MATH43032/63032, May 2014 Exam, Solutions[1] and Feedback

Solutions

A1.

$$\theta \mid\!\sim_{\vec{s}} \phi \iff \begin{cases} \forall i \; s_i \cap S_\theta = \emptyset & \text{or} \\ \exists i \; s_i \cap S_\theta \neq \emptyset & \text{and for the least such } i,\ s_i \cap S_\theta \subseteq S_\phi. \end{cases}$$

The Representation Theorem for Rational Consequence Relations: Every rational consequence relation on SL is of the form $\mid\!\sim_{\vec{s}}$ for some $\vec{s} = s_1, s_2, \ldots, s_m \subseteq \mathrm{At}^L$, and conversely every $\mid\!\sim_{\vec{s}}$ is a rational consequence relation. Bookwork 4 marks

(i) True, (ii) Not true (iii) True. Similar seen 2 marks each

A2. (a) Assume $\theta \mid\!\sim \psi$ and $\theta \not\mid\!\sim \neg\phi$. Then by RMO, $\theta \wedge \phi \mid\!\sim \psi$ and by CON, $\theta \mid\!\sim (\phi \to \psi)$. By REF, $\psi \mid\!\sim \psi$ and since $\psi \models \phi \to \psi$, by RWE, $\psi \mid\!\sim (\phi \to \psi)$. By DIS then with $\theta \mid\!\sim (\phi \to \psi)$ we get $\theta \vee \psi \mid\!\sim (\phi \to \psi)$. Similar seen 7 marks

(b) As usual let $\alpha_1 = p \wedge q, \alpha_2 = p \wedge \neg q, \alpha_3 = \neg p \wedge q, \alpha_4 = \neg p \wedge \neg q$. Running the Z-algorithm we get:

$A_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}$,
$K_0 = K = \{ q \mid\!\sim p,\ p \mid\!\sim \neg q \}$.
$u_1 = A_0 \cap S_{\neg q \vee p} \cap S_{\neg p \vee \neg q}$
$\quad = \{\alpha_1, \alpha_2, \alpha_4\} \cap \{\alpha_2, \alpha_3, \alpha_4\} = \{\alpha_2, \alpha_4\}$.

$A_1 = A_0 - u_1 = \{ \alpha_1, \alpha_3 \}$,
$K_1 = \{ q \mid\!\sim p \}$, since $S_p \cap u_1 = \{ \alpha_1, \alpha_2 \} \cap \{ \alpha_2, \alpha_4 \} \neq \emptyset$,
$S_q \cap u_1 = \{ \alpha_1, \alpha_3 \} \cap \{ \alpha_2, \alpha_4 \} = \emptyset$,
$u_2 = A_1 \cap S_{p \vee \neg q} = \{ \alpha_1, \alpha_3 \} \cap \{ \alpha_1, \alpha_2, \alpha_4 \} = \{ \alpha_1 \}$.

$A_2 = A_1 - u_2 = \{ \alpha_3 \}$,
$K_2 = \emptyset$ since $u_2 \cap S_q = \{ \alpha_1 \} \neq \emptyset$,
$u_3 = A_2 = \{ \alpha_3 \}$.

All the atoms have now been used up so the rational closure of K is $\mid\!\sim_{\vec{u}}$ Similar seen 7 marks

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

where

$$\vec{u} = u_1, u_2, u_3 = \{ \alpha_2, \alpha_4 \}, \ \{ \alpha_1 \}, \ \{ \alpha_3 \}.$$

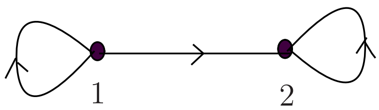
A3. *The Completeness Theorem for S_4*: For $\Gamma \subseteq SML$, $\theta \in SML$,

$$\Gamma \vdash^{S_4} \theta \iff \text{For all reflexive, transitive frames } \langle W, E, V \rangle \text{ and } i \in W, \text{ if } \langle W, E, V \rangle, i \models \phi \text{ for all } \phi \in \Gamma \text{ then } \langle W, E, V \rangle, i \models \theta.$$

Bookwork
2 marks

(i) Suppose that $\langle W, E, V \rangle$ is a reflexive, transitive frame and $i \in W$, $i \models \Diamond\Box\Diamond\theta$ (leaving the $\langle W, E, V \rangle$ implicit here). Then for some $\langle i, j \rangle \in E$, $j \models \Box\Diamond\theta$. Since the frame is reflexive, $\langle j, j \rangle \in E$ so also $j \models \Diamond\theta$. Hence for some $\langle j, k \rangle \in E$, $k \models \theta$. Since the frame is transitive and $\langle i, j \rangle, \langle j, k \rangle \in E$, $\langle i, k \rangle \in E$, so $i \models \Diamond\theta$, as required.

(ii) In the S_4 frame



$1 \models \Diamond p$ but $2 \not\models \Diamond p$ so $1 \not\models \Box\Diamond p$ and $2 \not\models \Box\Diamond p$. Hence $1 \not\models \Diamond\Box\Diamond p$. It follows that $\Diamond p \not\models^{S_4} \Diamond\Box\Diamond p$ and hence by the Completeness Theorem that $\Diamond p \nvdash^{S_4} \Diamond\Box\Diamond p$.

Similar seen
5 marks each part

A4. Writing as usual $x \vee y$ for $F_\vee(x, y)$,

(D1) $0 \vee 0 = 0, \quad 1 \vee 0 = 0 \vee 1 = 1$,

(D2) $\vee$ is continuous,

(D3) $\vee$ is increasing (not necessarily strictly) in each coordinate.

(D4) $\vee$ is associative, i.e. $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in [0, 1]$.

The classification is expressed by the Mostert-Shields Theorem for a continuous T-conorm: Let $F_\vee$ satisfy (D1-4) and let $A = \{ x \in [0, 1] \mid F_\vee(x, x) = x \}$.

Then for $x \in A$ and $0 \le z \le x \le y \le 1$,

$$F_\vee(z,y) = F_\vee(y,z) = z = \max\{y,z\},$$

and if $a < b$, $a,b \in A$ and $(a,b) \cap A = \emptyset$ then on $[a,b]$ either

$$\langle [a,b], F_\vee, < \rangle \cong \langle [0,1], x+y-xy, < \rangle$$

or

$$\langle [a,b], F_\vee, < \rangle \cong \langle [0,1], \min\{1, x+y\}, < \rangle,$$

with the latter of these holding just if $F_\vee(c,c) = b$ for some $a < c < b$. Bookwork 7 marks

Let $\vee$ satisfy (D1-4) and let $x \in [0,1]$. The function $0 \vee t$ is continuous by (D2) and $0 = 0 \vee 0 \le x \le 0 \vee 1 = 1$ by (D1) so by the Intermediate Value Theorem, $x = 0 \vee t$ for some $0 \le t \le 1$. Hence

$$0 \vee x = 0 \vee (0 \vee t) = (0 \vee 0) \vee t \text{ (by (D4))} \ = 0 \vee t \text{ (by (D1))} \ = x.$$

Similar seen 5 marks

A5. McNaughton's Theorem for $L = \{p\}$: A function $F : [0,1] \longrightarrow [0,1]$ is of the form F_θ for some $\theta \in SL$ iff there exist some $0 = \gamma_1 < \gamma_2 < \gamma_3 < ... < \gamma_{n-1} < \gamma_n = 1$ and $n_i, m_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n-1$, such that on each $[\gamma_i, \gamma_{i+1}]$, $F(x) = m_i + n_i x$ ($\in [0,1]$). Bookwork 3 marks

Let w be the $[0,1]$-valuation such that $w(p) = 1/n$ where $0 < n \in \mathbb{N}$ and let $F : [0,1] \to [0,1]$ be defined by

$$F(x) = \begin{cases} 1 - nx & \text{for } 0 \le x \le 1/n, \\ 0 & \text{for } 1/n \le x \le 1 - 1/n, \\ 1 - n + nx & \text{for } 1 - 1/n \le x \le 1. \end{cases}$$

By McNaughton's Theorem there is a $\phi \in SL$ such that $F_\phi = F$, so $w(\phi) = F_\phi(1/n) = F(1/n) = 0$. Also $F(0) = F(1) = 1 = F_\phi(0) = F_\phi(1)$ so $V_0(\phi) = 1 = V_1(\phi)$ for the valuations V_0, V_1 which give p truth values 0,1 respectively. But in this case V_0, V_1 are $\{0,1\}$-valuations in the sense of the Propositional Calculus and give the same value to ϕ in both Propositional Logic and Łukasiewicz Logic. Since these are the only two $\{0,1\}$-valuations on this language this means that ϕ is a tautology. New 5 marks

B6. By the Representation Theorem for Rational Consequence Relations it is enough to show that if $\theta \mid\!\sim_{\vec{s}} \phi \to \psi$ and $\theta \wedge \neg\phi \mid\!\sim_{\vec{s}} \neg\theta$ then $\theta \wedge \phi \mid\!\sim_{\vec{s}} \psi$, where $\vec{s} = s_1, s_2, \ldots, s_m \subseteq At^L$. So assume that $\theta \mid\!\sim_{\vec{s}} \phi \to \psi$ † and $\theta \wedge \neg\phi \mid\!\sim_{\vec{s}} \neg\theta$ ‡.

If $s_i \cap S_{\theta\wedge\phi} = \emptyset$ for all $i = 1, 2, \ldots, m$ we have $\theta \wedge \phi \mathrel{|\!\sim_{\vec{s}}} \psi$.

Otherwise let k be minimal such that $s_k \cap S_{\theta\wedge\phi} \neq \emptyset$. Clearly $s_k \cap S_\theta \supseteq s_k \cap S_{\theta\wedge\phi} \neq \emptyset$ since $S_{\theta\wedge\phi} = S_\theta \cap S_\phi$. If k is also minimal such that $s_k \cap S_\theta \neq \emptyset$ then $s_k \cap S_\theta \subseteq S_{\phi\to\psi}$ from † so

$$s_k \cap S_{\theta\wedge\phi} = s_k \cap S_\theta \cap S_\phi \subseteq S_{\phi\to\psi} \cap S_\phi = S_{\phi\wedge(\phi\to\psi)} \subseteq S_\psi$$

since $\phi \wedge (\phi \to \psi) \models \psi$.

On the other hand if the minimal j such that $s_j \cap S_\theta \neq \emptyset$ is less than k then, by choice of k, $s_j \cap S_{\theta\wedge\phi} = \emptyset$. Hence $s_j \cap S_\theta \cap S_\phi = \emptyset$ so $s_j \cap S_\theta \subseteq S_{\neg\phi}$ and $\emptyset \neq s_j \cap S_\theta \cap S_{\neg\phi} = s_j \cap S_{\theta\wedge\neg\phi}$. Clearly j is also minimal such that $\emptyset \neq s_j \cap S_{\theta\wedge\neg\phi}$ so by ‡ $s_j \cap S_{\theta\wedge\neg\phi} \subseteq S_{\neg\theta}$. But then

$$\emptyset \neq s_j \cap S_{\theta\wedge\neg\phi} = s_j \cap S_{\theta\wedge\neg\phi} \cap S_{\neg\theta} \subseteq S_\theta \cap S_{\neg\theta} = \emptyset,$$

contradiction.

So k must be minimal such that $s_k \cap S_{\theta\wedge\phi} \neq \emptyset$ and the result follows. Similar seen 9 marks

To show that the rule

$$\frac{\theta \mathrel{|\!\sim} \phi \to \psi}{\theta \wedge \phi \mathrel{|\!\sim} \psi}$$

fails for some rational consequence relation and choice of θ, ϕ, ψ let these sentences be the propositional variables p, q, r respectively and let

$$\vec{s} = \{p \wedge \neg q \wedge r\}, \{p \wedge q \wedge \neg r\}.$$

Then for the rational consequence relation $\mathrel{|\!\sim_{\vec{s}}}$, $p \mathrel{|\!\sim_{\vec{s}}} q \to r$ but not $p \wedge q \mathrel{|\!\sim} r$. Similar seen 3 marks

B7. A *formal proof* in the modal logic K is a finite sequence of sequents, $\Gamma_1 \,|\, \phi_1, \Gamma_2 \,|\, \phi_2, \ldots, \Gamma_m \,|\, \phi_m$, where the $\phi_i \in SML$ and the Γ_i are finite subsets of SML, such that for each $1 \leq i \leq m$ either $\Gamma_i \,|\, \phi_i$ is an instance of one of the axioms of K or there are some $j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1} \,|\, \phi_{j_1}, \Gamma_{j_2} \,|\, \phi_{j_2}, \ldots, \Gamma_{j_s} \,|\, \phi_{j_s}}{\Gamma_i \,|\, \phi_i}$$

is an instance of one of the rules of K. Bookwork 2 marks

Suppose that $\Gamma \vdash^K \theta$, say $\Gamma_1 \,|\, \phi_1, \Gamma_2 \,|\, \phi_2, \ldots, \Gamma_m \,|\, \phi_m$ is a proof of this, so $\phi_m = \theta$ and $\Gamma_m \subseteq \Gamma$. To show that $\Gamma \vdash^H \theta$ it is enough to show that the K

axiom can be derived in H and that the rule NEC can be derived in H since we can then replace any uses of these in the proof $\Gamma_1 \mid \phi_1, \Gamma_2 \mid \phi_2, \ldots, \Gamma_m \mid \phi_m$ by a derivation of them in H and hence obtain a proof of $\Gamma \vdash^H \theta$.

The K axiom can be derived in H by:

1	$\theta \to \phi, \theta$	|	$\theta \to \phi$	REF
2	$\theta \to \phi, \theta$	|	θ	REF
3	$\theta \to \phi, \theta$	|	ϕ	MP, 1, 2
4	$\Box(\theta \to \phi), \Box\theta$	|	$\Box\phi$	$H, 3$
5	$\Box(\theta \to \phi)$	|	$\Box\theta \to \Box\phi$	IMR, 3, 4

NEC, i.e.

$$\frac{|\theta}{|\Box\theta}$$

can derived in H since it is already a special case of the H rule. New marks 5

In the other direction it is similarly enough to show that any instance of the H rule,

$$\frac{\theta_1, \theta_2, \ldots, \theta_n \mid \phi}{\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_n \mid \Box\phi}$$

can be derived in K. The proof of this is by induction on n. If $n = 0$ the instance of the H rule is NEC so we already have it. Suppose now that $n > 0$ and we have the result for $n-1$. Then we can proceed to show the induction step via:

1	$\theta_1, \theta_2, \ldots, \theta_n$	|	ϕ	
2	$\theta_1, \theta_2, \ldots, \theta_{n-1}$	|	$\theta_n \to \phi$	IMR, 1
3	$\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_{n-1}$	|	$\Box(\theta_n \to \phi)$	IH
4	$\Box(\theta_n \to \phi)$	|	$\Box\theta_n \to \Box\phi$	K
5		|	$\Box(\theta_n \to \phi) \to (\Box\theta_n \to \Box\phi)$	IMR, 4
6	$\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_{n-1}$	|	$\Box\theta_n \to \Box\phi$	MP, 3, 5
7	$\Box\theta_n$	|	$\Box\theta_n$	REF
8	$\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_n$	|	$\Box\phi$	MP, 6, 7.

New marks 5

B8. (a) Assume that $\vdash^{\text{Ł}} \phi \to \chi$ and $\vdash^{\text{Ł}} \chi \to \xi$. By Ł2,

$$\vdash^{\text{Ł}} (\phi \to \chi) \to ((\chi \to \xi) \to (\phi \to \xi))$$

so by MP with $\vdash^{\text{Ł}} \phi \to \chi$,

$$\vdash^{\text{Ł}} (\chi \to \xi) \to (\phi \to \xi).$$

Again by MP with $\vdash^{\text{Ł}} \chi \to \xi$, $\vdash^{\text{Ł}} \phi \to \xi$.

(b) $\vdash^{\text{Ł}} \neg\psi \to (\neg\theta \to \neg\psi)$ follows directly from Ł1.

(c) By Ł3, $\vdash^{\text{Ł}} (\neg\theta \to \neg\psi) \to (\psi \to \theta)$ so with (b) & (a) $\vdash^{\text{Ł}} \neg\psi \to (\psi \to \theta)$.

(d) By Ł1, $\vdash^{\text{Ł}} \psi \to ((\theta \to \psi) \to \psi)$ so by MP and the given fact that $\vdash^{\text{Ł}} \psi$ we get that $\vdash^{\text{Ł}} (\theta \to \psi) \to \psi$.

(e) By Ł4,

$$\vdash^{\text{Ł}} ((\theta \to \psi) \to \psi) \to ((\psi \to \theta) \to \theta)$$

so by MP and (d), $\vdash^{\text{Ł}} (\psi \to \theta) \to \theta$.

(f) By (c), (e) & (a), $\vdash^{\text{Ł}} \neg\psi \to \theta$.

(g) By Ł1,

$$\vdash^{\text{Ł}} (\neg\psi \to \theta) \to ((\theta \to (\neg\psi \to \theta)) \to (\neg\psi \to \theta))$$

so by (f) and MP,

$$\vdash^{\text{Ł}} (\theta \to (\neg\psi \to \theta)) \to (\neg\psi \to \theta).$$

(h) By Ł4,

$$\vdash^{\text{Ł}} ((\theta \to (\neg\psi \to \theta)) \to (\neg\psi \to \theta)) \to (((\neg\psi \to \theta) \to \theta) \to \theta)$$

so by (g) and MP, $\vdash^{\text{Ł}} ((\neg\psi \to \theta) \to \theta) \to \theta$, i.e. $\vdash^{\text{Ł}} (\neg\psi \underline{\vee} \theta) \to \theta$. Similar seen 1 or 2 marks each part

Feedback

Generally the exam was done very well, clearly most students had put a lot of effort into this course. Some comments on individual questions:

A1 In parts (i)-(iii) it was enough just to say 'true' or 'false', spending time writing out a reason for your answer couldn't help you gain extra marks.

A2 Both parts were well done with many students scoring full marks.

A3 I was concerned that maybe this question was too hard but in the event most students 'breezed it' without even the need for any rough work.

A4 Also well done, clearly a question on a Mostert-Shields Theorem wasn't too much of a surprise. Indeed I had the impression that some students had learnt the version for $F_\wedge$ by heart because there was a tendency for the solution to slip back into this version rather than persisting with $F_\vee$!

A5 Whilst nearly all students knew what McNaughton's Theorem said far fewer actually understood its significance with only 2 or 3 students getting out the last part. If you weren't one of them have a glance at the model answer (and kick yourself!).

B6 Pretty much every student chose this question from Section B (well, it was hardly a surprise) but as usual many of the purported solutions were 'muddled'. In general the safest way to approach a question like this where you have to show $\theta \wedge \phi \mathrel{|\!\sim_{\vec{s}}} \psi$ is, after you've disposed of the trivial case $\forall i\, s_i \cap S_{\theta\wedge\phi} = \emptyset$, to consider the least i such that $s_i \cap S_{\theta\wedge\phi} \neq \emptyset$. This approach may not always give the slickest solution but it helps avoid the tangle of assumptions and cases that some students lumbered into.

A few students didn't read the question ('By using the Representation Theorem . . .) and threw away any marks by trying to give a derivation from the GM Rules.

The last part, the counter-example, was well done though many students wasted time showing in detail that their counter-example worked.

B7 There were a fair number of takers for this question though in many cases the student didn't appreciate what was required and (memories of previous years' questions?) somehow attempted to prove a Completeness Theorem for H.

B8 Despite this material appearing late on in the course a pleasing number of students scored well on this question, sometimes introducing interesting innovations on the anticipated model answer route. Sadly some students didn't take any notice of the 'without using the Completeness Theorem for

Ł' and threw away marks and time by giving semantic arguments to show $\models^{Ł}$ rather than using Proposition 8 to show $\vdash^{Ł}$ directly.