

Two and a Half Hours

THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

29 May 2014

14.00 – 16.30

Answer ALL FIVE questions in Section A (56 marks in all).

Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted,
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

Electronic calculators are not permitted

SECTION A

Answer ALL FIVE questions

A1. Explain (without proof) how a finite sequence $\vec{s} = s_1, s_2, \dots, s_m$ of subsets of At^L determines a rational consequence relation $\vdash_{\vec{s}}$.

State the *Representation Theorem for Rational Consequence Relations*.

In the case where $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{\neg p \wedge \neg q \wedge \neg r\}, \\ s_2 &= \{p \wedge q \wedge \neg r, p \wedge \neg q \wedge \neg r\}, \\ s_3 &= \{p \wedge q \wedge r\}, \end{aligned}$$

which of the following are true? [You need not justify your answers.]

- (i) $\neg(q \rightarrow p) \vdash_{\vec{s}} r$.
- (ii) $p \vee r \vdash_{\vec{s}} q$.
- (iii) $r \vdash_{\vec{s}} p \rightarrow q$.

[10 marks]

A2. (a) By giving a direct derivation from the *GM* rules show that the rule

$$\frac{\theta \vdash \psi, \quad \theta \not\vdash \neg\phi}{\theta \vee \psi \vdash \phi \rightarrow \psi}$$

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[7 marks]

(b) Use the Z-algorithm to find the rational closure of $K = \{q \vdash p, p \vdash \neg q\}$.

[7 marks]

A3. State the Completeness Theorem for S_4 . Show that:

- (i) $\diamond \Box \diamond \theta \vDash^{S_4} \diamond \theta$.
- (ii) $\diamond p \not\vdash^{S_4} \diamond \Box \diamond p$.

[12 marks]

A4. State the properties (D1)-(D4).

Explain the sense in which a function F_{\vee} satisfying (D1)-(D4) can be classified as a chimera of the functions $\max\{x, y\}$, $\min\{1, x + y\}$ and $x + y - xy$.

Show that if F_{\vee} satisfies (D1)-(D4) then $F_{\vee}(0, x) = x$ for $x \in [0, 1]$.

[12 marks]

A5. Let $L = \{p\}$. State McNaughton's Theorem for L .

Suppose that w is the $[0, 1]$ -valuation on L such that $w(p) = 1/n$, where $2 \leq n \in \mathbb{N}$. Show that there is a tautology $\phi \in SL$ (i.e. $\models \phi$) such that $w(\phi) = 0$.

[8 marks]

SECTION B

Answer TWO of the THREE questions

B6. By using the Representation Theorem for Rational Consequence Relations show that the rule

$$\frac{\theta \sim \phi \rightarrow \psi, \quad \theta \wedge \neg\phi \sim \neg\theta}{\theta \wedge \phi \sim \psi}$$

holds for all rational consequence relations but that the rule

$$\frac{\theta \sim \phi \rightarrow \psi}{\theta \wedge \phi \sim \psi}$$

fails for some rational consequence relation and choice of θ, ϕ, ψ .

[12 marks]

B7. Define what is meant by a *formal proof* in the modal logic K .Let H be the modal logic which results by replacing the rule NEC and the axiom $\Box(\theta \rightarrow \phi) \mid \Box\theta \rightarrow \Box\phi$ of K by the rule

$$\frac{\theta_1, \theta_2, \dots, \theta_n \mid \phi}{\Box\theta_1, \Box\theta_2, \dots, \Box\theta_n \mid \Box\phi}$$

Sketch a proof that for any $\theta \in SML, \Gamma \subseteq SML$,

$$\Gamma \vdash^K \theta \iff \Gamma \vdash^H \theta.$$

[12 marks]

B8. Let $\psi \in SL$ be such that $\vdash^{\mathbf{L}} \psi$. Show, without using the Completeness Theorem for \mathbf{L} , that:

- (a) If $\vdash^{\mathbf{L}} \phi \rightarrow \chi$ and $\vdash^{\mathbf{L}} \chi \rightarrow \xi$ then $\vdash^{\mathbf{L}} \phi \rightarrow \xi$.
- (b) $\vdash^{\mathbf{L}} \neg\psi \rightarrow (\neg\theta \rightarrow \neg\psi)$.
- (c) $\vdash^{\mathbf{L}} \neg\psi \rightarrow (\psi \rightarrow \theta)$.
- (d) $\vdash^{\mathbf{L}} (\theta \rightarrow \psi) \rightarrow \psi$.
- (e) $\vdash^{\mathbf{L}} (\psi \rightarrow \theta) \rightarrow \theta$.
- (f) $\vdash^{\mathbf{L}} \neg\psi \rightarrow \theta$.
- (g) $\vdash^{\mathbf{L}} (\theta \rightarrow (\neg\psi \rightarrow \theta)) \rightarrow (\neg\psi \rightarrow \theta)$ (i.e. $\theta \underline{\vee} (\neg\psi \rightarrow \theta)$.)
- (h) $\vdash^{\mathbf{L}} (\neg\psi \underline{\vee} \theta) \rightarrow \theta$.

[12 marks]

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: REF $\theta | \theta$

Rules of Proof:

$\text{AND} \quad \frac{\Gamma \theta, \Delta \phi}{\Gamma, \Delta \theta \wedge \phi}$	$\text{ANL} \quad \frac{\Gamma, \theta, \phi \psi}{\Gamma, \theta \wedge \phi \psi}$
$\text{ORR} \quad \frac{\Gamma \theta}{\Gamma \theta \vee \phi} \quad \frac{\Gamma \phi}{\Gamma \theta \vee \phi}$	$\text{DIS} \quad \frac{\Gamma, \theta \psi, \Delta, \phi \psi}{\Gamma, \Delta, \theta \vee \phi \psi}$
$\text{IMR} \quad \frac{\Gamma, \theta \phi}{\Gamma \theta \rightarrow \phi}$	$\text{MP} \quad \frac{\Gamma \theta, \Delta \theta \rightarrow \phi}{\Gamma, \Delta \phi}$
$\text{NIN} \quad \frac{\Gamma, \theta \phi, \Delta, \theta \neg \phi}{\Gamma, \Delta \neg \theta}$	$\text{NNO} \quad \frac{\Gamma \neg \neg \theta}{\Gamma \theta}$
$\text{MON} \quad \frac{\Gamma \theta}{\Gamma, \Delta \theta}$	$\text{AO} \quad \frac{\Gamma \theta \wedge \phi}{\Gamma \theta} \quad \frac{\Gamma \theta \wedge \phi}{\Gamma \phi}$

GM Rules

REF, $\theta \sim \theta$, together with:

Left Logical Equivalence	$\frac{\theta \equiv \phi, \theta \sim \psi}{\phi \sim \psi}$	LLE
Right Weakening	$\frac{\theta \sim \phi, \phi \models \psi}{\theta \sim \psi}$	RWE
Cautious Monotonicity	$\frac{\theta \sim \phi, \theta \sim \psi}{\theta \wedge \phi \sim \psi}$	CMO
And On Right	$\frac{\theta \sim \phi, \theta \sim \psi}{\theta \sim \phi \wedge \psi}$	AND
Disjunction, Or Left	$\frac{\theta \sim \psi, \phi \sim \psi}{\theta \vee \phi \sim \psi}$	DIS
Rational Monotonicity	$\frac{\theta \sim \psi, \theta \not\sim \neg \phi}{\theta \wedge \phi \sim \psi}$	RMO

Modal LogicsAxioms:

$$K := \text{REF} + \Box(\theta \rightarrow \phi) \mid \Box\theta \rightarrow \Box\phi$$

$$T := K + \Box\theta \mid \theta$$

$$D := K + \Box\theta \mid \Diamond\theta$$

$$B := K + \theta \mid \Box\Diamond\theta$$

$$S_4 := T + \Box\theta \mid \Box\Box\theta$$

$$S_5 := T + \Diamond\theta \mid \Box\Diamond\theta$$

Rules of Proof: All the rules of *SC* plus NEC $\frac{\mid \theta}{\mid \Box\theta}$

Lukasiewicz Logic, LAxioms: REF together with

$$L1 : \mid \theta \rightarrow (\phi \rightarrow \theta)$$

$$L2 : \mid (\theta \rightarrow \phi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\theta \rightarrow \psi))$$

$$L3 : \mid (\neg\theta \rightarrow \neg\phi) \rightarrow (\phi \rightarrow \theta)$$

$$L4 : \mid ((\theta \rightarrow \phi) \rightarrow \phi) \rightarrow ((\phi \rightarrow \theta) \rightarrow \theta) \quad \text{i.e.} \quad \mid (\theta \vee \phi) \rightarrow (\phi \vee \theta)$$

$$L5 : \mid (\theta \rightarrow \phi) \vee (\phi \rightarrow \theta)$$

Rules of Proof: Only MP**END OF EXAMINATION PAPER**