MATH43032/63032 Solutions, 2008-09

1. For $\theta, \phi \in SL$,

$\theta \mid\!\sim_{\vec{s}} \phi \iff \forall i \; s_i \cap S_\theta = \emptyset$ or $\exists i \; s_i \cap S_\theta \neq \emptyset$ and for the least such i, $s_i \cap S_\theta \subseteq S_\phi$.

Representation Theorem for Rational Consequence Relations:
Every rcr is of the form $\mid\!\sim_{\vec{s}}$ for some $\vec{s}$ and conversely, for every $\vec{s}$, $\mid\!\sim_{\vec{s}}$ is a rcr.

(i) True (ii) False (iii) True

2. From $\neg\phi \mid\!\sim \psi$ and $\neg\phi \not\mid\!\sim \neg\theta$ we have by RMO $\neg\phi \wedge \theta \mid\!\sim \psi$. Consequently, since $\psi \models \phi \vee \psi$, by RWE,

$$\neg\phi \wedge \theta \mid\!\sim \phi \vee \psi \qquad (*)$$

By REF $\phi \wedge \theta \mid\!\sim \phi \wedge \theta$ and since $\phi \wedge \theta \models \phi \vee \psi$ by RWE,

$$\phi \wedge \theta \mid\!\sim \phi \vee \psi \qquad (\dagger)$$

By * and † and DIS,

$$(\phi \wedge \theta) \vee (\neg\phi \wedge \theta) \mid\!\sim \phi \vee \psi$$

and finally $\theta \mid\!\sim \phi \vee \psi$ by LLE since

$$(\phi \wedge \theta) \vee (\neg\phi \wedge \theta) \equiv \theta.$$

3. By the Representation Theorem it is enough to show that the rule holds for $\mid\!\sim_{\vec{s}}$. So assume $\theta \vee \phi \mid\!\sim_{\vec{s}} \neg\theta$. If $s_i \cap S_{\psi\vee\phi} = \emptyset$ for all i then we are done. Otherwise let i be minimal such that $s_i \cap S_{\psi\vee\phi} \neq \emptyset$. So $s_j \cap S_\phi \subseteq s_j \cap S_{\psi\vee\phi} = \emptyset$ for $j < i$. Suppose $s_i \not\subseteq S_{\neg\theta}$, i.e. $s_i \cap S_\theta \neq \emptyset$. Let j be minimal such that $S_j \cap S_{\theta\vee\phi} \neq \emptyset$. Certainly $j \leq i$ since $\emptyset \neq s_i \cap S_\theta \subseteq s_i \cap S_{\theta\vee\phi}$. If $j < i$ then

$$\emptyset \neq s_j \cap S_{\theta\vee\phi} = s_j \cap (S_\theta \cap S_\phi) = (s_j \cap S_\theta) \cup (s_j \cap S_\phi) = s_j \cap S_\theta$$

since $s_j \cap S_\phi = \emptyset$. Either way then $s_j \cap S_\theta \neq \emptyset$. But this is impossible since from $\theta \vee \phi \mid\!\sim_{\vec{s}} \neg\theta$ we have

$$\emptyset \neq s_j \cap S_\theta \subseteq s_j \cap S_{\theta\vee\phi} \subseteq S_{\neg\theta} = At^L - S_\theta.$$

Consequently, $s_i \subseteq S_{\neg\theta}$ so $s_i \cap S_{\psi\vee\phi} \subseteq S_{\neg\theta}$ and $\psi \vee \phi \mid\!\sim_{\vec{s}} \neg\theta$.

For the second part let $L = \{p, q, r\}$, $s_1 = \{p \wedge \neg q \wedge \neg r\}, s_2 = \{p \wedge q \wedge r\}$, $\vec{s} = s_1, s_2$. Then $q \vee p \mid\!\sim_{\vec{s}} \neg q$ but $r \wedge p \not\mid\!\sim_{\vec{s}} \neg q$ so the 'rule'

$$\frac{\theta \vee \phi \mid\!\sim \neg\theta}{\psi \wedge \phi \mid\!\sim \neg\theta}$$

fails for $\mid\!\sim_{\vec{s}}$ with $\theta = q,\ \phi = p,\ \psi = r.$

4. For $\Gamma \subseteq SML, \theta \in SML$,

$$\Gamma \models^{S_4} \theta \iff \text{for all reflexive and transitive frames } \langle W, E, V\rangle \text{ and } i \in W, \text{ if } i \models \Gamma \text{ then } i \models \theta.$$

Completeness Theorem for S_4: For $\Gamma \subseteq SML, \theta \in SML$,

$$\Gamma \models^{S_4} \theta \iff \Gamma \vdash^{S_4} \theta.$$

(i) Let $\langle W, E, V\rangle$ be a reflexive and transitive frame and let $i \in W$. If $i \not\models \Diamond\theta$ then for any $(i,k) \in E$, $k \models \neg\theta$. Let $(i,j),(j,k) \in E$. By transitivity $(i,k) \in E$ so $k \models \neg\theta$. Since this holds for any $k \in W$ with $(j,k) \in E$, $j \models \Box\neg\theta$. Again since this must hold for any such $j \in W$ with $(i,j) \in E$, $i \models \Box\Box\neg\theta$. Either way then $i \models \Diamond\theta \vee \Box\Box\neg\theta$ so $\models^{S_4} \Diamond\theta \vee \Box\Box\neg\theta$

(ii) Consider the transitive and reflexive frame $\langle W, E, V\rangle$ with $W = \{0,1\}$, $E = \{(0,0),(0,1),(1,1)\}$ and $V_0(p) = 0$, $V_1(p) = 1$. Then $0 \not\models p$, since $V_0(p) = 0$. Also $1 \not\models \Box\neg p$ since $(1,1) \in E$ and $1 \not\models \neg p$, so $0 \not\models \Box\Box\neg p$. Hence $0 \not\models p \vee \Box\Box\neg p$ so $\not\models^{S_4} p \vee \Box\Box\neg p$.

5. A *proof* in B is a finite sequence of sequents

$$\Gamma_1 |\, \theta_1, \ldots, \Gamma_m |\, \theta_m,$$

where $\theta_i \in SML$ and $\Gamma_i \subseteq SML$ are finite, such that for each $i = 1, \ldots, m$, either $\Gamma_i |\, \theta_i$ is an instance of an axiom of B, or for some $j_1, \ldots, j_s < i$

$$\frac{\Gamma_{j_1} |\, \theta_{j_1} \quad \ldots \quad \Gamma_{j_s} |\, \theta_{j_s}}{\Gamma_i |\, \theta_i}$$

is an instance of one of the rules NEC or AND-MON from SC.

$\Gamma \vdash^B \theta \iff$ there is a proof $\Gamma_1 |\, \theta_1, \ldots, \Gamma_m |\, \theta_m$ in B such that $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$.

Formal proof of $\vdash^B \Box\theta \to \Box\Box\Diamond\theta$:

1. $\theta \,|\, \Box\Diamond\theta$ $\quad$ B axiom
2. $|\, \theta \to \Box\Diamond\theta$ $\quad$ IMR from 1.
3. $|\, \Box(\theta \to \Box\Diamond\theta)$ $\quad$ NEC from 2.
4. $\Box(\theta \to \Box\Diamond\theta) \,|\, \Box\theta \to \Box\Box\Diamond\theta$ $\quad$ K axiom
5. $|\, \Box(\theta \to \Box\Diamond\theta) \to (\Box\theta \to \Box\Box\Diamond\theta)$ $\quad$ IMR from 4.
6. $|\, \Box\theta \to \Box\Box\Diamond\theta$ $\quad$ MP from 3. and 5.

6. Call a frame $\langle W, E, V\rangle$ an H-frame if it satisfies that whenever $(i,j) \in E$ then $(j,j) \in E$. Say that $\Gamma \models^H \theta$ if whenever $\langle W, E, V\rangle$ is a H-frame, $i \in W$ and $i \models \Gamma$ then $i \models \theta$.

Let $\Gamma_1 \,|\, \theta_1, \dots, \Gamma_m \,|\, \theta_m$ be a proof in H. We show by induction on k, for $k = 1, \dots, m$, that $\Gamma_i \models^H \theta_i$. All the steps in this proof are the same as for K except when the justification is via the H axiom so all there is left to prove is that $\Diamond\theta \models^H \Diamond\Diamond\theta$. To this end suppose $\langle W, E, V\rangle$ is an H-frame, $i \in W$ and $i \models \Diamond\theta$. Then for some $j \in W$, $(i,j) \in E$ and $j \models \theta$. Since this is a H-frame $(j,j) \in E$ so $j \models \Diamond\theta$. Hence $i \models \Diamond\Diamond\theta$, as required.

Now consider the H-frame: $W = \{0,1\}$, $E = \{(0,1),(1,1)\}$ with $V_0(p) = 0, V_1(p) = 1$. In this frame $0 \models \Diamond p$ but $0 \not\models p$ so $\Diamond p \not\models^H p$ and $\Diamond p \not\vdash^H p$ by the above argument.

7.
(C1) $F_\wedge(0,1) = F_\wedge(1,0) = 0, \quad F_\wedge(1,1) = 1$

(C2) $F_\wedge$ is continuous.

(C3) $F_\wedge$ is increasing (not necessarily strictly) in each coordinate.

(C4) $F_\wedge$ is associative.

Since $0 < F_\wedge(x,x) < x$ for $x \in (0,1)$ the only values of $x \in [0,1]$ for which $F_\wedge(x,x) = x$ are 0,1. Hence by the Mostert-Shields Theorem $\langle [0,1], F_\wedge, <\rangle$ is either isomorphic to $\langle [0,1], \times, <\rangle$ or to $\langle [0,1], \max\{x+y-1, 0\}, <\rangle$. However it cannot be the latter since, e.g. $\max\{1/2 + 1/2 - 1, 0\} = 0$, so there would be $0 < x < 1$ such that $F_\wedge(x,x) = 0$. Hence it must be that

$$\langle [0,1], F_\wedge, <\rangle \cong \langle [0,1], \times, <\rangle.$$

8. For $\Gamma \subseteq SL, \theta \in SL$,

$\Gamma \models^{\text{Ł}} \theta \iff$ for all $[0,1]$-valuations w (in Ł), if $w(\phi) = 1$ for all $\phi \in \Gamma$ then $w(\theta) = 1$.

(i) Not true. Firstly, $p \models^{\text{Ł}} p \wedge p$, since whenever the $[0,1]$-valuation w is such that $w(p) = 1$ then $w(p \wedge p) = \max\{2w(p) - 1, 0\} = 1$. However for the valuation w with $w(p) = 1/2$, $w(p \wedge p) = 0$ and

$$w(p \to (p \wedge p)) = \min\{1 - w(p) + w(p \to p), 1\} = 1/2 \neq 1$$

so $\not\models^{\text{Ł}} p \to (p \wedge p)$.

(ii) True. Let w be any $[0,1]$-valuation. If $w(\theta \to \phi) = 1$ then

$$w((\phi \to \theta) \to \phi) \leq w(\theta \to \phi)$$

so

$$w(((\phi \to \theta) \to \phi) \to (\theta \to \phi)) = 1.$$

Otherwise $w(\theta \to \phi) < 1$ so

$$(\dagger). \qquad w(\theta \to \phi) = 1 - w(\theta) + w(\phi) \quad \text{and} \quad w(\phi) < w(\theta)$$

Therefore $w(\phi \to \theta) = 1$ so

$$w((\phi \to \theta) \to \phi) = \min\{1 - 1 + w(\phi), 1\} = w(\phi) \leq w(\theta \to \phi)$$

from ($\dagger$) since $1 - w(\theta) \geq 0$. Hence again

$$w(((\phi \to \theta) \to \phi) \to (\theta \to \phi)) = 1,$$

as required.