# MATH43032/63032 Exam Solutions, 2013[1]

**A1.**

$$\theta \mid\!\sim_{\vec{s}} \phi \iff \begin{cases} \forall i \;\; s_i \cap S_\theta = \emptyset & \text{or} \\ \exists i \;\; s_i \cap S_\theta \neq \emptyset & \text{and for the least such } i,\; s_i \cap S_\theta \subseteq S_\phi. \end{cases}$$

*The Representation Theorem for Rational Consequence Relations*: Every rational consequence relation on SL is of the form $\mid\!\sim_{\vec{s}}$ for some $\vec{s} = s_1, s_2, \ldots, s_m \subseteq \mathrm{At}^L$, and conversely every $\mid\!\sim_{\vec{s}}$ is a rational consequence relation.

(i) Not true, (ii) True (iii) True.

**A2.** (a) Assume $\theta \wedge \phi \mid\!\sim \neg\psi$ and $\theta \not\mid\!\sim \neg\psi$. From $\theta \wedge \phi \mid\!\sim \neg\psi$ by CON, $\theta \mid\!\sim \phi \to \neg\psi$ and with $\theta \not\mid\!\sim \neg\psi$ and RMO, $\theta \wedge \psi \mid\!\sim \phi \to \neg\psi$. Since $\theta \wedge \psi \models \psi$, $\theta \wedge \psi \mid\!\sim \psi$ by SCL. Hence, with $\theta \wedge \psi \mid\!\sim \phi \to \neg\psi$. and AND, $\theta \wedge \psi \mid\!\sim \psi \wedge (\phi \to \neg\psi)$, and, since $\psi \wedge (\phi \to \neg\psi) \models \neg\phi$, with RWE, $\theta \wedge \psi \mid\!\sim \neg\phi$, as required.

(b) As usual let $\alpha_1 = p \wedge q, \alpha_2 = p \wedge \neg q, \alpha_3 = \neg p \wedge q, \alpha_4 = \neg p \wedge \neg q$. Running the Z-algorithm we get:

$A_0 = \{\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \,\}$,
$K_0 = K = \{\, p \vee q \mid\!\sim \neg p,\ \neg q \mid\!\sim p \,\}$.
$u_1 = A_0 \cap S_{\neg(p\vee q)\vee\neg p} \cap S_{\neg\neg q \vee p} = A_0 \cap S_{\neg p} \cap S_{q \vee p}$
$\quad = \{\alpha_3, \alpha_4\} \cap \{\alpha_1, \alpha_2, \alpha_3\} = \{\alpha_3\}$.

$A_1 = A_0 - u_1 = \{\, \alpha_1, \alpha_2, \alpha_4 \}$,
$K_1 = \{\, \neg q \mid\!\sim p, \}$, since $S_{p\vee q} \cap u_1 = \{\, \alpha_1, \alpha_2, \alpha_3 \,\} \cap \{\, \alpha_3 \,\} \neq \emptyset$,
$S_{\neg q} \cap u_1 = \{\, \alpha_2, \alpha_4 \,\} \cap \{\, \alpha_3 \,\} = \emptyset$,
$u_2 = A_1 \cap S_{\neg\neg q \vee p} = \{\, \alpha_1, \alpha_2, \alpha_4 \,\} \cap \{\, \alpha_1, \alpha_2, \alpha_3 \,\} = \{\, \alpha_1, \alpha_2 \,\}$.

$A_2 = A_1 - u_2 = \{\, \alpha_4 \}$,
$K_2 = \emptyset$ since $u_2 \cap S_{\neg q} = \{\, \alpha_2 \,\} \neq \emptyset$.
$u_3 = A_2 = \{\, \alpha_4 \,\}$

Since all the atoms have now been used up we must have $u_4 = \emptyset$ so the rational closure of $K$ is $\mid\!\sim_{\vec{u}}$ where

$$\vec{u} = u_1, u_2, u_3 = \{\, \alpha_3 \,\},\ \{\, \alpha_1, \alpha_2 \,\},\ \{\, \alpha_4 \,\}.$$

**A3.** For $\Gamma \subseteq SML, \theta \in SML$, $\Gamma \models^K \theta$ iff for all frames $\langle W, E, V\rangle$ and $i \in W$, if $\langle W, E, V\rangle, i \models \phi$ for all $\phi \in \Gamma$ then $\langle W, E, V\rangle, i \models \theta$.

(i) Let $\langle W, E, V\rangle$ be a frame, $i \in W$ and suppose that (in $\langle W, E, V\rangle$) $i \models \neg\Box\Diamond\theta$. Then there must be some $j \in W$ such that $\langle i, j\rangle \in E$ and $j \not\models \Diamond\theta$. If $i \models \Box\theta$ then $j \models \theta$ so $i \models \Diamond\theta$ and $i \models \neg\Box\theta \vee \Diamond\theta$. On the other hand if $i \not\models \Box\theta$ then $i \models \neg\Box\theta$ and again $i \models \neg\Box\theta \vee \Diamond\theta$. Either way then $i \models \neg\Box\theta \vee \Diamond\theta$ and $\neg\Box\Diamond\theta \models^K \neg\Box\theta \vee \Diamond\theta$ follows.

(ii) Let $\langle W, E, V\rangle$ be the frame with $W = \{0, 1, 2\}$, $E = \{\langle 0, 1\rangle, \langle 0, 2\rangle\}$, $V_0(p) = V_0(q) = 1$, $V_1(p) = 0, V_1(q) = 1$, $V_2(p) = 1, V_2(q) = 0$. Then $0 \not\models \Box q$, since $\langle 0, 2\rangle \in E$ and $2 \not\models q$, so

$$1 \models \neg\Box q. \tag{1}$$

[1] As usual these are more detailed than I would necessarily require from the students since they are also intended to serve an instructional purpose for the students.

Also $0 \models \Box(p \vee q)$, since $1 \models q$, $2 \models p$ so at both vertices accessible from 0, $p \vee q$ holds, and $0 \not\models \Box p$ since $\langle 0, 1 \rangle \in E$ and $1 \not\models p$. So

$$0 \not\models \Box(p \vee q) \to \Box p.$$

Putting this together with (1) gives that $\neg\Box q \not\models^K \Box(p \vee q) \to \Box p$.

**A4.** A *proof* in $D$ is a sequence of sequents $\Gamma_1|\theta_1, \Gamma_2|\theta_2, \ldots, \Gamma_m|\theta_m$, where the $\Gamma_i$ are finite subsets of $SML$ and the $\theta_i \in SML$, such that for each $i = 1, 2, \ldots, m$, either $\Gamma_i|\theta_i$ is an instance of an axiom of $D$ or $\exists j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1}|\theta_{j_1}, \Gamma_{j_2}|\theta_{j_2}, \ldots, \Gamma_{j_s}|\theta_{j_s}}{\Gamma_i|\theta_i}$$

is an instance of a rule of proof of $D$.

$\Gamma \vdash^D \theta$ if $\exists$ a proof in $D$, $\Gamma_1|\theta_1, \Gamma_2|\theta_2, \ldots, \Gamma_m|\theta_m$, such that $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$.

Proof in $D$ of $\vdash^D \Box\Box\theta \to \Box\Diamond\theta$:

| | | | |
|---|---|---|---|
| 1 | $\Box\theta$ | $\mid \Diamond\theta$ | $D$ axiom |
| 2 | | $\mid (\Box\theta \to \Diamond\theta)$ | IMR, 1 |
| 3 | | $\mid \Box(\Box\theta \to \Diamond\theta)$ | NEC, 2 |
| 4 | $\Box(\Box\theta \to \Diamond\theta)$ | $\mid \Box\Box\theta \to \Box\Diamond\theta$ | $K$ axiom |
| 5 | | $\mid \Box(\Box\theta \to \Diamond\theta) \to (\Box\Box\theta \to \Box\Diamond\theta)$ | IMR 4 |
| 6 | | $\mid \Box\Box\theta \to \Box\Diamond\theta$ | MP 3, 5. |

**A5.** Let $\wedge$ ($= F_\wedge$) satisfy (C1-C4) and let $A = \{\, x \in [0,1] \mid x \wedge x = x \,\}$. Then for $x \in A$ and $0 \le z \le x \le y \le 1$,

$$z \wedge y = y \wedge z = z = \min\{y, z\},$$

and if $a < b$, $a, b \in A$ and $(a, b) \cap A = \emptyset$ then on $[a, b]$ either $\langle [a,b], \wedge, < \rangle \cong \langle [0,1], \times, < \rangle$ or $\langle [a,b], \wedge, < \rangle \cong \langle [0,1], \max\{0, x+y-1\}, < \rangle$.

Let $G : [0,1]^2 \to [0,1]$ be defined by

$$G(x, y) = \min\{x, y, 2xy\}.$$

and assume that satisfies C1-C4. Notice that for $x \in [0,1]$, $2x^2 < x$ iff $0 < x < \frac{1}{2}$, so

$$G(x, x) = \min\{x, x, 2x^2\} = x \iff x = 0 \ \text{ or } \ \tfrac{1}{2} \le x \le 1$$

and

$$A = \{x \in [0,1] \mid G(x,x) = x\} = \{0\} \cup [\tfrac{1}{2}, 1].$$

By the above result then $G(x, y) = \min\{x, y\}$ whenever either $x \in [\frac{1}{2}, 1]$ or $y \in [\frac{1}{2}, 1]$. On $[0, \frac{1}{2}]$, $\langle [0, \frac{1}{2}], G, < \rangle$ is isomorphic to either $\langle [0,1], \times, < \rangle$ or to $\langle [0,1], \max\{0, x+y-1\}, < \rangle$ and since $G(x,x) = 2x^2 > 0$ for $x \in (0, \frac{1}{2}]$ it must be the former of these since the corresponding property fails for $\langle [0,1], \max\{0, x+y-1\}, < \rangle$.

**A6.** For $\Gamma \subseteq SL$, $\theta \in SL$, $\Gamma \models^{\text{Ł}} \theta$ if for all $[0,1]$-valuations $w$, if $w(\phi) = 1$ for all $\phi \in \Gamma$ then $w(\theta) = 1$.

The *Completeness Theorem* for Ł states that for $\Gamma \subseteq SL$, $\Gamma$ finite, and $\theta \in SL$, $\Gamma \models^{\text{Ł}} \theta$ iff $\Gamma \vdash^{\text{Ł}} \theta$.

(i) Let $w(p) = \frac{1}{2}$, $w(q) = \frac{1}{3}$, so $w(\neg p) = 1 - \frac{1}{2} = \frac{1}{2}$. Recalling that $w(\theta \to \phi) = 1 \iff w(\theta) \leq w(\phi)$ we have that $w(p \to q) < 1$, $w(\neg p \to q) < 1$ so

$$w((p \to q)\underline{\vee}(\neg p \to q)) = \max\{w(p \to q), (\neg p \to q)\} < 1$$

Hence $\not\models^{\text{Ł}} (p \to q)\underline{\vee}(\neg p \to q)$ and by the Completeness Theorem $\nvdash^{\text{Ł}} (p \to q)\underline{\vee}(\neg p \to q)$ .

(ii) Let $w$ be a $[0,1]$-valuation. Suppose that

$$w((p \to q) \vee (\neg p \to q)) = \min\{w(p \to q) + w(\neg p \to q), 1\} < 1.$$

Then

$$w((p \to q) \vee (\neg p \to q)) = w(p \to q) + w(\neg p \to q) < 1 \tag{2}$$

and both $w(p \to q), w(\neg p \to q) < 1$, so

$$w(p \to q) = 1 - w(p) + w(q),$$

$$w(\neg p \to q) = 1 - w(\neg p) + w(q) = 1 - (1 - w(p)) + w(q) = w(p) + w(q).$$

But then

$$w(p \to q) + w(\neg p \to q) = (1 - w(p) + w(q)) + (w(p) + w(q)) = 1 + 2w(q) \geq 1,$$

contradicting (2). Hence we must have $w((p \to q) \vee (\neg p \to q)) = 1$ and since $w$ was arbitrary,

$$\models^{\text{Ł}} (p \to q) \vee (\neg p \to q).$$

By the Completeness Theorem then

$$\vdash^{\text{Ł}} (p \to q) \vee (\neg p \to q).$$

**B7.** Let $\vec{s} = s_1, s_2, \ldots, s_m$ and assume that $\theta \mathrel{|\!\sim_{\vec{s}}} \phi$ and $\phi \wedge \neg\psi \mathrel{|\!\sim_{\vec{s}}} \psi$. By the Representation Theorem it is enough to show that $\theta \mathrel{|\!\sim_{\vec{s}}} \psi$. If $s_i \cap S_\theta = \emptyset$ for all $i = 1, \ldots, m$ then $\theta \mathrel{|\!\sim_{\vec{s}}} \psi$. So assume that for some $i$, $s_i \cap S_\theta \neq \emptyset$, say $i$ is the least such. Then from $\theta \mathrel{|\!\sim_{\vec{s}}} \phi$, $s_i \cap S_\theta \subseteq S_\phi$.

Suppose that $\theta \mathrel{|\!\not\sim_{\vec{s}}} \psi$. Then $s_i \cap S_\theta \subsetneq S_\psi$ so for some $\alpha \in s_i \cap S_\theta$, $\alpha \notin S_\psi$. $\therefore$ $\alpha \in S_{\neg\psi}$. Also $\alpha \in S_\phi$ since $s_i \cap S_\theta \subseteq S_\phi$ so $\alpha \in S_\phi \cap S_{\neg\psi} = S_{\phi\wedge\neg\psi}$. Hence $s_i \cap S_{\phi\wedge\neg\psi} \neq \emptyset$. Let $j$ be minimal such that $s_j \cap S_{\phi\wedge\neg\psi} \neq \emptyset$. Then from $\phi \wedge \neg\psi \mathrel{|\!\sim_{\vec{s}}} \psi$, $s_j \cap S_{\phi\wedge\neg\psi} \subseteq S_\psi$. But clearly $s_j \cap S_{\phi\wedge\neg\psi} \subseteq S_{\neg\psi}$ so we have

$$\emptyset \neq s_j \cap S_{\phi\wedge\neg\psi} \subseteq S_\psi \cap S_{\neg\psi} = S_\psi \cap (At^L - S_\psi) = \emptyset,$$

contradiction. We conclude that $\theta \mathrel{|\!\not\sim_{\vec{s}}} \psi$ is false, i.e. $\theta \mathrel{|\!\sim_{\vec{s}}} \psi$, as required.

For the second part let $L = \{p, q, r\}$, $\theta = p$, $\phi = q$, $\psi = r$ and $\vec{s} = \{\neg p \wedge q \wedge r\}, \{p \wedge q \wedge \neg r\}$. Then $\theta \wedge \neg\phi \mathrel{|\!\sim_{\vec{s}}} \phi$ and $\phi \mathrel{|\!\sim_{\vec{s}}} \psi$ but $\theta \mathrel{|\!\not\sim_{\vec{s}}} \psi$ so the rule fails for this choice of rcr and $\theta, \phi, \psi$.

**B8.** The proof is by induction on $n \in \mathbb{N}$ such that $\theta \in SML_n$. If $n = 0$ then $\theta = p$ for some $p \in L$ and

$$\begin{aligned}
\langle W, E, V\rangle,\ w \models \theta &\iff V_w(p) = 1 \\
&\iff V'_w(p) = 1, \quad \text{since } V_w = V'_w, \\
&\iff \langle W', E', V'\rangle, w \models \theta.
\end{aligned}$$

Now suppose that $\theta \in SML_{n+1} - SML_n$ and the result holds for $\phi \in SML_n$. If $\theta = (\phi \wedge \psi)$ with $\phi, \psi \in SML_n$ then

$$\begin{aligned}
\langle W, E, V\rangle, w \models \theta &\iff \langle W, E, V\rangle, w \models \phi \text{ and } \langle W, E, V\rangle, w \models \psi \\
&\iff \langle W', E', V'\rangle, w \models \phi \text{ and } \langle W', E', V'\rangle, w \models \psi \\
&\qquad \text{by Inductive Hypothesis,} \\
&\iff \langle W', E', V'\rangle, w \models \theta.
\end{aligned}$$

The cases for $\theta = \neg\phi, (\phi \vee \psi), (\phi \to \psi)$ are exactly similar.

Finally if $\theta = \Box\phi$ with $\phi \in SML_n$ then

$$\begin{aligned}
\langle W, E, V\rangle, w \models \theta &\iff \forall y \in W \text{ with } \langle w, y\rangle \in E,\ \langle W, E, V\rangle, y \models \phi \\
&\iff \forall y \in W \text{ with } \langle w, y\rangle \in E,\ \langle W', E', V'\rangle, y \models \phi, \\
&\qquad \text{by Inductive Hypothesis,} \\
&\iff \forall y \in W' \text{ with } \langle w, y\rangle \in E',\ \langle W', E', V'\rangle, y \models \phi, \\
&\qquad \text{since for } w \in W,\ \{y \in W \mid \langle w, y\rangle \in E\} = \{y \in W' \mid \langle w, y\rangle \in E'\}, \\
&\iff \langle W', E', V'\rangle, w \models \theta,
\end{aligned}$$

which concludes the proof that for $w \in W$, $\theta \in SML$,

$$\langle W, E, V\rangle,\ w \models \theta \iff \langle W', E', V'\rangle, w \models \theta. \tag{3}$$

To show the last part suppose that $\models^D \Diamond\theta$ but not $\models^D \theta$. Then there is a $D$-frame $\langle W, E, V\rangle$ (i.e. serial frame) and $w \in W$ such that

$$\langle W, E, V\rangle,\ w \not\models \theta.$$

Let $v \notin W$ and $\langle W', E', V'\rangle$ be as above (with $V'_v$ chosen arbitrarily). Then $\langle W', E', V'\rangle$ is serial so since $\vdash^D \Diamond\theta$, by the Completeness Theorem for $D$,

$$\langle W', E', V'\rangle, v \models \Diamond\theta.$$

But since $w$ is the only vertex in $W'$ accessible from $v$ it must be the case that

$$\langle W', E', V'\rangle, w \models \theta,$$

and hence by (3),

$$\langle W, E, V\rangle, w \models \theta,$$

– contradiction. Hence we must have $\vdash^D \theta$, as required.

**B9.** (i) If $\psi$ is an axiom of $\models^{Ł}$ then $\emptyset \mid \psi$ is a proof, so $\vdash^{Ł} \psi$.

(ii) Suppose that $\vdash^{Ł} \psi$ and $\vdash^{Ł} (\psi \to \eta)$, say $\Delta_1 \mid \phi_1, \Delta_2 \mid \phi_2, \ldots, \Delta_m \mid \phi_m$ and $\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_r \mid \theta_r$ are, respectively, proofs of these, so $\Delta_m = \Gamma_r = \emptyset$, $\phi_m = \psi$, $\theta_r = (\psi \to \eta)$. Then

$$\Delta_1 \mid \phi_1, \Delta_2 \mid \phi_2, \ldots, \Delta_m \mid \phi_m, \Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_r \mid \theta_r, \emptyset \mid \eta$$

is a proof of $\emptyset \models^{Ł} \eta$, this last sequent $\emptyset \mid \eta$ being justified from $\Delta_m \mid \phi_m$ $(= \emptyset \mid \psi)$ and $\Gamma_r \mid \theta_r$ $(= \emptyset \mid (\psi \to \eta))$ by MP.

(iii) From axiom Ł1 and (i)

$$\vdash^{Ł} (\theta \to \theta) \to ((\theta \to (\theta \to \theta)) \to (\theta \to \theta)).$$

From this and the instance $\vdash^{Ł} (\theta \to \theta)$ of (a1), using (ii),

$$\vdash^{Ł} (\theta \to (\theta \to \theta)) \to (\theta \to \theta). \tag{4}$$

(iv) From axiom Ł4, using (i),

$$\vdash^{Ł} ((\theta \to (\theta \to \theta)) \to (\theta \to \theta)) \to (((\theta \to \theta) \to \theta) \to \theta)$$

so with (4) using (ii),

$$\vdash^{Ł} (\theta \to \theta) \to \theta) \to \theta)). \tag{5}$$

## Feedback

**A1.** Very well done, almost everyone got full marks.

**A2.** Rather few students got out part (a). What should have been a hint to you was having $\theta \not\mid\!\sim \neg\psi$ in the premises. The only way you can use this is with RMO and another premiss of the form $\theta \mid\!\sim \zeta$ for some $\zeta$. At the start we don't have such a premiss, but we can obtain one by applying CON to $\theta \wedge \phi \mid\!\sim \neg\psi$, giving $\theta \mid\!\sim \phi \to \neg\psi$, and now applying RMO gives $\theta \wedge \psi \mid\!\sim \phi \to \neg\psi$. We're now nearly there since $\theta \wedge \psi \mid\!\sim \psi$ (via SCL) and with AND,

$$\theta \wedge \psi \mid\!\sim \psi \wedge (\phi \to \neg\psi) \models \neg\phi \quad \text{etc.}$$

Part (b) was well done, most scored highly on this.

**A3.** The definition was done well but part (i) proved tricky. The point was that the left hand side holding at $i$ in a frame meant that there must be some $j \in W$ such that $\langle i, j \rangle \in E$. Then if $j \models \theta$ we have $i \models \Diamond\theta$ whilst if $j \models \neg\theta$ then $i \models \Diamond\neg\theta$, equivalently $i \models \neg\Box\theta$.

Part (ii) was generally well done except that some students didn't make explicit what their proposed frame, neither giving it as $\langle W, E, V \rangle$ nor putting in the arrows on the edges in their graph.

**A4.** A lot of students lost marks by omitting to say that the $\Gamma_i$ need to be finite in the definition of a proof. Apart from that the question was mostly done well, though an error which occurred a few times was to apply NEC to a sequent with a non-empty left hand side. The NEC rule doesn't allow this.

**A5.** Not very well done, no one got full marks. Several students wrote down C1-C4. This wasn't asked for in the question, so was just a waste of time. Most students did know what the Mostert-Shields Theorem said but then failed to correctly work out the set of $x \in [0,1]$ for which $G(x,x) = x$, usually taking it, incorrectly, to be $\{0, \frac{1}{2}, 1\}$.

At this point a common error was to go on to say that the structure $\langle [0, \frac{1}{2}], G, < \rangle$ *was* either $\langle [0, \frac{1}{2}], \times, < \rangle$ or $\langle [0, \frac{1}{2}], \max\{0, x+y-1\}, < \rangle$. What they should have said was that the structure $\langle [0, \frac{1}{2}], G, < \rangle$ *was isomorphic to* either $\langle [0,1], \times, < \rangle$ or $\langle [0,1], \max\{0, x+y-1\}, < \rangle$. Overall then whilst most students knew the words in the Mostert-Shields Theorem they often didn't actually understand it.

**A6.** Average mark for this question was around 6 out of 10. A surprisingly common error in the definition of $\vdash^{Ł}$ was to start with 'There exists a $[0,1]$-valuation $w$ . . . ' rather than 'For all $[0,1]$-valuations $w$ . . . '. In the statement of the Completeness Theorem for Ł the requirement that $\Gamma$ be finite was frequently omitted.

Part (i) was done well, and so too was part (ii) although the choice of cases wasn't always the most efficient.

**B7.** Most students scored well on this question but few actually scored full marks. The common error was to arrive at the point where you had $i$ minimal such that $s_i \cap S_\theta \neq \emptyset$, so $s_i \cap S_\theta \subseteq S_\phi$, and you supposed that $s_i \cap S_\theta \not\subseteq S_\psi$. Then there must be some $\alpha \in s_i \cap S_\theta$ such that $\alpha \in S_{\neg\psi}$ and of course $\alpha \in S_\phi$ so $\alpha \in s_i \cap S_\phi \cap S_{\neg\psi}$. The mistake now was to assume that this same $i$ was minimal such that $s_i \cap S_\phi \cap S_{\neg\psi} \neq \emptyset$, so from the premise $\phi \wedge \neg\psi \mid\!\sim \psi$ this gave

$$\emptyset \neq s_i \cap S_\phi \cap S_{\neg\psi} \subseteq S_\psi,$$

which as you noted is impossible, so the assumption $s_i \cap S_\theta \not\subseteq S_\psi$ must be wrong. What you should have argued was that since $s_i \cap S_\phi \cap S_{\neg\psi} \neq \emptyset$ there must be a least $j$ such that $s_j \cap S_\phi \cap S_{\neg\psi} \neq \emptyset$ and from that

$$\emptyset \neq s_j \cap S_\phi \cap S_{\neg\psi} \subseteq S_\psi,$$

again giving the required contradiction.

The second part of the question was well done, except some students used sentences $\theta \wedge \phi \wedge \neg\psi$ as elements of the $s_i$ where they should have been using atoms such as $p \wedge q \wedge \neg r$.

**B8.** The first part was mostly well done although some students wrote $V_w(\theta) = 1$ instead of $i \models \theta$ (etc.) even when $\theta$ is not a propositional variable. In modal logic $V_w$ only gets defined on propositional variables. Whilst I was vexed by this misuse I didn't take any marks off for it. In the second part some students wrote rather a lot without actually getting anywhere – if you don't see how to answer a question it is best to go to another question rather than just ramble on like a monkey at a typewriter hoping you'll say something that I'm looking for!

**B9.** Parts (iii),(iv) were done rather well but unfortunately almost no one understood what was being asked for in (i) and (ii) – essentially I wanted you to give the proof of Proposition 8 in the Real Valued Logics notes. So instead of saying that if $\psi$ is an instance of an axiom then $\mid \psi$ is a proof of $\vdash^{Ł} \psi$ I got answers involving the Completeness Theorem – whose proof is even too long and complicated to be included in this course!