

# MATH43001/63001, January 2014 Exam, Solutions<sup>1</sup>

**A1.** (i)  $f(x_1, f(x_1, x_1)) \in TL$  since  $x_1 \in TL$ , by Te1, so  $f(x_1, x_1) \in TL$  by Te3, and in turn  $f(x_1, f(x_1, x_1)) \in TL$  by Te3 again.

(ii)  $f((f(x_1, x_2), x_1)) \notin TL$  since we can show by induction on  $|t|$  that any  $t \in TL$  contains the same number of left parentheses as function symbols, whilst in the case of  $f((f(x_1, x_2), x_1))$  there are 3 left parentheses but only 2 function symbols.

(iii)  $\forall w_1 (R(x_1, x_1) \vee R(x_1, x_1)) \in FL$  since  $R(x_1, x_1) \in FL$  by L1, so  $(R(x_1, x_1) \vee R(x_1, x_1)) \in FL$  by L2. Since  $w_2$  does not occur in this formula, by L3 the result of replacing all occurrences of  $x_2$  in it by  $w_2$  and prefixing with  $\forall w_1$ , yields  $\forall w_1 (R(x_1, x_1) \vee R(x_1, x_1)) \in FL$ . [Notice that there's no requirement in L3 that  $x_2$  actually occurs in  $(R(x_1, x_1) \vee R(x_1, x_1))$ .]

(iv)  $\forall x_1 (R(x_1, x_1) \vee \neg R(x_1, x_1)) \notin FL$  since we can prove by induction on  $|\theta|$  for  $\theta \in FL$  that an occurrence of  $\forall$  in  $\theta$  is never immediately followed by  $x_1$  (or any *free* variable).

(v) False. It is enough to just say this (similarly in (vi),(vii)), but for the record,

$$\begin{aligned} M \models \forall w_1 \exists w_2 R(w_2, w_1) &\Leftrightarrow \text{for all } n \in |M|, \text{ there exists } m \in |M|, \langle m, n \rangle \in R^M \\ &\Leftrightarrow \text{for all } n \in |M|, \text{ there exists } m \in |M|, m < n, \end{aligned}$$

which is *not true*, there is no such  $m$  when  $n = 2$ .

(vi) True.

$$\begin{aligned} M \models \forall w_1 \forall w_2 (\exists w_3 R(f(w_1, w_3), f(w_2, w_3)) \rightarrow R(w_1, w_2)) \\ \Leftrightarrow \text{for each } n, m \in |M|, \text{ if there is } k \in |M| \text{ such} \\ \text{that } \langle f^M(n, k), f^M(m, k) \rangle \in R^M \text{ then } \langle n, m \rangle \in R^M \\ \Leftrightarrow \text{for each } n, m \in |M|, \text{ if there is } k \in |M| \text{ such} \\ \text{that } \langle n \times k, m \times k \rangle \in R^M \text{ then } \langle n, m \rangle \in R^M \\ \Leftrightarrow \text{for each } n, m \in |M|, \text{ if there is } k \in |M| \text{ such} \\ \text{that } n \times k < m \times k \text{ then } n < m, \end{aligned}$$

which is true since if  $k \in |M|$  then  $k > 0$  and so if  $n \times k < m \times k$  then  $n < m$ .

(vii) True.

$$\begin{aligned} M \models \exists w_1 \forall w_2 (R(w_2, w_1) \rightarrow R(f(w_2, w_2), w_1)) \\ \Leftrightarrow \text{there is } n \in |M| \text{ such that for any } m \in |M| \text{ if} \\ \langle m, n \rangle \in R^M \text{ then } \langle f^M(m, m), n \rangle \in R^M, \\ \Leftrightarrow \text{there is } n \in |M| \text{ such that for any } m \in |M| \text{ if} \\ m < n \text{ then } m^2 < n. \end{aligned}$$

which is *true* since for  $n = 2$  there is no  $m \in |M|$  such that  $m < n$  so for every  $m \in |M|$ ,  $m < n \Rightarrow m^2 < n$  is true.

There are lots of possible choices here, for example:

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<sup>1</sup>These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

$$\begin{aligned}
\theta_1(x_1, x_2) &= (\neg R(x_1, x_2) \wedge \neg R(x_2, x_1)) \\
\theta_2(x_1) &= \neg \exists w_1 R(w_1, x_1) \\
\theta_3(x_1, x_2) &= R(f(x_1, x_1), x_2) \\
\theta_4(x_1) &= \exists w_1 (\neg \exists w_2 R(w_2, w_1) \wedge (R(w_1, x_1) \wedge R(x_1, f(w_1, w_1))))
\end{aligned}$$

Again lots of choices here but an easy one, since  $K$  has a multiplicative identity (i.e. 1) but  $M$  does not is

$$\phi = \exists w_1 \forall w_2 \neg R(w_2, f(w_1, w_2)).$$

**A2.** It is enough to simply write down a formula in PNF logically equivalent to

$$(\exists w_1 P(w_1) \rightarrow \neg(\forall w_1 P(w_1) \wedge \exists w_1 Q(w_1))) \quad \dagger$$

but for the record here is one such derivation (there are many of course). By Lemma 1 and the ‘Useful Logical Equivalentents’,

$$(\forall w_1 P(w_1) \wedge \exists w_1 Q(w_1)) \equiv (\forall w_2 P(w_2) \wedge \exists w_3 Q(w_3)) \equiv \forall w_2 \exists w_3 (P(w_2) \wedge Q(w_3)).$$

Hence

$$\neg(\forall w_1 P(w_1) \wedge \exists w_1 Q(w_1)) \equiv \neg \forall w_2 \exists w_3 (P(w_2) \wedge Q(w_3)) \equiv \exists w_2 \forall w_3 \neg(P(w_2) \wedge Q(w_3)).$$

So

$$\begin{aligned}
(\exists w_1 P(w_1) \rightarrow \neg(\forall w_1 P(w_1) \wedge \exists w_1 Q(w_1))) \\
&\equiv (\exists w_1 P(w_1) \rightarrow \exists w_2 \forall w_3 \neg(P(w_2) \wedge Q(w_3))) \\
&\equiv \forall w_1 \exists w_2 \forall w_3 (P(w_1) \rightarrow \neg(P(w_2) \wedge Q(w_3))),
\end{aligned}$$

this last being in PNF and logically equivalent to  $\dagger$ .

**A3.** We prove this by induction on  $|\theta|$ .

If  $\theta = R(x_{i_1}, x_{i_2}, \dots, x_{i_r})$  for some  $r$ -ary relation symbol of  $L$  then  $\theta^* = \theta \in FL$ .

Now assume that  $\phi^* \in FL$  for ever  $\phi \in FL$  with  $|\phi| < |\theta|$ . We need to consider various cases:

If  $\theta = \neg\phi$  for some  $\phi \in FL$  then by the IH  $\phi^* \in FL$  (since  $|\phi| < |\theta|$ ) and  $\theta^* = \phi^*$  so  $\theta^* \in FL$ .

If  $\theta = (\phi \wedge \psi)$ , then by the IH  $\phi^*, \psi^* \in FL$  (since  $|\phi|, |\psi| < |\theta|$ ) and so

$$\theta^* = (\phi \wedge \psi)^* = (\phi^* \wedge \psi^*) \in FL.$$

The cases for  $\vee$  and  $\rightarrow$  are exactly similar.

If  $\theta = \exists w_i \psi(w_i/x_j)$  then  $|\psi| < |\theta|$  so  $\psi^* \in FL$  by the IH and

$$\theta^* = (\exists w_i \psi(w_i/x_j))^* = \exists w_i \psi^*(w_i/x_j) \in FL.$$

The case for  $\forall$  is exactly similar and this concludes the proof that  $\theta^* \in FL$  for all  $\theta \in FL$ .

**A4.** A (formal) *proof* (in PC) is a sequence of sequents

$$\Gamma_1 \mid \phi_1, \Gamma_2 \mid \phi_2 \dots, \Gamma_m \mid \phi_m$$

where the  $\Gamma_i$  are finite subsets of  $FL$ , the  $\phi_i \in FL$  and for  $i = 1, 2, \dots, m$ , either  $\Gamma_i \mid \phi_i$  is an instance of REF or there are some  $j_1, j_2, \dots, j_s < i$  such that

$$\frac{\Gamma_{j_1} \mid \phi_{j_1}, \Gamma_{j_2} \mid \phi_{j_2}, \dots, \Gamma_{j_s} \mid \phi_{j_s}}{\Gamma_i \mid \phi_i}$$

is an instance of one of the rules of proof.

A formal proof of  $\exists w_1 (\theta(w_1) \rightarrow \phi) \vdash \forall w_1 \theta(w_1) \rightarrow \phi$  where  $w_1$  does not occur in  $\phi$ :

- |   |  |                 |
|---|--|-----------------|
| 1 | $\forall w_1 \theta(w_1), \theta(x_1) \rightarrow \phi \mid \theta(x_1) \rightarrow \phi$  | REF             |
| 2 | $\forall w_1 \theta(w_1), \theta(x_1) \rightarrow \phi \mid \forall w_1 \theta(w_1)$       | REF             |
| 3 | $\forall w_1 \theta(w_1), \theta(x_1) \rightarrow \phi \mid \phi$                          | MP, 1, 2        |
| 4 | $\theta(x_1) \rightarrow \phi \mid \forall w_1 \theta(w_1) \rightarrow \phi$               | IMR, 3          |
| 5 | $\exists w_1 (\theta(w_1) \rightarrow \phi) \mid \forall w_1 \theta(w_1) \rightarrow \phi$ | $\exists I$ , 4 |

[We may assume wlog that  $x_1$  does not appear in any formula on line 1 of this proof.]

**A5.** Completeness Theorem: For  $\Gamma \subseteq FL$  and  $\theta \in FL$ ,  $\Gamma \vdash \theta \iff \Gamma \models \theta$ .

(a) Let  $M$  be the structure for  $L$  with  $|M| = \{0, 1\}$ ,  $P^M = \{0\}$ ,  $g^M(0) = g^M(1) = 0$ . Then since  $g^M(0), g^M(1) \in P^M$  and  $|M| = \{0, 1\}$ ,  $g^M(a) \in P^M$  for every  $a \in |M|$ , and  $M \models \forall w_1 P(g(w_1))$ . However  $1 \notin P^M$  so  $M \not\models P(1)$  and  $M \not\models \forall w_1 P(w_1)$ . Hence  $\forall w_1 P(g(w_1)) \not\models \forall w_1 P(w_1)$ .

(b) Let  $M$  be the structure for  $L$  such that

$$M \models \forall w_1 (P(w_1) \rightarrow \neg P(g(w_1))) \quad \star$$

Let  $a \in |M|$ . If  $M \models \neg P(a)$  then  $M \models \exists w_1 \neg P(w_1)$ . On the other hand if  $M \models P(a)$  then from  $\star$ ,

$$M \models P(a) \rightarrow \neg P(g(a))$$

and hence  $M \models \neg P(g(a))$ . By Lemma 16 (or directly) then  $M \models \neg P(g^M(a))$  so  $M \models \exists w_1 \neg P(w_1)$ . Either way then  $M \models \exists w_1 \neg P(w_1)$ . This shows that

$$\forall w_1 (P(w_1) \rightarrow \neg P(g(w_1))) \models \exists w_1 \neg P(w_1)$$

so by the Completeness Theorem,

$$\forall w_1 (P(w_1) \rightarrow \neg P(g(w_1))) \vdash \exists w_1 \neg P(w_1).$$

It is not the case that  $P(x_1) \rightarrow \neg P(g(x_1)), P(x_2) \vdash \neg P(g(x_2))$ . To show this it is enough, by the Completeness Theorem, to show  $P(x_1) \rightarrow P(g(x_1)), P(x_2) \not\models \neg P(g(x_2))$ . To this end let  $M$  be the structure as in (a) above and consider the assignment  $x_1 \mapsto 1, x_2 \mapsto 0$ . Then with this interpretation  $P(x_1) \rightarrow \neg P(g(x_1))$  is true (since  $P(x_1)$  is false) and  $P(x_2)$  is true but  $\neg P(g(x_2))$  is false (since  $M \models P(0)$ ).

**B6.** (a) By the Completeness Theorem for Normal Structures it is enough to show that if  $M$  is a normal structure for  $L$  satisfying

$$M \models \forall w_1, w_2 (\theta(w_1, w_2) \rightarrow w_1 = w_2) \quad \S$$

then

$$M \models \forall w_1, w_2 (\theta(w_1, w_2) \rightarrow \theta(w_2, w_1)).$$

So assume  $\xi$  and let  $a, b \in |M|$ . Then if  $M \models \theta(a, b)$ , from  $\xi$ ,  $M \models a = b$ , and hence  $a = b$  since  $M$  is normal. So, from  $M \models \theta(a, b)$  we obtain  $M \models \theta(b, a)$ . This shows that

$$M \models \theta(a, b) \rightarrow \theta(b, a)$$

and hence

$$M \models \forall w_1 \forall w_2 (\theta(w_1, w_2) \rightarrow \theta(w_2, w_1))$$

since  $a, b$  were arbitrary, as required.

(b) A formal proof of  $EqL, \phi(x_1), \neg\phi(x_2) \vdash \neg x_1 = x_2$  :

|    |  |                |
|----|--|----------------|
| 1  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \phi(x_1)$   | REF            |
| 2  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \neg\phi(x_2)$   | REF            |
| 3  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid x_1 = x_2$   | REF            |
| 4  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \forall w_1 \forall w_2 (w_1 = w_2 \rightarrow (\phi(w_1) \leftrightarrow \phi(w_2)))$ | Eq7            |
| 5  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \forall w_2 (x_1 = w_2 \rightarrow (\phi(x_1) \leftrightarrow \phi(w_2)))$             | $\forall O, 4$ |
| 6  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid (x_1 = x_2 \rightarrow (\phi(x_1) \leftrightarrow \phi(x_2)))$                         | $\forall O, 5$ |
| 7  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \phi(x_1) \leftrightarrow \phi(x_2)$   | MP, 3, 6       |
| 8  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \phi(x_1) \rightarrow \phi(x_2)$   | AO, 7          |
| 9  | $x_1 = x_2, \phi(x_1), \neg\phi(x_2) \mid \phi(x_2)$   | MP, 1, 8       |
| 10 | $\phi(x_1), \neg\phi(x_2) \mid \neg x_1 = x_2$   | NIN, 2, 9      |

**B7.** Suppose that  $\vdash \theta$ , say

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \dots, \Gamma_m \mid \theta_m$$

is a proof of this, so  $\Gamma_m = \emptyset$  and  $\theta_m = \theta$ . We shall show that

$$\underline{\Gamma}_1 \mid \underline{\theta}_1, \underline{\Gamma}_2 \mid \underline{\theta}_2, \dots, \underline{\Gamma}_m \mid \underline{\theta}_m \quad \blackbox$$

where

$$\underline{\Gamma} = \{\underline{\theta} \mid \theta \in \Gamma\}.$$

This suffices since  $\underline{\Gamma}_m = \emptyset$  and  $\underline{\theta}_m = \underline{\theta}$ .

We need to show that for each  $1 \leq i \leq m$ ,  $\underline{\Gamma}_i \mid \underline{\theta}_i$  is ‘justified’ in the sense of either being an instance of REF or by following from earlier sequents by one of the rules of proof. The argument depends on the justification for  $\Gamma_i \mid \theta_i$  being in  $\blackbox$ .

If  $\Gamma_i \mid \theta_i$  is an instance of REF then  $\theta_i \in \Gamma_i$ , so  $\underline{\theta}_i \in \underline{\Gamma}_i$  and  $\underline{\Gamma}_i \mid \underline{\theta}_i$  is also justified by REF. Otherwise there are some  $j_1, \dots, j_s < i$  such that

$$\frac{\Gamma_{j_1} \mid \theta_{j_1}, \dots, \Gamma_{j_s} \mid \theta_{j_s}}{\Gamma_i \mid \theta_i}$$

is an instance of one of the rules. In that case it is easy to see that, apart from the  $\exists I$  rule,

$$\frac{\underline{\Gamma}_{j_1} \mid \underline{\theta}_{j_1}, \dots, \underline{\Gamma}_{j_s} \mid \underline{\theta}_{j_s}}{\underline{\Gamma}_i \mid \underline{\theta}_i}$$

is also an instance of that same rule, and so  $\underline{\Gamma}_i \mid \underline{\theta}_i$  is likewise justified.

In the case that the rule is  $\exists I$  it looks like

$$\frac{\Gamma_{j_1} \mid \theta_{j_1}}{\Gamma_{j_1} \mid \exists w_j \theta'_{j_1}}$$

where  $\Gamma_{j_1} \mid \exists w_j \theta'_{j_1} = \Gamma_i \mid \theta_i$  and  $\theta'_{j_1}$  is the result of replacing any number of occurrences of the term  $t(\vec{x})$  in  $\theta_{j_1}$  by  $w_j$ . But then, again,

$$\frac{\underline{\Gamma}_{j_1} \mid \underline{\theta}_{j_1}}{\underline{\Gamma}_{j_1} \mid \exists w_j \underline{\theta}'_{j_1}}$$

is again an instance of  $\exists I$ , when we replace exactly these *corresponding* copies of the term  $\underline{t}(\vec{x})$  (with the obvious meaning). [The point here is that by replacing  $c$  by  $d$  we might produce new copies of this term  $\underline{t}(\vec{x})$  in  $\underline{\theta}_{j_1}$ , for example if  $t(\vec{x}) = c$ , whence  $\underline{t}(\vec{x}) = d$ , but we only substitute  $w_j$  for those which were substituted in the original instance of  $\exists I$ .]

[An alternative proof here is to use the Completeness Theorem and argue semantical about interpretations in which  $c$  and  $d$  are assigned the same value and hence for which  $\theta, \underline{\theta}$  get the same truth value.]

If only some occurrences of  $c$  are replaced by  $d$  the the result does not hold in general, for example for  $P$  a unary relation symbol,  $\vdash P(c) \vee \neg P(c)$  but  $\not\vdash P(c) \vee \neg P(d)$ .

**B8.** A suitable set of sentences of  $L$  is:

$$\Gamma = \{ \exists w_1, w_2, \dots, w_n \left( \bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \right) \mid 1 \leq n \in \mathbb{N} \}.$$

Suppose that  $\Gamma' \subseteq SL$  also has this property,  $\theta \in \Gamma'$  but for any finite subset  $\Delta$  of  $\Gamma$ ,

$$EqL, \Delta \not\vdash \theta \quad \ddagger.$$

Then consider the set of sentences

$$\{ \neg \theta \} \cup \Gamma \cup EqL.$$

If  $\Psi$  is a finite subset of  $\{ \neg \theta \} \cup \Gamma \cup EqL$  then for  $\Delta = \Psi \cap \Gamma$ ,  $\Delta$  is a finite subset of  $\Gamma$  and  $\Psi \subseteq \Delta \cup \{ \neg \theta \} \cup EqL$ . Since by assumption  $EqL, \Delta \not\vdash \theta$ ,  $\Delta \cup \{ \neg \theta \} \cup EqL$  is consistent, so in turn  $\Psi \cup EqL$  is consistent, and satisfiable in a normal structure by the Completeness Theorem for Normal Structures. By the Compactness Theorem for Normal Structures then  $\{ \neg \theta \} \cup \Gamma$  is satisfiable in a normal structure,  $M$  say. Then  $M \models \neg \theta$  and  $M \models \Gamma$ , so by the property of  $\Gamma$ ,  $|M|$  must be infinite. Therefore by the assumption on  $\Gamma'$ ,  $M \models \Gamma'$ . Hence  $M \models \theta$  since  $\theta \in \Gamma'$  – contradiction. We conclude that  $\ddagger$  holds for some finite  $\Delta \subseteq \Gamma$ .

Now suppose that  $\Gamma'$  was finite, say  $\Gamma' = \{ \theta_1, \theta_2, \dots, \theta_m \}$ . Then for each  $\theta_i \in \Gamma'$  there would be a finite  $\Delta_i \subseteq \Gamma$  such that  $EqL, \Delta_i \vdash \theta_i$ . Hence  $\Delta = \bigcup_{i=1}^m \Delta_i$  is finite and  $EqL, \Delta \vdash \theta_i$  for each  $\theta_i \in \Gamma'$ . By the Completeness Theorem for Normal Structures it follows that if  $M$  is normal and  $M \models \Delta$  then  $M \models \Gamma'$  so  $|M|$  must be infinite. Now since  $\Delta \subseteq \Gamma$  is finite there must be  $k \in \mathbb{N}$  such that

$$\Delta \subseteq \{ \exists w_1, w_2, \dots, w_n \left( \bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \right) \mid 1 \leq n \leq k \}.$$

Hence if  $K$  is a normal model of this right hand side set then it must be a model of  $EqL$  and  $\Delta$  and hence be infinite. But the normal structure  $K$  for  $L$  with  $|K| = \{1, 2, \dots, k\}$  is a model of this right hand side but clearly is not infinite. We conclude that  $\Gamma'$  must be infinite.

## Feedback

For feedback on the questions common to, or largely overlapping with, those for MATH33001 (i.e. A1,A3,A5) see the feedback for that course. For the other questions:

**A2** As usual quite well done.

**A4** Most answers were correct, the commonest error was to apply an incorrect version of the  $\exists O$  rule,

$$\frac{\Gamma \mid \exists w_i \theta(w_i)}{\Gamma \mid \theta(x_j)}$$

and an incorrect version of the  $\forall I$  rule

$$\frac{\Gamma \mid (\theta(x_i) \rightarrow \phi)}{\Gamma \mid (\forall w_j \theta(w_j) \rightarrow \phi)}$$

Obviously the advantage of inventing rules like this is that it can make it much easier to prove things. On the other hand it carries with it the disadvantage that you don't get any marks for doing so.

**B6** In part (a) I had expected students to argue directly that by the Completeness Theorem for Normal Structures it was enough to show

$$\forall w_1 \forall w_2 (\theta(w_1, w_2) \rightarrow w_1 = w_2) \models \forall w_1 \forall w_2 (\theta(w_1, w_2) \rightarrow \theta(w_2, w_1))$$

and then just consider a normal structure in which the left hand side held. Instead all the answers given used the standard Completeness Theorem and produced a formal proof, a very much longer way to get there.

In part (b) you really were required to give a formal proof and it was almost universally well done.

**B7** The obvious way to do this was by induction on the length of proof of  $\vdash \theta$  (it just amounts to replacing each formulae  $\phi$  in the proof by  $\underline{\phi}$ ). Instead many students tried to prove it by induction on  $|\theta|$ , a rather bad idea because the provability of  $\theta$  has little to do with the length of  $\theta$ . Strangely most students got the last part of this question right whereas I'd thought it was rather tricky!

**B8** Very few takers for this question, but most who did got reasonable marks.