sin(ax+b)) &= a\cos(ax+b) \\
\frac{d}{dx}(\cos(ax+b)) &= -a\sin(ax+b) \\
\frac{d}{dx}(\tan(ax+b)) &= a\sec^2(ax+b)
\end{aligned}$$

Notice that since $a = e^{\ln(a)}$, $a^x = e^{x\ln(a)}$ so

$$\frac{d(a^x)}{dx} = \frac{d(e^{x\ln(a)})}{dx} = \ln(a)e^{x\ln(a)} = \ln(a)a^x.$$

More examples

1. Differentiate $y = (x^2+7)^3$

Write $u = x^2 + 7$. Then $y = u^3$ and $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{dx} = 2x$ so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \times 2x = 3(x^2+7)^2 \times 2x = 6x(x^2+7)^2.$$

2. Differentiate $\ln(\cos(x))$. Put $u = \cos(x)$ so $y = \ln(u)$, $dy/du = 1/u$, $du/dx = -\sin(x)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (-\sin(x)) = \frac{-\sin(x)}{\cos(x)} = -\tan(x).$$

3. To differentiate $y = \log_x(a) = \dfrac{\log_e(a)}{\log_e(x)}$ (by the change of base rule) put $u = \log_e(x) = \ln(x)$ so $y = \dfrac{\ln(a)}{u}$. Then

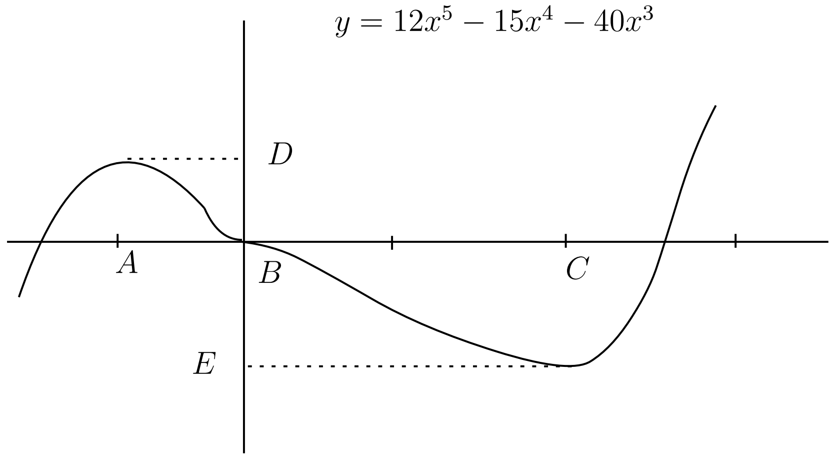
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left(\frac{\ln(a)}{u}\right) \cdot \frac{du}{dx} = \frac{-\ln(a)}{u^2} \cdot \frac{1}{x} =$$

$$-\frac{\ln(a)}{x(\ln(x))^2} = -\frac{\log_x(a)}{x\ln(x)}.$$

Maxima and Minima

Consider the graph of

$$f(x) = 12x^5 - 15x^4 - 40x^3$$



The gradient is zero when $x = A, B, C$. Clearly the function has a (local) maximum when $x = A$, a (local) minimum when $x = C$. When $x = B$ it is neither. Since

$$\begin{aligned} f'(x) = 60x^4 - 60x^3 - 120x^2 = 0 &\iff 60x^2(x^2 - x - 2) = 0 \\ &\iff x^2(x+1)(x-2) = 0 \\ &\iff x = -1 \text{ or } x = 0 \text{ or } x = 2 \end{aligned}$$

so $A = -1$, $B = 0$, $C = 2$ (and $D = f(-1) = 13,\ E = f(2) = -176$).

We can recognize C as a minimum point since the gradient of $f(x)$, i.e. $f'(x)$ is increasing at $x = C$, equivalently the derivative of $f'(x)$,

$$\frac{d}{dx}\left(\frac{df}{dx}\right), \quad \text{written} \quad \frac{d^2 f}{dx^2} \quad \text{or} \quad f''(x)$$

and called the *second derivative* of f, is positive at $x = C$.

Similarly we can recognize A as a maximum point because the gradient of f is decreasing at A, i.e. $f''(x) < 0$ at A.

At B $f'(x) = 0$ but it is neither a maximum nor a minimum point. We call such a point a *point of inflection.*

In this case

$$f''(x) = 240x^3 - 180x^2 - 240x$$

so at C where $x = 2$,

$$f''(2) = 240 \times 8 - 180 \times 4 - 240 \times 2 = 720 > 0,$$

and at A where $x = -1$,

$$f''(-1) = 240 \times (-1) - 180 \times 1 - 240 \times (-1) = -180 < 0,$$

whilst at B where $x = 0$

$$f''(0) = 240 \times 0 - 180 \times 0 + 240 \times 0 = 0.$$

This special case analysis holds in general for a function f:

$x = b$ (or just b) is a *stationary point* of function f if $f'(b) = 0$ (i.e. the value of the derivative of f at b is zero).

$x = b$ (or just b) is a (local) *maximum point* of f if $f(b) \geq f(b \pm \epsilon)$ for all ϵ sufficiently close to zero. Similarly for a (local) *minimum point* with $\leq$ in place of $\geq$.

Any maximum or minimum point of f is a stationary point of f.

If b is a stationary point of f and $f''(b) < 0$ then b is a maximum point of f.

If b is a stationary point of f and $f''(b) > 0$ then b is a minimum point of f.

Note: If $f''(b) = 0$ we'd have to investigate further, it could be a maximum or minimum point or a point of inflection. The converse to these last two does not necessarily hold, e.g. $f(x) = x^4$ has a minimum point at $x = 0$ but $f''(0) = 0$ whilst $f(x) = x^3$ has a point of inflection at $x = 0$ and $f''(0) = 0$.

Example Let

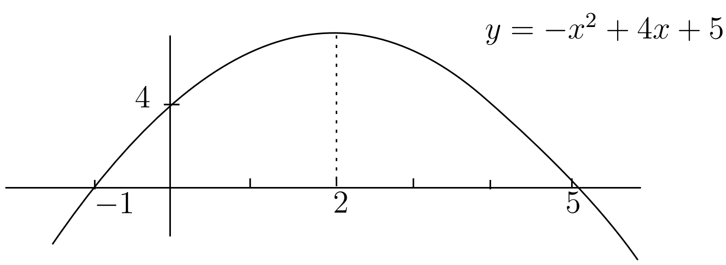
$$y = f(x) = -x^2 + 4x + 5$$

Then $f'(x) = -2x + 4$ so $f'(x) = 0$ just if $-2x + 4 = 0$, i.e. $x = 2$, and this is the only stationary point.

$$f''(x) = -2$$

so certainly $f''(2) = -2 < 0$ and $x = 2$ is a maximum point, the value of $f(x)$ there being $-(2)^2 + 4(2) + 5 = 9$. Since there are no other maximum or minimum points and f we can sketch the graph[7] of f as

[7] In the exam you would not be expected to do this but in this case we could also indicate the points where it crosses the x and y axes – since $f(x)$ factorizes as $f(x) = -(x - 5)(x + 1)$ so it's value is zero, i.e. crosses the x-axis when $x = 5, -1$. Similarly the point where it crosses the y-axis is when $x = 0$, i.e. $y = -0^2 + 5.0 + 4 = 4$.



Note that the graph cannot go up again to the left of $x = 2$ or up again to the right of $x = 2$ otherwise it would have a second maximum or minimum (so a second stationary pint)

Another example Find and classify the stationary points of $f(x) = x^3 - 12x + 1$. Sketch the graph of this function and using this indicate why the equation

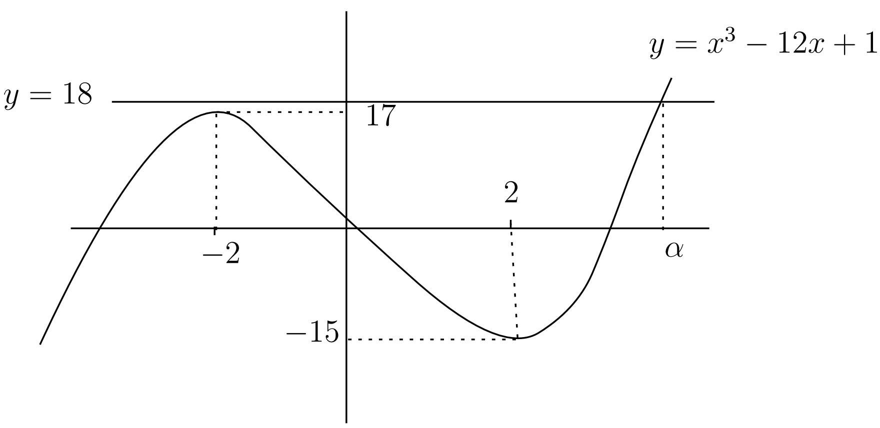
$$x^3 - 12x + 1 = 18$$

has only one solution.

$f'(x) = 3x^2 - 12$ so

$$f'(x) = 0 \iff 3x^2 - 12 = 0 \iff x^2 = 4 \iff x = 2, -2.$$

So the stationary points of f are $2, -2$. $f''(x) = 6x$ so $f''(2) = 6 \times 2 > 0$ – so 2 is a minimum point, and $f''(-2) = 6 \times (-2) = -12 < 0$ – so -2 is a maximum point. The values of f at these points are $2^3 - 12 \times 2 + 1 = -15$ and $(-2)^3 - 12(-2) + 1 = 17$.



Since the line $y = 18$ crosses the graph at just one point the equation $18 = x^3 - 12x + 1$ will have just one solution ($x = \alpha$ on the graph).

Summary of useful formulae

Powers

For a, b a real numbers and r, s rational numbers such that a^r, a^s etc. are defined (and even for real numbers r, s when $a, b \geq 0$):

[P1] $a^1 = a, \quad a^0 = 1$

[P2] $a^r a^s = a^{r+s}$

[P3] $a^r / a^s = a^{r-s}, \quad a^{-s} = 1/a^s$

[P4] $(a^r)^s = a^{rs}$

[P5] $(ab)^r = a^r b^r, \quad (a/b)^r = a^r / b^r$

Logarithms

For $a > 0$, $a \neq 1$:

[L1] $a^x = b \iff x = \log_a b$

[L2] $\log_a a = 1$ and $\log_a 1 = 0$

[L3] $\log_a (a^x) = x$

[L4] $a^{\log_a b} = b$

[L5] $\log_a bc = \log_a b + \log_a c$

[L6] $\log_a (b/c) = \log_a b - \log_a c$

[L7] $\log_a b^y = y \log_a b$

[L8] $\log_b c = \dfrac{\log_a c}{\log_a b}$

Solving Equations

The solutions of

$$ax^2 + bx + c = 0$$

are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad \text{or} \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{for short.}$$

Trigonometric Identities

radians	degrees	cos	sin
0	$0°$	1	0
2π	$360°$	1	0
$\pi/2$	$90°$	0	1
π	$180°$	-1	0
$3\pi/2$	$270°$	0	-1
$\pi/4$	$45°$	$1/\sqrt{2}$	$1/\sqrt{2}$
$3\pi/4$	$135°$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$-\pi/4$	$-45°$	$1/\sqrt{2}$	$-1/\sqrt{2}$
$7\pi/4$	$315°$	$1/\sqrt{2}$	$-1/\sqrt{2}$
3π	$540°$	-1	0
$\pi/3$	$60°$	$1/2$	$\sqrt{3}/2$
$\pi/6$	$30°$	$\sqrt{3}/2$	$1/2$

$$\cos(-x) = \cos(x), \qquad \sin(-x) = -\sin(x)$$

$$\cos^2(x) + \sin^2(x) = 1$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\cos(2A) = 2\cos^2(A) - 1$$
$$\sin(2A) = 2\sin(A)\cos(A)$$

Differentiation

For real r and constants a, b,

Basic version	General version
$\frac{d}{dx}(b) = 0$	
$\frac{d}{dx}(x^r) = rx^{r-1} \quad (r \neq 0)$	$\frac{d}{dx}((ax+b)^r) = ar(ax+b)^{r-1}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	$\frac{d}{dx}(\ln(ax+b)) = \frac{a}{ax+b}$
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\cos(ax+b)) = -a\sin(ax+b)$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\tan(ax+b)) = a\sec^2(ax+b)$

Rules for Differentiation

	Function $f(x)$	Derivative $f'(x)$	
(h)	$cg(x)$	$cg'(x)$	constant $\times$ function
(i)	$g(x)+h(x)$	$g'(x)+h'(x)$	sum of functions
(j)	$g(x)h(x)$	$g'(x)h(x)+g(x)h'(x)$	product of functions
(k)	$\frac{g(x)}{h(x)}$	$\frac{g'(x)h(x)-g(x)h'(x)}{h(x)^2}$	quotient of functions

The Chain Rule If y is a function of u, $y = f(u)$, where u is a function of x, $u = g(x)$ (so $y = f(u) = f(g(x))$), the rule for differentiation is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

0C1/1C1 Practice 1st In-Class Test, 2012

ID Number .... NAME ....

Answer all 8 questions in the spaces provided
The test lasts 35 minutes
THE USE OF ANY ELECTRONIC DEVICE DURING THIS TEST IS PROHIBITED

1. Evaluate $(14 - 17)/3^2$ using the rules and conventions described in the lectures. Express your answer as an integer or fraction (that is in the form $\frac{p}{q}$ where p is an integer and q is a natural number) in its simplest form.

2. Evaluate $31/(7+11)-2$ using the rules and conventions described in the lectures. Express your answer as an integer or fraction in its simplest form.

3. Multiply out the brackets in $(2 - (a - b))(b - a)$ and collect terms.

4. Multiply out the brackets in $(1 - x)(2x + 1)(1 + x)$ and collect terms.

5. Put the $\dfrac{x^3y^{10}}{x^7y^2}$ in the form x^my^n where m, n are fractions or integers.

6. Put the $\left(\dfrac{y^{1/3}}{xy^{-3/4}}\right)^{-1}$ in the form a^mb^n where m, n are fractions or integers.

7. Express $\ln\left(\sqrt{u^{-3}v^{4/3}}\right)$ in terms of $\ln u$ and $\ln v$.

8. Given that $b = \log_8(3)$ express $\log_2(3)$ in terms of b.

Practice 1st In-Class Test Solutions

1. $(14-17)/3^2 = -3/9 = -1/3$

2. $31/(7+11) - 2 = 31/18 - 2/1 = (31 \times 1 - 18 \times 2)/(18 \times 1) = -5/18$

3. $(2-(a-b))(b-a) = (2-a+b)(b-a) = 2b - 2a - ab + a^2 + b^2 - ba = a^2 + b^2 - 2ab - 2a + 2b$

4. $(1-x)(2x+1)(1+x) = (1-x)(2x+2x^2+1+x) = (1-x)(2x^2+3x+1) = 2x^2+3x+1-2x^3-3x^2-x = -2x^3-x^2+2x+1$

5. $\dfrac{x^3y^{10}}{x^7y^2} = x^{3-7}y^{10-2} = x^{-4}y^8$

6. $\left(\dfrac{y^{1/3}}{xy^{-3/4}}\right)^{-1} = \left(y^{(1/3-(-3/4))}x^{-1}\right)^{-1} =$
   $\left(x^{-1}y^{(1/3+3/4)}\right)^{-1} = x^{(-1)(-1)}y^{-13/12} = xy^{-13/12}$

7. $\ln\left(\sqrt{u^{-3}v^{4/3}}\right) = (\frac{1}{2}) \times \ln(u^{-3}v^{4/3}) = \frac{1}{2}(\ln(u^{-3}) + \ln(v^{4/3})) =$
   $\frac{1}{2}(-3\ln(u) + (4/3)\ln(v)) = -(3/2)\ln u + (2/3)\ln v$

8. By the change of base rule $\log_2(3) = \dfrac{\log_8(3)}{\log_8(2)}$, and since $2 = 8^{1/3}$, $\log_8(2) = 1/3$ and we get that $\log_2(3) = 3b$

0C1/1C1 Practice 2nd In-Class Test, 2012

ID Number .... NAME ....

Answer all 7 questions in the spaces provided
The test lasts 35 minutes
THE USE OF ANY ELECTRONIC DEVICE DURING THIS TEST IS PROHIBITED

1. Find all solutions to the equation $x^2 - 3x - 1 = 0$

2. Find all solutions to the equation $\dfrac{2}{x+3} + \dfrac{1}{2x+9} = \dfrac{2}{x+4}$

3. A right angled triangle has hypothenuse of length h and an angle A with $\cos(A) = 2/3$. Find the length of the other side adjacent to the angle A.

4. Find $\cos(2A)$ for the triangle in question 3 above.

5. Find the point of intersection of the lines $y = 3x - 1$ and $y = -2x + 9$.

6. Find the equation of the line through the points $(-1, 2)$ and $(3, 6)$.

7. Find the equation of the line through the point $(3, 2)$ which is normal (i.e. perpendicular) to the line $y = 4 - x$.

Practice 2nd In-Class Test Solutions

1. Using the formula the solutions are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 - 4 \times (-1)}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$

2.

$$\begin{aligned}
\frac{2}{x+3} + \frac{1}{2x+9} = \frac{2}{x+4} \quad &\Longleftrightarrow \quad 2(x+4)(2x+9) + (x+4)(x+3) = 2(x+3)(2x+9) \quad \text{(multiply both} \\
&\qquad \text{sides by } (x+4)(x+3)(2x+9)) \\
&\Longleftrightarrow \quad 4x^2 + 34x + 72 + x^2 + 7x + 12 = 4x^2 + 30x + 54 \\
&\Longleftrightarrow \quad x^2 + 11x + 30 = 0 \\
&\Longleftrightarrow \quad (x+5)(x+6) = 0 \\
&\Longleftrightarrow \quad x = -5, -6.
\end{aligned}$$

3. $\dfrac{2}{3} = \cos(A) = \dfrac{\text{Adjacent}}{\text{Hypothenuse}}$, so the length of the adjacent side is $2h/3$.

4. $\cos(2A) = 2\cos^2(A) - 1 = 2(2/3)^2 - 1 = (8/9) - 1 = -1/9$

5. At the point (x, y) at which the lines $y = 3x - 1$ and $y = -2x + 9$ cross both these equations must hold. Hence $3x - 1 = -2x + 9$, i.e. $x = 2$ and substituting this value for x gives $y = 3 \times 2 - 1 = 5$. So the point of intersection is $(2, 5)$.

6. If the line $y = mx + c$ goes through the points $(-1, 2)$ and $(3, 6)$ we must have $2 = -m + c$ and $6 = 3m + c$. Subtracting the first of these from the second gives $4 = 4m$ so $m = 1$ and then substituting this value of m into the first equation gives $2 = -1 + c$ so $c = 3$. Thus the required line is $y = x + 3$.

7. Since the gradient of the line $y = 4 - x$ is -1 if the equation of this normal is $y = mx + c$ we must have $(-1) \times m = -1$, i.e. $m = 1$. Also if this line is to pass through the point $(3, 2)$ we must have

$$2 = 3m + c = 3 + c(\text{since } m = 1),$$

so $c = -1$ and the equation of the required normal is $y = x - 1$.