THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

20 May 2013
9.45 – 12.15

Answer **ALL** six questions in Section A (60 marks in all) and **TWO** questions in Section B (20 marks in all).

If you answer more than two questions from Section B only the first two appearing in your answer book will be marked.

A list of axioms and rules of proof is appended to this examination paper

The use of calculators is not permitted
SECTION A

Answer ALL five questions

A1. Explain (without proof) how a finite sequence \( \vec{s} = s_1, s_2, ..., s_m \) of subsets of \( At^L \) determines a rational consequence relation \( \models_{\vec{s}} \).

State the Representation Theorem for Rational Consequence Relations.

In the case where \( L = \{p,q,r\} \), \( \vec{s} = s_1, s_2, s_3 \) and

\[
\begin{align*}
    s_1 &= \{p \land q \land r, \ p \land q \land \neg r\}, \\
    s_2 &= \{p \land \neg q \land r, \ \neg p \land q \land r\}, \\
    s_3 &= \{\neg p \land q \land \neg r\},
\end{align*}
\]

which of the following are true? [You need not justify your answers.]

(i) \((p \rightarrow q) \models_{\vec{s}} r\).
(ii) \(\neg p \lor \neg q \models_{\vec{s}} r\).
(iii) \(\neg p \land \neg q \models_{\vec{s}} p \lor q\).

[10 marks]

A2. (a) By giving a direct derivation from the GM rules show that the rule

\[
\frac{\theta \land \phi \models \neg \psi \quad \theta \not\models \neg \psi}{\theta \land \psi \models \neg \phi}
\]

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[5 marks]

(b) Use the Z-algorithm to find the rational closure of \( K = \{p \lor q \models \neg p, \ \neg q \models p\} \).

[8 marks]

A3. How is the relation \( \models^K \) defined?

Show that:

(i) \(\neg \Box \Diamond \theta \models^K \neg \Box \theta \lor \Diamond \theta\).
(ii) \(\neg \Box q \not\models^K \Box(p \lor q) \rightarrow \Box p\).

[10 marks]
A4. What is meant by a proof in $D$? [You need not explicitly state the rules or axioms.]
What does it mean to say that $\Gamma \vdash^D \theta$?

Without using the Completeness Theorem for $D$ show that

$$\vdash^D \square \square \theta \to \square \diamond \theta.$$ 

[10 marks]

A5. Explain the sense in which a function $F$ satisfying C1-C4 can be classified as a chimera of the functions $\min\{x, y\}$, $\max\{0, x + y - 1\}$ and $xy$.

Assuming that the function $G : [0, 1]^2 \to [0, 1]$ defined by

$$G(x, y) = \min\{x, y, 2xy\}$$

satisfies C1-C4 how does $G$ fit into this classification?

[7 marks]

A6. How is the relation $\models^L$ defined?

State the Completeness Theorem for $L$.

Show that:

(i) $\not\models^L (p \to q) \lor (\neg p \to q)$.

(ii) $\models^L (p \to q) \lor (\neg p \to q)$.

[10 marks]
SECTION B

Answer TWO of the three questions

B7. By using the Representation Theorem for Rational Consequence Relations show that the rule

\[
\frac{\theta \models \phi \quad \phi \land \neg \psi \models \psi}{\theta \models \psi}
\]

holds for all rational consequence relations but that the rule

\[
\frac{\theta \land \neg \phi \models \phi \quad \phi \models \psi}{\theta \models \psi}
\]

fails for some rational consequence relation and choice of \( \theta, \phi, \psi \).

[10 marks]

B8. Let \( \langle W, E, V \rangle, \langle W', E', V' \rangle \) be frames such that \( W' = W \cup \{ u \} \) for some \( u \notin W \), \( E' = E \cup \{ (u, v) \} \) for some \( v \in W \), and \( V_w = V'_w \) for all \( w \in W \). Outline a proof that for \( w \in W \) and \( \theta \in SML \)

\[ \langle W, E, V \rangle, w \models \theta \iff \langle W', E', V' \rangle, w \models \theta. \]

Hence show that if \( \models^D \diamond \theta \) then \( \models^D \theta \).

[10 marks]

B9. Explain briefly why it is that

(i) If \( \psi \) is an axiom of \( L \) then \( \vdash_L \psi \).

(ii) If \( \vdash_L \psi \) and \( \vdash_L (\psi \rightarrow \eta) \) then \( \vdash_L \eta \).

Hence show that

(iii) \( \vdash_L (\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta) \).

(iv) \( \vdash_L ((\theta \rightarrow \theta) \rightarrow \theta) \rightarrow \theta \).

You may assume the axioms and rules of \( L \), and

(a1) \( \vdash_L \theta \rightarrow \theta \),

but you may not assume the Completeness Theorem for \( L \).

[10 marks]
Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: \( \text{REF} \theta \vdash \theta \)

Rules of Proof:

<table>
<thead>
<tr>
<th>Rule</th>
<th>premise</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>AND</td>
<td>( \Gamma</td>
<td>\theta, \Delta</td>
</tr>
<tr>
<td>ORR</td>
<td>( \Gamma</td>
<td>\theta \lor \phi )</td>
</tr>
<tr>
<td>IMR</td>
<td>( \Gamma, \theta</td>
<td>\phi )</td>
</tr>
<tr>
<td>NIN</td>
<td>( \Gamma, \theta</td>
<td>\phi, \Delta, \theta</td>
</tr>
<tr>
<td>MON</td>
<td>( \Gamma</td>
<td>\theta )</td>
</tr>
</tbody>
</table>

GM Rules

\( \text{REF}, \theta \vdash \theta \), together with:

<table>
<thead>
<tr>
<th>Rule</th>
<th>premise</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left Logical Equivalence</td>
<td>( \theta \equiv \phi ), ( \theta \vdash \psi )</td>
<td>( \phi \vdash \psi )</td>
</tr>
<tr>
<td>Right Weakening</td>
<td>( \theta \vdash \phi ), ( \phi \models \psi )</td>
<td>( \theta \vdash \psi )</td>
</tr>
<tr>
<td>Cautious Monotonicity</td>
<td>( \theta \vdash \phi ), ( \theta \vdash \psi )</td>
<td>( \theta \land \phi \vdash \psi )</td>
</tr>
<tr>
<td>And On Right</td>
<td>( \theta \vdash \phi ), ( \theta \vdash \psi )</td>
<td>( \theta \vdash \phi \land \psi )</td>
</tr>
<tr>
<td>Disjunction, Or Left</td>
<td>( \theta \vdash \psi ), ( \phi \vdash \psi )</td>
<td>( \theta \lor \phi \vdash \psi )</td>
</tr>
<tr>
<td>Rational Monotonicity</td>
<td>( \theta \vdash \psi ), ( \theta \vdash \neg \phi )</td>
<td>( \theta \land \phi \vdash \psi )</td>
</tr>
</tbody>
</table>
Modal Logics

Axioms:

\[ K := \text{REF} + \Box(\theta \rightarrow \phi) | \Box \theta \rightarrow \Box \phi \]

\[ T := K + \Box \theta | \theta \]

\[ D := K + \Box \theta | \Diamond \theta \]

\[ B := K + \theta | \Box \Diamond \theta \]

\[ S_4 := T + \Box \theta | \Box \Box \theta \]

\[ S_5 := T + \Diamond \theta | \Box \Diamond \theta \]

Rules of Proof: All the rules of SC plus NEC
\[ \frac{}{\Box \theta} \]

Łukasiewicz Logic, L

Axioms: \text{REF} together with

\[ L_1 : | \theta \rightarrow (\phi \rightarrow \theta) \]

\[ L_2 : | (\theta \rightarrow \phi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\theta \rightarrow \psi)) \]

\[ L_3 : | (\neg \theta \rightarrow \neg \phi) \rightarrow (\phi \rightarrow \theta) \]

\[ L_4 : | ((\theta \rightarrow \phi) \rightarrow \phi) \rightarrow ((\phi \rightarrow \theta) \rightarrow \theta) \quad \text{i.e.} \quad | (\theta \lor \phi) \rightarrow (\phi \lor \theta) \]

\[ L_5 : | (\theta \rightarrow \phi) \lor (\phi \rightarrow \theta) \]

Rules of Proof: Only MP