MATH43032/63032 First Coursework, 2014, Problem, Solution and Feedback

The Problem

1. By using the Representation Theorem for Rational Consequence Relations show that the rule

$$\frac{\psi \mathrel{|\!\sim} \varphi, \quad \psi \wedge \neg\theta \mathrel{|\!\not\sim} \varphi}{\psi \wedge \theta \mathrel{|\!\sim} \varphi}$$

holds for all rational consequences relations but that the rule

$$\frac{\psi \mathrel{|\!\sim} \varphi}{\psi \wedge \theta \mathrel{|\!\sim} \varphi}$$

fails for some rational consequence relation (and choice of ψ, θ, φ).

The Solution

For the first part we need to show that if $\mathrel{|\!\sim}$ is a rational consequence relation (rcr) such that

$$\psi \mathrel{|\!\sim} \varphi, \tag{1}$$

$$\psi \wedge \neg\theta \mathrel{|\!\not\sim} \varphi, \tag{2}$$

then

$$\psi \wedge \theta \mathrel{|\!\sim} \varphi. \tag{3}$$

By the Representation Theorem we may assume that $\mathrel{|\!\sim} = \mathrel{|\!\sim}_{\vec{s}}$ where $\vec{s} = s_1, s_2, \ldots, s_m$.

If $s_i \cap S_{\psi\wedge\theta} = \emptyset$ for all $i = 1, \ldots, m$ then by definition of $\mathrel{|\!\sim}_{\vec{s}}$, $\psi \wedge \theta \mathrel{|\!\sim} \varphi$, giving (3) as required.

Otherwise let j be minimal such that $s_j \cap S_{\psi\wedge\theta} \neq \emptyset$. Then since

$$s_j \cap S_{\psi\wedge\theta} = s_j \cap S_\psi \cap S_\varphi \subseteq s_j \cap S_\psi,$$

$s_j \cap S_\psi \neq \emptyset$. If this j is minimal such that $s_j \cap S_\psi \neq \emptyset$ then from (1) we have that

$$s_j \cap S_{\psi\wedge\theta} \subseteq s_j \cap S_\psi \subseteq s_j \cap S_\varphi$$

and $\psi \wedge \theta \mathrel{|\!\sim} \varphi$.

Otherwise there must be a $k < j$ such that $s_k \cap S_\psi \neq \emptyset$. Let k be the least such, so from (1) we have that $s_k \cap S_\psi \subseteq S_\varphi$. Also since $k < j$, and j was minimal such that $s_j \cap S_{\psi\wedge\theta} \neq \emptyset$ it must be that $s_k \cap S_{\psi\wedge\theta} = \emptyset$. So

$$s_k \cap S_{\psi\wedge\neg\theta} = (s_k \cap S_{\psi\wedge\neg\theta}) \cup (s_k \cap S_{\psi\wedge\theta}) = s_k \cap S_{(\psi\wedge\neg\theta)\vee(\psi\wedge\theta)} = s_k \cap S_\psi \neq \emptyset.$$

But then k is minimal such that $s_k \cap S_{\psi\wedge\neg\theta} \neq \emptyset$ and $s_k \cap S_{\psi\wedge\neg\theta} = s_k \cap S_\psi \subseteq S_\varphi$ from (1) whilst $s_k \cap S_{\psi\wedge\neg\theta} \subsetneq S_\varphi$ from (2), contradiction. We conclude that this case is impossible and the required conclusion (3) is proved.

For the last part let $L = \{p, q, r\}$, $\psi = p$, $\theta = q$, $\varphi = r$ and

$$\vec{s} = \{p \wedge \neg q \wedge r\}, \{p \wedge q \wedge \neg r\}.$$

Then $p \mid\!\sim_{\vec{s}} r$ but $p \wedge q \mid\!\not\sim_{\vec{s}}$ so $\mid\!\sim_{\vec{s}}$ is an rcr for which this rule fails.

The Feedback

Generally very well done. One, somewhat inexplicable, error was to write $\psi \wedge \neg\theta \mid\!\sim \varphi$ rather than the correct $\psi \wedge \neg\theta \mid\!\not\sim \varphi$. This didn't cost any marks however because it was always used as the correct version.

Some students did lose a mark though in the last part by, for example, giving the counter-example $\mid\!\sim_{\vec{s}}$ as

$$\vec{s} = s_1, s_2 = \{\psi \wedge \neg\theta \wedge \varphi\},\ \{\psi \wedge \theta \wedge \neg\varphi\}.$$

The error here is that θ, φ, ψ are by convention sentences which means that $\psi \wedge \neg\theta \wedge \varphi$, for example, isn't necessarily an atom. To avoid this problem it would be enough to say 'let $\psi = p, \varphi = q, \theta = r$' (recall the question does say 'for *some* choice of θ, φ, ψ').