Two and a half hours

THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

20 May 2013

9.45 – 12.15

Answer **ALL** six questions in Section A (60 marks in all) and
TWO questions in Section B (20 marks in all).

If you answer more than two questions from Section B only the first two appearing in your answer book will be marked.

A list of axioms and rules of proof is appended to this examination paper

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The use of calculators is not permitted

SECTION A

Answer **ALL** five questions

A1. Explain (without proof) how a finite sequence $\vec{s} = s_1, s_2, ..., s_m$ of subsets of At^L determines a rational consequence relation $\mid\!\sim_{\vec{s}}$.

State the *Representation Theorem for Rational Consequence Relations.*

In the case where $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{p \wedge q \wedge r,\ p \wedge q \wedge \neg r\}, \\ s_2 &= \{p \wedge \neg q \wedge r,\ \neg p \wedge q \wedge r\}, \\ s_3 &= \{\neg p \wedge q \wedge \neg r\}, \end{aligned}$$

which of the following are true? [You need not justify your answers.]

(i) $(p \to q) \mid\!\sim_{\vec{s}} r$.

(ii) $\neg p \vee \neg q \mid\!\sim_{\vec{s}} r$.

(iii) $\neg p \wedge \neg q \mid\!\sim_{\vec{s}} p \vee q$.

[10 marks]

A2. (a) By giving a direct derivation from the *GM* rules show that the rule

$$\frac{\theta \wedge \phi \mid\!\sim \neg\psi \qquad \theta \not\mid\!\sim \neg\psi}{\theta \wedge \psi \mid\!\sim \neg\phi}$$

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[5 marks]

(b) Use the Z-algorithm to find the rational closure of $K = \{\, p \vee q \mid\!\sim \neg p,\ \neg q \mid\!\sim p \,\}$.

[8 marks]

A3. How is the relation $\models^K$ defined?

Show that:

(i) $\neg\Box\Diamond\theta \models^K \neg\Box\theta \vee \Diamond\theta$.

(ii) $\neg\Box q \not\models^K \Box(p \vee q) \to \Box p$.

[10 marks]

A4. What is meant by a *proof* in D? [You need not explicitly state the rules or axioms.] What does it mean to say that $\Gamma \vdash^D \theta$?

Without using the Completeness Theorem for D show that

$$\vdash^D \Box\Box\theta \to \Box\Diamond\theta.$$

[10 marks]

A5. Explain the sense in which a function $F_\wedge$ satisfying C1-C4 can be classified as a chimera of the functions $\min\{x, y\}$, $\max\{0, x + y - 1\}$ and xy.

Assuming that the function $G : [0, 1]^2 \to [0, 1]$ defined by

$$G(x, y) = \min\{x, y, 2xy\}$$

satisfies C1-C4 how does G fit into this classification?

[7 marks]

A6. How is the relation $\models^{Ł}$ defined?

State the *Completeness Theorem* for Ł.

Show that:

(i) $\nvdash^{Ł} (p \to q)\underline{\vee}(\neg p \to q)$.

(ii) $\vdash^{Ł} (p \to q) \vee (\neg p \to q)$.

[10 marks]

SECTION B

Answer **TWO** of the three questions

B7. By using the Representation Theorem for Rational Consequence Relations show that the rule

$$\frac{\theta \mathrel{|\!\sim} \phi \qquad \phi \wedge \neg\psi \mathrel{|\!\sim} \psi}{\theta \mathrel{|\!\sim} \psi}$$

holds for all rational consequence relations but that the rule

$$\frac{\theta \wedge \neg\phi \mathrel{|\!\sim} \phi \qquad \phi \mathrel{|\!\sim} \psi}{\theta \mathrel{|\!\sim} \psi}$$

fails for some rational consequence relation and choice of θ, ϕ, ψ.

[10 marks]

B8. Let $\langle W, E, V\rangle$, $\langle W', E', V'\rangle$ be frames such that $W' = W \cup \{u\}$ for some $u \notin W$, $E' = E \cup \{\langle u, v\rangle\}$ for some $v \in W$, and $V_w = V'_w$ for all $w \in W$. Outline a proof that for $w \in W$ and $\theta \in SML$

$$\langle W, E, V\rangle,\ w \models \theta \iff \langle W', E', V'\rangle,\ w \models \theta.$$

Hence show that if $\models^D \Diamond\theta$ then $\models^D \theta$.

[10 marks]

B9. Explain briefly why it is that

(i) If ψ is an axiom of Ł then $\vdash^{Ł} \psi$.

(ii) If $\vdash^{Ł} \psi$ and $\vdash^{Ł} (\psi \to \eta)$ then $\vdash^{Ł} \eta$.

Hence show that

(iii) $\vdash^{Ł} (\theta \to (\theta \to \theta)) \to (\theta \to \theta)$.

(iv) $\vdash^{Ł} ((\theta \to \theta) \to \theta) \to \theta$.

You may assume the axioms and rules of Ł, and

(a1) $\vdash^{Ł} \theta \to \theta$,

but you may not assume the Completeness Theorem for Ł.

[10 marks]

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: REF $\theta \,|\, \theta$

Rules of Proof:

AND $\dfrac{\Gamma \,|\, \theta, \quad \Delta | \phi}{\Gamma, \Delta \,|\, \theta \wedge \phi}$

ANL $\dfrac{\Gamma, \theta, \phi \,|\, \psi}{\Gamma, \theta \wedge \phi \,|\, \psi}$

ORR $\dfrac{\Gamma \,|\, \theta}{\Gamma \,|\, \theta \vee \phi} \quad \dfrac{\Gamma \,|\, \phi}{\Gamma \,|\, \theta \vee \phi}$

DIS $\dfrac{\Gamma, \theta \,|\, \psi, \quad \Delta, \phi \,|\, \psi}{\Gamma, \Delta, \theta \vee \phi \,|\, \psi}$

IMR $\dfrac{\Gamma, \theta \,|\, \phi}{\Gamma \,|\, \theta \to \phi}$

MP $\dfrac{\Gamma \,|\, \theta, \quad \Delta \,|\, \theta \to \phi}{\Gamma, \Delta \,|\, \phi}$

NIN $\dfrac{\Gamma, \theta \,|\, \phi, \quad \Delta, \theta \,|\, \neg\phi}{\Gamma, \Delta \,|\, \neg\theta}$

NNO $\dfrac{\Gamma | \neg\neg\theta}{\Gamma \,|\, \theta}$

MON $\dfrac{\Gamma \,|\, \theta}{\Gamma, \Delta \,|\, \theta}$

AO $\dfrac{\Gamma \,|\, \theta \wedge \phi}{\Gamma \,|\, \theta} \quad \dfrac{\Gamma \,|\, \theta \wedge \phi}{\Gamma \,|\, \phi}$

GM Rules

REF, $\theta \mathrel{|\!\sim} \theta$, together with:

Left Logical Equivalence $\dfrac{\theta \equiv \phi, \quad \theta \mathrel{|\!\sim} \psi}{\phi \mathrel{|\!\sim} \psi}$ LLE

Right Weakening $\dfrac{\theta \mathrel{|\!\sim} \phi, \quad \phi \models \psi}{\theta \mathrel{|\!\sim} \psi}$ RWE

Cautious Monotonicity $\dfrac{\theta \mathrel{|\!\sim} \phi, \quad \theta \mathrel{|\!\sim} \psi}{\theta \wedge \phi \mathrel{|\!\sim} \psi}$ CMO

And On Right $\dfrac{\theta \mathrel{|\!\sim} \phi, \quad \theta \mathrel{|\!\sim} \psi}{\theta \mathrel{|\!\sim} \phi \wedge \psi}$ AND

Disjunction, Or Left $\dfrac{\theta \mathrel{|\!\sim} \psi, \quad \phi \mathrel{|\!\sim} \psi}{\theta \vee \phi \mathrel{|\!\sim} \psi}$ DIS

Rational Monotonicity $\dfrac{\theta \mathrel{|\!\sim} \psi, \quad \theta \mathrel{|\!\not\sim} \neg\phi}{\theta \wedge \phi \mathrel{|\!\sim} \psi}$ RMO

Modal Logics

Axioms:

K := $\text{REF} + \square(\theta \to \phi) \,|\, \square\theta \to \square\phi$

T := $K + \square\theta \,|\, \theta$

D := $K + \square\theta \,|\, \diamond\theta$

B := $K + \theta \,|\, \square\diamond\theta$

S_4 := $T + \square\theta \,|\, \square\square\theta$

S_5 := $T + \diamond\theta \,|\, \square\diamond\theta$

Rules of Proof: All the rules of SC plus NEC $\dfrac{|\,\theta}{|\,\square\theta}$

Łukasiewicz Logic, Ł

Axioms: REF together with

Ł1 : $|\,\theta \to (\phi \to \theta)$

Ł2 : $|\,(\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$

Ł3 : $|\,(\neg\theta \to \neg\phi) \to (\phi \to \theta)$

Ł4 : $|\,((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $|\,(\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$

Ł5 : $|\,(\theta \to \phi) \underline{\vee} (\phi \to \theta)$

Rules of Proof: Only MP

END OF EXAMINATION PAPER