Two and a Half Hours

THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

29 May 2014

14.00 – 16.30

Answer ALL FIVE questions in Section A (56 marks in all).
Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted,
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

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Electronic calculators are not permitted

SECTION A

Answer ALL FIVE questions

A1. Explain (without proof) how a finite sequence $\vec{s} = s_1, s_2, ..., s_m$ of subsets of At^L determines a rational consequence relation $\mid\!\sim_{\vec{s}}$.

State the *Representation Theorem for Rational Consequence Relations.*

In the case where $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{\neg p \wedge \neg q \wedge \neg r\}, \\ s_2 &= \{p \wedge q \wedge \neg r,\ p \wedge \neg q \wedge \neg r\}, \\ s_3 &= \{p \wedge q \wedge r\}, \end{aligned}$$

which of the following are true? [You need not justify your answers.]

(i) $\neg(q \to p) \mid\!\sim_{\vec{s}} r$.

(ii) $p \vee r \mid\!\sim_{\vec{s}} q$.

(iii) $r \mid\!\sim_{\vec{s}} p \to q$.

[10 marks]

A2. (a) By giving a direct derivation from the *GM* rules show that the rule

$$\frac{\theta \mid\!\sim \psi, \quad \theta \not\mid\!\sim \neg\phi}{\theta \vee \psi \mid\!\sim \phi \to \psi}$$

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[7 marks]

(b) Use the Z-algorithm to find the rational closure of $K = \{\, q \mid\!\sim p,\ \ p \mid\!\sim \neg q \,\}$.

[7 marks]

A3. State the Completeness Theorem for S_4. Show that:

(i) $\Diamond\Box\Diamond\theta \models^{S_4} \Diamond\theta$.

(ii) $\Diamond p \not\models^{S_4} \Diamond\Box\Diamond p$.

[12 marks]

A4. State the properties (D1)-(D4).

Explain the sense in which a function $F_\vee$ satisfying (D1)-(D4) can be classified as a chimera of the functions $\max\{x, y\}$, $\min\{1, x + y\}$ and $x + y - xy$.

Show that if $F_\vee$ satisfies (D1)-(D4) then $F_\vee(0, x) = x$ for $x \in [0, 1]$.

[12 marks]

A5. Let $L = \{p\}$. State McNaughton's Theorem for L.

Suppose that w is the $[0, 1]$-valuation on L such that $w(p) = 1/n$, where $2 \leq n \in \mathbb{N}$. Show that there is a tautology $\phi \in SL$ (i.e. $\models \phi$) such that $w(\phi) = 0$.

[8 marks]

SECTION B

Answer TWO of the THREE questions

B6. By using the Representation Theorem for Rational Consequence Relations show that the rule

$$\frac{\theta \mid\!\sim \phi \to \psi, \quad \theta \wedge \neg\phi \mid\!\sim \neg\theta}{\theta \wedge \phi \mid\!\sim \psi}$$

holds for all rational consequence relations but that the rule

$$\frac{\theta \mid\!\sim \phi \to \psi}{\theta \wedge \phi \mid\!\sim \psi}$$

fails for some rational consequence relation and choice of θ, ϕ, ψ.

[12 marks]

B7. Define what is meant by a *formal proof* in the modal logic K.

Let H be the modal logic which results by replacing the rule NEC and the axiom $\Box(\theta \to \phi) \mid \Box\theta \to \Box\phi$ of K by the rule

$$\frac{\theta_1, \theta_2, \ldots, \theta_n \mid \phi}{\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_n \mid \Box\phi}$$

Sketch a proof that for any $\theta \in SML$, $\Gamma \subseteq SML$,

$$\Gamma \vdash^K \theta \iff \Gamma \vdash^H \theta.$$

[12 marks]

B8. Let $\psi \in SL$ be such that $\vdash^{Ł} \psi$. Show, without using the Completeness Theorem for Ł, that:

(a) If $\vdash^{Ł} \phi \to \chi$ and $\vdash^{Ł} \chi \to \xi$ then $\vdash^{Ł} \phi \to \xi$.

(b) $\vdash^{Ł} \neg\psi \to (\neg\theta \to \neg\psi)$.

(c) $\vdash^{Ł} \neg\psi \to (\psi \to \theta)$.

(d) $\vdash^{Ł} (\theta \to \psi) \to \psi$.

(e) $\vdash^{Ł} (\psi \to \theta) \to \theta$.

(f) $\vdash^{Ł} \neg\psi \to \theta$.

(g) $\vdash^{Ł} (\theta \to (\neg\psi \to \theta)) \to (\neg\psi \to \theta)$ (i.e. $\theta \underline{\vee} (\neg\psi \to \theta)$.)

(h) $\vdash^{Ł} (\neg\psi \underline{\vee} \theta) \to \theta$.

[12 marks]

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, *SC*

Axioms: REF $\theta \mid \theta$

Rules of Proof:

AND $\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma, \Delta \mid \theta \wedge \phi}$ ANL $\dfrac{\Gamma, \theta, \phi \mid \psi}{\Gamma, \theta \wedge \phi \mid \psi}$

ORR $\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \quad \dfrac{\Gamma \mid \phi}{\Gamma \mid \theta \vee \phi}$ DIS $\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma, \Delta, \theta \vee \phi \mid \psi}$

IMR $\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$ MP $\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma, \Delta \mid \phi}$

NIN $\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma, \Delta \mid \neg\theta}$ NNO $\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$

MON $\dfrac{\Gamma \mid \theta}{\Gamma, \Delta \mid \theta}$ AO $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \quad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$

GM Rules

REF, $\theta \mid\!\sim \theta$, together with:

Left Logical Equivalence $\dfrac{\theta \equiv \phi, \quad \theta \mid\!\sim \psi}{\phi \mid\!\sim \psi}$ LLE

Right Weakening $\dfrac{\theta \mid\!\sim \phi, \quad \phi \models \psi}{\theta \mid\!\sim \psi}$ RWE

Cautious Monotonicity $\dfrac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \wedge \phi \mid\!\sim \psi}$ CMO

And On Right $\dfrac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \mid\!\sim \phi \wedge \psi}$ AND

Disjunction, Or Left $\dfrac{\theta \mid\!\sim \psi, \quad \phi \mid\!\sim \psi}{\theta \vee \phi \mid\!\sim \psi}$ DIS

Rational Monotonicity $\dfrac{\theta \mid\!\sim \psi, \quad \theta \not\mid\!\sim \neg\phi}{\theta \wedge \phi \mid\!\sim \psi}$ RMO

Modal Logics

Axioms:

$$
\begin{aligned}
\mathrm{K} &:= \quad \mathrm{REF} + \Box(\theta \to \phi) \mid \Box\theta \to \Box\phi \\
\mathrm{T} &:= \quad K + \Box\theta \mid \theta \\
\mathrm{D} &:= \quad K + \Box\theta \mid \Diamond\theta \\
\mathrm{B} &:= \quad K + \theta \mid \Box\Diamond\theta \\
\mathrm{S}_4 &:= \quad T + \Box\theta \mid \Box\Box\theta \\
\mathrm{S}_5 &:= \quad T + \Diamond\theta \mid \Box\Diamond\theta
\end{aligned}
$$

Rules of Proof: All the rules of SC plus NEC $\dfrac{\mid \theta}{\mid \Box\theta}$

Łukasiewicz Logic, Ł

Axioms: REF together with

Ł1 : $\mid \theta \to (\phi \to \theta)$

Ł2 : $\mid (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$

Ł3 : $\mid (\neg\theta \to \neg\phi) \to (\phi \to \theta)$

Ł4 : $\mid ((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $\mid (\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$

Ł5 : $\mid (\theta \to \phi) \underline{\vee} (\phi \to \theta)$

Rules of Proof: Only MP

END OF EXAMINATION PAPER