MATH43001/63001, January 2013 Exam, Solutions[1]

A1. (i) $f(w_1) \notin TL$ since this word contains a bound variable (w_1) and we can prove by induction on $|t|$ that no term t of L can contain a bound variable. [Not necessary to give the proof but for the record: Clearly true if t is a constant or free variable x_i and if $t = f(t_1)$ and no bound variable occurs in t_1 then none will occur in t either.]

(ii) $f)x_1(\notin TL$ since no term can end with (. [Not necessary to give the proof but for the record: Clearly true if t is a free variable x_i and if $t = f(t_1)$ then t ends in), so not in (. Hence the assertion is true for all $t \in TL$ by induction on $|t|$.]

(iii) $\exists w_2(R(w_2, x_1) \to \forall w_1 R(w_1, x_1)) \in FL$ since $R(x_2, x_1) \in FL$ by L1, so $\forall w_1\, R(w_1, x_1) \in FL$ by L3. By L2 then $(R(x_2, x_1) \to \forall w_1\, R(w_1, x_1)) \in FL$ and finally by L3 $\exists w_2((R(w_2, x_1) \to \forall w_1 R(w_1, x_1)) \in FL$.

(iv) $(\neg \exists w_1\, R(x_1, x_1)) \notin FL$ since we can prove by induction on $|\theta|$ for $\theta \in FL$ that the number of left round brackets '(' in θ equals the number of relation, function and binary connective (i.e. $\wedge, \vee, \to$) symbols occurring in θ and this is not the case for $(\neg \exists w_1\, R(x_1, x_1))$. [Again it is not necessary to prove this but, for the record, such a proof could go as follows: We first prove it for terms $t \in TL$ (where of course there are are no relation symbols nor connectives) by induction on $|t|$. Moving on to formulae it is clearly true for $R(t_1, t_2)$ since it is true for t_1, t_2 and along with R we introduce one new '('. Finally, by inspection we can see that if it holds for $\phi, \psi \in FL$ then it holds for $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$, $\exists w_j\, \psi(w_j/x_i)$ and $\forall w_j\, \psi(w_j/x_i)$ (assuming here of course that w_j does not already occur in ψ).]

(v)
$$\begin{aligned} M \models \forall w_1\, R(w_1, f(w_1)) &\iff \text{for all } n \in \mathbb{N}^+,\ \langle n, f^M(n)\rangle \in R^M \\ &\iff \text{for all } n \in \mathbb{N}^+,\ n | f^M(n) \\ &\iff \text{for all } n \in \mathbb{N}^+,\ n | n+1 \end{aligned}$$

which is *not true*, for example $2 \nmid (2+1)$. [In your exam script it is enough to simply give an answer 'true'/'false', similarly with parts (vi),(vii).]

(vi) $M \models \forall w_1 \exists w_2 (R(w_1, w_2) \wedge \neg R(w_2 \wedge w_1))$

$\iff$ for each $n \in \mathbb{N}^+$ there is an $m \in \mathbb{N}^+$ such that $\langle n, m\rangle \in R^M$ and $\langle m, n\rangle \notin R^M$

$\iff$ for each $n \in \mathbb{N}^+$ there is an $m \in \mathbb{N}^+$ such that $n|m$ and $m \nmid n$

which is *true* since for each $n \in \mathbb{N}^+$, $n|2n$ but $2n \nmid n$.

(vii) $M \models \exists w_1 \forall w_2 \forall w_3\, (R(w_2, f(w_1)) \wedge R(w_3, f(w_1))) \to (R(w_2, w_3) \vee R(w_3, w_2))$

$\iff$ there is $n \in \mathbb{N}^+$ such that for any $m, k \in \mathbb{N}^+$ if $\langle m, f^M(n)\rangle$, $\langle k, f^M(n)\rangle \in R^M$ then either $\langle m, k\rangle \in R^M$ or $\langle k, m\rangle \in R^M$

$\iff$ there is $n \in \mathbb{N}^+$ such that for any $m, k \in \mathbb{N}^+$ if $m|(n+1)$ and $k|(n+1)$ then either $m|k$ or $k|m$.

which is *true* since for $n = 1$ it is the case that for any two divisors m, k of $1 + 1 = 2$, either $m|k$ or $k|m$.

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

$\theta_1(x_1) = \forall w_1\, R(x_1, w_1)$

$\theta_2(x_1, x_2) = (R(x_1, x_2) \wedge R(x_2, x_1))$

$\theta_3(x_1) = \exists w_2\, (\forall w_1\, R(w_2, w_1) \wedge \neg R(f(w_2), x_1))$

$\theta_4(x_1) = \exists w_2\, (\forall w_1\, R(w_2, w_1) \wedge \forall w_3\, (R(w_3, x_1) \to (R(w_3, w_2) \vee R(f(w_2), w_3))))$

$\phi = \forall w_1\, R(w_1, f(w_1))$ (since always $n \leq n+1$, so this holds in K, but by (v) does not hold in M).

A2. A suitable logical equivalent (there are many possibilities here) in PNF is

$$\exists w_3 \exists w_1 \forall w_2\, (R(w_1, w_2) \to R(w_3, w_3)).$$

It is enough to just write this down for the marks but for the record we could argue: Since $\exists w_1 R(w_1, w_1) \equiv \exists w_3 R(w_3, w_3)$, by the 'Useful Equivalents' (UEs for short),

$$(\forall w_1 \exists w_2\, R(w_1, w_2) \to \exists w_1 R(w_1, w_1)) \equiv (\forall w_1 \exists w_2\, R(w_1, w_2) \to \exists w_3 R(w_3, w_3)). \tag{1}$$

Again by the UEs,

$$(\forall w_1 \exists w_2\, R(w_1, w_2) \to R(x_3, x_3)) \equiv \exists w_1\, (\exists w_2\, R(w_1, w_2) \to R(x_3, x_3)) \tag{2}$$

$$(\exists w_2\, R(x_1, w_2) \to R(x_3, x_3)) \equiv \forall w_2\, (R(x_1, w_2) \to R(x_3, x_3)). \tag{3}$$

From (2),(3) resp. and the UEs,

$$\exists w_3\, (\forall w_1 \exists w_2\, R(w_1, w_2) \to R(w_3, w_3)) \equiv \exists w_3 \exists w_1\, (\exists w_2\, R(w_1, w_2) \to R(w_3, w_3)) \tag{4}$$

$$\exists w_3 \exists w_1\, (\exists w_2\, R(w_1, w_2) \to R(w_3, w_3)) \equiv \exists w_3 \exists w_1 \forall w_2\, (R(w_1, w_2) \to R(w_3, w_3)) \tag{5}$$

Putting together (1),(4),(5) with the transitivity of $\equiv$ gives the stated PNF.

Clearly we could have altered the order in which we 'moved out' the quantifiers here to give logically equivalent, but formally different, PNF's, for example

$$\exists w_3 \exists w_1 \forall w_2\, (R(w_1, w_2) \to R(w_3, w_3)) \equiv \exists w_1 \forall w_2 \exists w_3\, (R(w_1, w_2) \to R(w_3, w_3)).$$

A3. A (formal) *proof* (in PC) is a sequence of sequents

$$\Gamma_1 \mid \phi_1,\ \Gamma_2 \mid \phi_2 \ldots, \Gamma_m \mid \phi_m$$

where the Γ_i are finite subsets of FL, the $\phi_i \in FL$ and for $i = 1, 2, \ldots, m$, either $\Gamma_i \mid \phi_i$ is an instance of REF or there are some $j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1} \mid \phi_{j_1},\ \Gamma_{j_2} \mid \phi_{j_2}, \ldots, \Gamma_{j_s} \mid \phi_{j_s}}{\Gamma_i \mid \phi_i}$$

is an instance of one of the rules of proof.

A formal proof of $\exists w_1\, P(w_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \vdash \exists w_1\, Q(w_1)$:

1	$P(x_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \mid P(x_1)$	REF
2	$P(x_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \mid \forall w_1\, (P(w_1) \to Q(w_1))$	REF
3	$P(x_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \mid (P(x_1) \to Q(x_1))$	$\forall$O, 2
4	$P(x_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \mid Q(x_1)$	MP, 1,3
5	$P(x_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \mid \exists w_1\, Q(x_1)$	$\exists$I, 4
6	$\exists w_1\, P(w_1),\ \forall w_1\, (P(w_1) \to Q(w_1)) \mid \exists w_1\, Q(x_1)$	$\exists$O, 5

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.

(a) Let M be a structure for L and suppose that

$$M \models \exists w_1 \forall w_2 \, (R(w_1, w_2) \vee R(w_2, w_1)),$$

so for some $a \in |M|$,

$$M \models \forall w_2 \, (R(a, w_2) \vee R(w_2, a)),$$

equivalently for all $b \in |M|$, $M \models R(a, b) \vee R(b, a)$. Taking $b = a$ here gives $M \models R(a, a) \vee R(a, a)$ so $M \models R(a, a)$ and $M \models \exists w_1 \, R(w_1, w_1)$. This shows that

$$\exists w_1 \forall w_2 \, (R(w_1, w_2) \vee R(w_2, w_1)) \models \exists w_1 \, R(w_1, w_1)$$

and so by the Completeness Theorem,

$$\exists w_1 \forall w_2 \, (R(w_1, w_2) \vee R(w_2, w_1)) \vdash \exists w_1 \, R(w_1, w_1).$$

(b) Let M be the structure for L such that $|M| = \{0, 1\}$, $R^M = \{ \langle 0, 1 \rangle, \langle 1, 0 \rangle \}$. Then $M \models R(0, 1)$ and $M \models R(1, 0)$ so $M \models \exists w_2 \, R(0, w_2)$ and $M \models \exists w_2 \, R(1, w_2)$, and hence since $|M| = \{0, 1\}$, $M \models \forall w_1 \exists w_2 \, R(w_1, w_2)$. However since $\langle 0, 0 \rangle, \langle 1, 1 \rangle \notin R^M$, $M \not\models R(0, 0)$ and $M \not\models R(1, 1)$ and hence $M \not\models \exists w_1 \, R(w_1, w_1)$. This shows that

$$\forall w_1 \exists w_2 \, R(w_1, w_2) \not\models \exists w_1 \, R(w_1, w_1)$$

and by the Completeness Theorem it follows that

$$\forall w_1 \exists w_2 \, R(w_1, w_2) \nvdash \exists w_1 \, R(w_1, w_1).$$

It is not the case that $R(x_1, x_1) \equiv R(x_2, x_2)$ since let M be the structure with $|M| = \{1, 2\}$ and $R^M = \{\langle 1, 1 \rangle\}$. Then for the assignment $x_1 \mapsto 1$, $x_2 \mapsto 2$ $R(x_1, x_1)$ is true in M but $R(x_2, x_2)$ is not. Hence $R(x_1, x_1) \not\equiv R(x_2, x_2)$.

A5. (i)+(ii) $\not\models$ (iii): Let M be the structure for L such that $|M| = \{0\}$ and $P^M = \{0\}$, $f^M(0) = 0$. Then (i) is true in M since $M \models P(f(0))$, so $M \models P(0) \to P(f(0))$. Also $M \models$ (ii) since $M \models P(f(0))$, so $M \models \neg P(0) \vee P(f(0))$. However (iii) fails to hold in M since $f(0) = 0$ and $M \models P(f(0))$ so $M \not\models \exists w_1 \, \neg P(f(w_1))$.

(i)+(iii) $\not\models$ (ii): Let M be the structure for L with $|M| = \{0, 1, 2\}$ and $P^M = \{1\}$, $f^M(0) = f^M(1) = 1$, $f(2) = 2$. Then $M \models$ (i) since $M \not\models P(0)$, $M \not\models P(2)$ so $M \models P(0) \to P(f(0))$, $M \models P(2) \to P(f(2))$, and $M \models P(f(1))$ so $M \models P(1) \to P(f(1))$. Also $M \models$ (iii) since $M \models \neg P(f(2))$. However $M \not\models$ (ii) since $M \not\models P(0)$ and $M \not\models \neg P(f(0))$ (because $f(0) = 1$ and $M \models P(1)$).

(ii)+(iii) $\not\models$ (i): Let M be the structure for L with $|M| = \{0, 1\}$ and $P^M = \{0\}$, $f^M(0) = f^M(1) = 1$. Then $M \models P(0)$ and $M \models \neg P(f(1))$ (since $f^M(1) = 1$ and $M \not\models P(1)$) so $M \models P(0) \vee \neg P(f(0))$ and $M \models P(1) \vee \neg P(f(1))$. Hence $M \models$ (ii). Also $M \models$ (iii) since $M \models \neg P(f(1))$. However $M \models P(0)$ and $M \not\models P(f(0))$ (since $f(0) = 1$) so $M \not\models P(0) \to P(f(0))$ and in turn $M \not\models$ (i).

B6. Let $\theta(x_1, x_2, \ldots, x_m) \in FL$ and assume that for all $\phi(x_1, x_2, \ldots, x_k) \in FL$ with $|\phi| < |\theta|$ and all $r_1, r_2, \ldots, r_k \in |M|$,

$$M \models \phi(r_1, r_2, \ldots, r_k) \iff M \models \phi(q(r_1), q(r_2), \ldots, q(r_k)) \tag{6}$$

There are several cases to consider:

$\underline{\theta = P(x_i)}$ for some i.
In this case if $M \models P(n_1)$ then $n_1 \in P^M$ so $q(n_1) = 1 \in P^M$ and $M \models P(q(n_1))$, whilst if $M \not\models P(n_1)$ then $n_1 \notin P^M$ so $q(n_1) = 0 \notin P^M$ and $M \not\models P(q(n_1))$. Combining these then,

$$M \models P(n_1) \iff M \models P(q(n_1)),$$

as required in this case.

$\underline{\theta(x_1, \ldots, x_n) = (\phi(x_1, \ldots, x_m) \wedge \psi(x_1, \ldots, x_m))}$
In this case $|\phi|, |\psi| < |\theta|$ and

$$\begin{aligned}
M \models \theta(n_1, \ldots, n_m) &\iff M \models \phi(n_1, \ldots, n_m) \text{ and } M \models \psi(n_1, \ldots, n_m) \\
&\iff M \models \phi(q(n_1), \ldots, q(n_m)) \text{ and } M \models \psi(q(n_1), \ldots, q(n_m)), \\
&\qquad \text{by the Inductive Hypothesis,} \\
&\iff M \models \phi(q(n_1), \ldots, q(n_m)) \wedge \psi(q(n_1), \ldots, q(n_m)) \\
&\iff M \models \theta(q(n_1), \ldots, q(n_m)).
\end{aligned}$$

The cases for the other connectives are exactly analogous.

$\underline{\theta(x_1, \ldots, x_m) = \exists w_j \, \phi(x_1, \ldots, x_m, w_j/x_{m+1})}$
(where, purely to simplify the notation, we have assumed that it is the variable x_{m+1} that is substituted by w_j)
In this case $|\phi(x_1, \ldots, x_{m+1})| < |\theta(x_1, \ldots, x_m)|$ and

$$\begin{aligned}
M \models \theta(n_1, \ldots, n_m) &\Rightarrow \exists n_{m+1} \in |M|, M \models \phi(n_1, \ldots, n_m, n_{m+1}) \\
&\Rightarrow M \models \phi(q(n_1), \ldots, q(n_m), q(n_{m+1})) \\
&\qquad \text{by the Inductive Hypothesis,} \\
&\Rightarrow M \models \exists w_j \, \phi(q(n_1), \ldots, q(n_m), w_j) \\
&\Rightarrow M \models \theta(q(n_1), \ldots, q(n_m)),
\end{aligned}$$

and in the other direction

$$\begin{aligned}
M \models \theta(q(n_1), \ldots, q(n_m)) &\Rightarrow \exists k \in |M|, M \models \phi(q(n_1), \ldots, q(n_m), k) \\
&\Rightarrow M \models \phi(qq(n_1), \ldots, qq(n_m), q(k)) \\
&\qquad \text{by the Inductive Hypothesis,} \\
&\Rightarrow M \models \phi(q(n_1), \ldots, q(n_m), q(k)), \\
&\qquad \text{since } qq(n) = q(n) \text{ for } n \in |M|, \\
&\Rightarrow M \models \phi(n_1), \ldots, n_m, k) \\
&\qquad \text{by the Inductive Hypothesis,} \\
&\Rightarrow M \models \theta(n_1, \ldots, n_m).
\end{aligned}$$

The case for $\forall$ is directly similar.

B7. (i) A formal proof of $Eq,\ \forall w_1\, c = w_1 \vdash P(c) \to \forall w_1\, P(w_1)$:

1	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid \forall w_1\, c = w_1$	REF
2	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid c = x_1$	$\forall$O, 1
3	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid \forall w_1, w_2(w_1 = w_2 \to (P(w_1) \to P(w_2)))$	Eq4
4	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid \forall w_2(c = w_2 \to (P(c) \to P(w_2)))$	$\forall$O, 3
5	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid (c = x_1 \to (P(c) \to P(x_1)))$	$\forall$O, 4
6	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid (P(c) \to P(x_1))$	MP, $2, 5$
7	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid P(c)$	REF
8	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid P(x_1)$	MP, $6, 7$
9	$Eq,\ P(c),\ \forall w_1\, c = w_1 \mid \forall w_1\, P(w_1)$	$\forall$I, 8
10	$Eq,\ \forall w_1\, c = w_1 \mid (P(c) \to \forall w_1\, P(w_1))$	IMR, 9

(ii) A formal proof of $\exists w_1\, P(w_1) \vdash \exists w_1 \exists w_2\, (P(w_1) \wedge P(w_2))$:

1	$P(x_1),\ P(x_2) \mid P(x_1)$	REF
2	$P(x_1),\ P(x_2) \mid P(x_2)$	REF
3	$P(x_1),\ P(x_2) \mid (P(x_1) \wedge P(x_2))$	REF
4	$P(x_1),\ P(x_2) \mid \exists w_2\, (P(x_1) \wedge P(w_2))$	$\exists$I, 3
5	$P(x_1),\ P(x_2) \mid \exists w_1 \exists w_2\, (P(x_1) \wedge P(w_2))$	$\exists$I, 4
6	$P(x_1),\ \exists w_1\, P(w_1) \mid \exists w_1 \exists w_2\, (P(x_1) \wedge P(w_2))$	$\exists$O, 5
7	$\exists w_1\, P(w_1) \mid \exists w_1 \exists w_2\, (P(x_1) \wedge P(w_2))$	$\exists$O, 6

[Notice that on the last line the repetition of $\exists w_1\, P(w_1)$ disappears because the left hand side of a sequent is actually a set!]

B8. The Compactness Theorem: For $\Gamma \subseteq FL$, Γ is satisfiable in a structure for L iff every finite subset of Γ is satisfiable in a structure for L.

Assume on the contrary that such a sentence θ did exist and consider the set of sentences of L:

$$\Gamma = \{\theta\} \cup \{\, \neg \exists w_1, \ldots, w_n \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, w_i, w_{n+2}) \mid n \in \mathbb{N}^+ \,\}.$$

We first show that every finite subset of Γ is satisfiable. Let $\Delta \subseteq \Gamma$ be finite. So there is an $m \in \mathbb{N}^+$ such that if

$$\neg \exists w_1, \ldots, w_n \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, w_i, w_{n+2})$$

appears in Δ then $n \leq m$. Let K be the finite structure for L with $|K| = \{1, 2, \ldots, m, m+1\}$ and

$$T^K = \{\langle i, j, j\rangle \mid 1 \leq i, j \leq m+1\}.$$

Clearly K is finitely separated, by the set $A = \{1, 2, \ldots, m+1\}$, and indeed this is the only set which effects the separation since for each $1 \leq j \leq m+1$ the only n for which $K \models T(j, n, j)$

is j itself. Hence $K \models \theta$ and

$$K \models \neg \exists w_1, \ldots, w_n \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, w_i, w_{n+2})$$

for each $n \leq m$. So $K \models \Delta$ and Δ is satisfiable.

By the Compactness Theorem then Γ has a model, M say. Since $M \models \theta$, M is finitely separated, by $A = \{a_1, a_2, \ldots, a_n\} \subseteq |M|$ say. Then

$$M \models \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, a_i, w_{n+2})$$

so

$$M \models \exists w_1, \ldots, w_n \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, w_i, w_{n+2}).$$

But this is a contradiction since $M \models \Gamma$ and

$$\neg \exists w_1, \ldots, w_n \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, w_i, w_{n+2}) \in \Gamma.$$

Hence no such θ can exist.