THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

29 May 2014
14.00 – 16.30

Answer ALL FIVE questions in Section A (56 marks in all).
Answer TWO of the THREE questions in Section B (24 marks in total).
If more than TWO questions from Section B are attempted,
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

Electronic calculators are not permitted
A1. Explain (without proof) how a finite sequence \( \vec{s} = s_1, s_2, ..., s_m \) of subsets of \( At^L \) determines a rational consequence relation \( \not\sim \). 

State the *Representation Theorem for Rational Consequence Relations*. 

In the case where \( L = \{p, q, r\} \), \( \vec{s} = s_1, s_2, s_3 \) and 

\[
\begin{align*}
  s_1 &= \{\neg p \land \neg q \land \neg r\}, \\
  s_2 &= \{p \land q \land \neg r, p \land \neg q \land \neg r\}, \\
  s_3 &= \{p \land q \land r\},
\end{align*}
\]

which of the following are true? [You need not justify your answers.]

(i) \( \neg(q \to p) \not\sim \vec{s} r \).

(ii) \( p \lor r \not\sim \vec{s} q \).

(iii) \( r \not\sim \vec{s} p \to q \).

[10 marks]

A2. (a) By giving a direct derivation from the \( GM \) rules show that the rule

\[
\begin{array}{c}
\theta \not\sim \psi, \quad \theta \not\sim \neg \phi \\
\hline
\theta \lor \psi \not\sim \phi \to \psi
\end{array}
\]

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[7 marks]

(b) Use the Z-algorithm to find the rational closure of \( K = \{q \not\sim p, \ p \not\sim \neg q\} \).

[7 marks]

A3. State the Completeness Theorem for \( S_4 \). Show that:

(i) \( \Box \Box \theta \vdash_{S_4} \theta \).

(ii) \( \Box p \not\vdash_{S_4} \Box \Box p \).

[12 marks]
**A4.** State the properties (D1)-(D4).

Explain the sense in which a function $F$ satisfying (D1)-(D4) can be classified as a chimera of the functions $\max\{x, y\}$, $\min\{1, x + y\}$ and $x + y - xy$.

Show that if $F$ satisfies (D1)-(D4) then $F(0, x) = x$ for $x \in [0, 1]$.  

[12 marks]

**A5.** Let $L = \{p\}$. State McNaughton’s Theorem for $L$.

Suppose that $w$ is the $[0, 1]$-valuation on $L$ such that $w(p) = 1/n$, where $2 \leq n \in \mathbb{N}$. Show that there is a tautology $\phi \in SL$ (i.e. \( \models \phi \)) such that $w(\phi) = 0$.

[8 marks]
B6. By using the Representation Theorem for Rational Consequence Relations show that the rule
\[
\frac{\theta \vdash \phi \rightarrow \psi, \quad \theta \land \neg \phi \vdash \neg \theta}{\theta \land \phi \vdash \psi}
\]
holds for all rational consequence relations but that the rule
\[
\frac{\theta \vdash \phi \rightarrow \psi}{\theta \land \phi \vdash \psi}
\]
fails for some rational consequence relation and choice of \(\theta, \phi, \psi\).

[12 marks]

B7. Define what is meant by a *formal proof* in the modal logic \(K\).

Let \(H\) be the modal logic which results by replacing the rule NEC and the axiom
\(\Box(\theta \rightarrow \phi) | \Box \theta \rightarrow \Box \phi\) of \(K\) by the rule
\[
\frac{\theta_1, \theta_2, \ldots, \theta_n | \phi}{\Box \theta_1, \Box \theta_2, \ldots, \Box \theta_n | \Box \phi}
\]
Sketch a proof that for any \(\theta \in SML, \Gamma \subseteq SML,\)
\[
\Gamma \vdash^K \theta \iff \Gamma \vdash^H \theta.
\]

[12 marks]

B8. Let \(\psi \in SL\) be such that \(\vdash^L \psi\). Show, without using the Completeness Theorem
for \(L\), that:

(a) If \(\vdash^L \phi \rightarrow \chi\) and \(\vdash^L \chi \rightarrow \xi\) then \(\vdash^L \phi \rightarrow \xi\).
(b) \(\vdash^L \neg \psi \rightarrow (\neg \theta \rightarrow \neg \psi)\).
(c) \(\vdash^L \neg \psi \rightarrow (\psi \rightarrow \theta)\).
(d) \(\vdash^L (\theta \rightarrow \psi) \rightarrow \psi\).
(e) \(\vdash^L (\psi \rightarrow \theta) \rightarrow \theta\).
(f) \(\vdash^L \neg \psi \rightarrow \theta\).
(g) \(\vdash^L (\theta \rightarrow (\neg \psi \rightarrow \theta)) \rightarrow (\neg \psi \rightarrow \theta)\) \(i.e. \theta \vdash^L (\neg \psi \rightarrow \theta)\).
(h) \(\vdash^L (\neg \psi \neg \theta) \rightarrow \theta\).

[12 marks]
Axioms and Rules of Proof

Sentential, or Propositional, Calculus, \( SC \)

**Axioms:** \( \text{REF} \quad \theta \vdash \theta \)

**Rules of Proof:**

\[
\begin{align*}
\text{AND} & \quad \frac{\Gamma \vdash \theta, \Delta \vdash \phi}{\Gamma, \Delta \vdash \theta \land \phi} \\
\text{ORR} & \quad \frac{\Gamma \vdash \theta \lor \phi}{\Gamma, \Delta \vdash \theta \lor \phi} \\
\text{IMR} & \quad \frac{\Gamma, \theta \vdash \phi}{\Gamma \vdash \theta \to \phi} \\
\text{NIN} & \quad \frac{\Gamma, \theta \vdash \phi, \Delta, \theta \vdash \neg \phi}{\Gamma, \Delta \vdash \neg \theta} \\
\text{MON} & \quad \frac{\Gamma \vdash \theta}{\Gamma, \Delta \vdash \theta}
\end{align*}
\]

\[
\begin{align*}
\text{ANL} & \quad \frac{\Gamma, \theta, \phi \vdash \psi}{\Gamma \vdash \theta \land \phi \vdash \psi} \\
\text{DIS} & \quad \frac{\Gamma, \theta \vdash \psi, \Delta, \phi \vdash \psi}{\Gamma, \Delta, \theta \lor \phi \vdash \psi} \\
\text{MP} & \quad \frac{\Gamma \vdash \theta, \Delta \vdash \theta \to \phi}{\Gamma \vdash \theta \land \phi} \\
\text{NNO} & \quad \frac{\Gamma \vdash \neg \theta}{\Gamma \vdash \theta}
\end{align*}
\]

**GM Rules**

\( \text{REF}, \theta \vdash \theta, \) together with:

\[
\begin{align*}
\text{Left Logical Equivalence} & \quad \frac{\theta \equiv \phi, \theta \vdash \psi}{\phi \vdash \psi} & \text{LLE} \\
\text{Right Weakening} & \quad \frac{\theta \vdash \phi, \phi \vdash \psi}{\theta \vdash \psi} & \text{RWE} \\
\text{Cautious Monotonicity} & \quad \frac{\theta \vdash \phi, \theta \vdash \psi}{\theta \land \phi \vdash \psi} & \text{CMO} \\
\text{And On Right} & \quad \frac{\theta \vdash \phi, \theta \vdash \psi}{\theta \vdash \phi \land \psi} & \text{AND} \\
\text{Disjunction, Or Left} & \quad \frac{\theta \vdash \psi, \phi \vdash \psi}{\theta \lor \phi \vdash \psi} & \text{DIS} \\
\text{Rational Monotonicity} & \quad \frac{\theta \vdash \psi, \theta \not\vdash \neg \phi}{\theta \land \phi \vdash \psi} & \text{RM0}
\end{align*}
\]
Modal Logics

Axioms:

\[ K := \text{REF} + \Box(\theta \rightarrow \phi) \]

\[ T := K + \Box \theta \]

\[ D := K + \Box \Diamond \theta \]

\[ B := K + \theta \rightarrow \Box \Diamond \theta \]

\[ S_4 := T + \Box \theta \rightarrow \Box \Box \theta \]

\[ S_5 := T + \Diamond \theta \rightarrow \Box \Box \theta \]

Rules of Proof: All the rules of SC plus NEC

\[ \frac{\theta}{\Box \theta} \]

Lukasiewicz Logic, L

Axioms: REF together with

\[ L_1 : | \theta \rightarrow (\phi \rightarrow \theta) \]

\[ L_2 : | (\theta \rightarrow \phi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\theta \rightarrow \psi)) \]

\[ L_3 : | (\neg \theta \rightarrow \neg \phi) \rightarrow (\phi \rightarrow \theta) \]

\[ L_4 : | ((\theta \rightarrow \phi) \rightarrow \phi) \rightarrow ((\phi \rightarrow \theta) \rightarrow \theta) \]

\[ \text{i.e. } | (\theta \lor \phi) \rightarrow (\phi \lor \theta) \]

\[ L_5 : | (\theta \rightarrow \phi) \lor (\phi \rightarrow \theta) \]

Rules of Proof: Only MP