MATH43001/63001, January 2012 Exam, Solutions[1]

A1 (i) $f(f(w_2)) \notin TL$ since this word contains a bound variable (w_2) and we can prove by induction on $|t|$ that no term t of L can contain a bound variable. [Not necessary to give the proof but for the record: Clearly true if t is a constant or free variable x_i and if $t = g(t_1, \ldots, t_n)$ and no bound variables occur in $t_1, \ldots, t_n$ then none will occur in t either.]

(ii) $f(f(f(f(x_1))) \notin TL$ since this word has different numbers of right ')' and left '(' round brackets and we can prove by induction on $|t|$ that any $t \in TL$ has the same number of each. [Not necessary to give the proof but for the record: Clearly true if t is a free variable x_i (when there are zero of either) and if $t = f(t_1)$ then the number of '(' in t equals 1 plus the number in t_1, equals 1 plus the number of ')' in t_1, by inductive hypothesis, which equals the number of ')' in t.]

(iii) $\forall w_1 \neg R(w_1, x_1, w_1) \in FL$ since $R(x_2, x_1, x_2) \in FL$ by L1, so $\neg R(x_2, x_1, x_2) \in FL$ by L2, and finally then $\forall w_1 \neg R(w_1, x_1, w_1) \in FL$ by L3.

(iv) $\forall w_1 R(w_1, x_1, w_1) \vee R(w_1, w_1, x_1) \notin FL$ since we can prove by induction on $|\theta|$ for $\theta \in FL$ that the number of left round brackets '(' in θ equals the number of relation, function and binary connective (i.e. $\wedge, \vee, \to$) symbols occurring in θ and this is not the case for $\forall w_1 R(w_1, x_1, w_1) \vee R(w_1, w_1, x_1)$. [Again it is not necessary to prove this but, for the record, such a proof could go as follows: We first prove it for terms $t \in TL$ (where of course there are are no relation symbols nor connectives) by induction on $|t|$. Moving on to formulae it is clearly true for $R(t_1, t_2, t_3)$ since it is true for t_1, t_2, t_3 and along with R we introduce one new '('. Finally, by inspection we can see that if it holds for $\phi, \psi \in FL$ then it holds for $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$, $\exists w_j \, \psi(w_j/x_i)$ and $\forall w_j \, \psi(w_j/x_i)$ (assuming here of course that w_j does not already occur in ψ).]

(v) $M \models \forall w_1 R(w_1, f(w_1), f(f(w_1))) \iff$

$$\begin{aligned} & \text{for all } n \in \mathbb{N}, \langle n, f^M(n), f^M(f^M(n)) \rangle \in R^M \\ \iff & \text{ for all } n \in \mathbb{N}, n < f^M(n) < f^M(f^M(n)) \\ \iff & \text{ for all } n \in \mathbb{N}, n < n+1 < (n+1)+1, \end{aligned}$$

which is *true*. [In your exam script it is enough to simply give an answer 'true'/'false', similarly with parts (vi),(vii).]

(vi) $M \models \exists w_1 \exists w_2 R(w_1, w_2, f(w_1)) \iff$

$$\begin{aligned} & \text{there are } n, m \in \mathbb{N} \text{ such that } \langle n, m, f^M(n) \rangle \in R^M \\ \iff & \text{ there are } n, m \in \mathbb{N} \text{ such that } n < m < n+1, \end{aligned}$$

which is *false* since there can be no natural number m between the natural numbers n and $n+1$.

(vii) $M \models \forall w_1 \forall w_2 \exists w_3 (R(w_1, w_3, f(f(w_2))) \vee R(w_2, w_3, f(f(w_1)))) \iff$

for all $n, m \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that either $\langle n, k, f^M(f^M(m)) \rangle \in R^M$ or $\langle m, k, f^M(f^M(n)) \rangle \in R^M$

$\iff$ for all $n, m \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that either $n < k < m+2$ or $m < k < n+2$.

This is *true* since if $n \leq m$ we can take $k = n+1$ and if $m < n$ we can take $k = m+1$.

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

$\theta_1(x_1, x_2) = R(x_1, x_2, f(x_2))$

$\theta_2(x_1, x_2) = \neg(\theta_1(x_1, x_2) \vee \theta_1(x_2, x_1)) = \neg(R(x_1, x_2, f(x_2)) \vee R(x_2, x_1, f(x_1)))$

$\theta_3(x_1, x_2) = \exists w_1\, (R(x_1, w_1, x_2) \vee R(x_2, w_1, x_1))$

$\theta_4(x_1) = \neg\exists w_1 \exists w_2\, R(w_1, x_1, w_2)$

$\phi = \exists w_1\, \neg R(w_1, f(w_1), f(f(w_1)))$ (since this holds in K when $w_1 = 0$ but by (v) does not hold in M).

$$\neg(\exists w_1\, P(w_1) \wedge \neg\exists w_1\, R(w_1)).$$

A2. A suitable logical equivalent (there are many possibilities here) in PNF is

$$\forall w_1 \exists w_2\, (\neg P(w_1) \vee R(w_2)).$$

It is enough to just write this down for the marks but for the record we could argue:

$$\neg(\exists w_1\, P(w_1) \wedge \neg\exists w_1\, R(w_1)) \equiv (\neg\exists w_1\, P(w_1) \vee \neg\neg\exists w_1\, R(w_1))) \tag{1}$$

by the 'Useful Equivalents' (UEs for short). Also by the UEs

$$\neg\exists w_1\, P(w_1) \equiv \forall w_1\, \neg P(w_1) \quad \text{and} \quad \neg\neg\exists w_1\, R(w_1) \equiv \exists w_1\, R(w_1)$$

so with Lemma 1 and (1),

$$\neg(\exists w_1\, P(w_1) \wedge \neg\exists w_1\, R(w_1)) \equiv (\forall w_1\, \neg P(w_1) \vee \exists w_1\, R(w_1)). \tag{2}$$

By the UEs $\exists w_1\, R(w_1) \equiv \exists w_2\, R(w_2)$ so by Lemma 1, (2) and the transitivity of $\equiv$ we get

$$\neg(\exists w_1\, P(w_1) \wedge \neg\exists w_1\, R(w_1)) \equiv (\forall w_1\, \neg P(w_1) \vee \exists w_2\, R(w_2)). \tag{3}$$

By the UEs again

$$(\forall w_1\, \neg P(x_1) \vee \exists w_2\, R(w_2)) \equiv \forall w_1\, (\neg P(w_1) \vee \exists w_2\, R(w_2)) \tag{4}$$

and

$$(\neg P(x_1) \vee \exists w_2\, R(w_2)) \equiv \exists w_2\, (\neg P(x_1) \vee R(w_2)), \tag{5}$$

By Lemma 1 with (5),

$$\forall w_1\, (\neg P(w_1) \vee \exists w_2\, R(w_2)) \equiv \forall w_1 \exists w_2\, (\neg P(w_1) \vee R(w_2))$$

and this with (4), (2) and the transitivity of $\equiv$ now gives the required result.

A3. A formal proof of

$$\exists w_1\, \theta(w_1) \to \phi \vdash \forall w_1\, (\theta(w_1) \to \phi)$$

where w_1 does not occur in ϕ.

1	$\theta(x_1),\ \exists w_1\, \theta(w_1) \to \phi \mid \theta(x_1)$	REF
2	$\theta(x_1),\ \exists w_1\, \theta(w_1) \to \phi \mid \exists w_1\, \theta(w_1)$	∃I, 1
3	$\theta(x_1),\ \exists w_1\, \theta(w_1) \to \phi \mid \exists w_1\, \theta(w_1) \to \phi$	REF
4	$\theta(x_1),\ \exists w_1\, \theta(w_1) \to \phi \mid \phi$	MP, 2, 3
5	$\exists w_1\, \theta(w_1) \to \phi \mid \theta(x_1) \to \phi$	IMR, 4
6	$\exists w_1\, \theta(w_1) \to \phi \mid \forall w_1\, (\theta(w_1) \to \phi)$	∀I, 5

where we may assume that the free variable x_1 is chosen not to occur in ϕ

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.
(a) Let M be the structure for L such that $|M| = \mathbb{N}$, $P^M = \{ n \in \mathbb{N} \,|\, n \text{ is even} \}$, $Q^M = \{ n \in \mathbb{N} \,|\, n \text{ is odd} \}$. Then

$$M \models \exists w_1 \, P(w_1) \tag{6}$$

since $M \vDash P(0)$. Similarly

$$M \models \exists w_1 \, Q(w_1) \tag{7}$$

since $M \vDash Q(1)$. However for any $n \in \mathbb{N}$ $M \nvDash P(n) \wedge Q(n)$ since n cannot be both even and odd. Hence $M \nvDash \exists w_1 \,(P(w_1) \wedge Q(w_1))$ so with (7)

$$M \nvDash \exists w_1 \, Q(w_1) \to \exists w_1 \,(P(w_1) \wedge Q(w_1))$$

and with (6) we obtain

$$\exists w_1 \, P(w_1) \nvDash \exists w_1 \, Q(w_1) \to \exists w_1 \,(P(w_1) \wedge Q(w_1))$$

and by the Completeness Theorem

$$\exists w_1 \, P(w_1) \nvdash \exists w_1 \, Q(w_1) \to \exists w_1 \,(P(w_1) \wedge Q(w_1)).$$

(b) Let M be a structure for L and suppose that

$$M \models \forall w_1 \,(P(w_1) \vee Q(w_1)) \qquad \star$$

but

$$M \nvDash \forall w_1 \, P(w_1) \vee \exists w_1 \, Q(w_1) \qquad \dagger$$

Then

$$M \nvDash \forall w_1 \, P(w_1) \quad \text{and} \quad M \nvDash \exists w_2 \, Q(w_2).$$

Hence for some $a \in |M|$, $M \nvDash P(a)$ and also $M \nvDash Q(a)$ since $M \nvDash \exists w_1 \, Q(w_1)$. Hence $M \nvDash P(a) \vee Q(a)$. But this contradicts $\star$. Hence given $\star$ $\dagger$ must fail, so

$$\forall w_1 \,(P(w_1) \vee Q(w_1)) \models \forall w_1 \, P(w_1) \vee \exists w_1 \, Q(w_1)$$

and by the Completeness Theorem

$$\forall w_1 \,(P(w_1) \vee Q(w_1)) \vdash \forall w_1 \, P(w_1) \vee \exists w_1 \, Q(w_1).$$

A5. (i)+(ii) $\nvDash$ (iii): Let M be the structure for L such that $|M| = \{0, 1\}$ and $R^M = \{\langle 0, 1\rangle, \langle 1, 1\rangle\}$. Then (i) is true in M since $M \nvDash R(0,0)$ and $M \nvDash R(1,0)$ so for each $n \in |M|$ there is an $m \in |M|$ such that $M \vDash \neg R(n,m)$. Also (ii) is true in M since $M \vDash R(1,1), R(0,1)$ so $M \vDash \forall w_2 \, R(w_2, 1)$. However (iii) fails to hold in M since $M \nvDash R(0,0) \vee R(0,0)$ so $M \nvDash \forall w_1 \forall w_2 \,(R(w_1,w_2) \vee R(w_2,w_1))$.
(i)+(iii) $\nvDash$ (ii): Let M be the structure for L with $|M| = \{0, 1, 2\}$ and

$$R^M = \{\langle 0, 1\rangle, \langle 0, 0\rangle, \langle 1, 2\rangle, \langle 1, 1, \rangle, \langle 2, 0\rangle, \langle 2, 2\rangle\}.$$

Then (i) is true in M since $R(0,2), R(1,0), R(2,1)$ all fail in M and (iii) is true in M since for each of the pairs $\langle n, m\rangle$, $\langle m, n\rangle$ with $n, m \in |M|$ at least one of them is in R^M. However (ii) fails since $\langle 0, 2\rangle \notin R^M$, $\langle 1, 0\rangle \notin R^M$,$\langle 2, 1\rangle \notin R^M$.

(ii)+(iii) $\not\models$ (i): Let M be the structure for L with $|M| = \{0\}$ and $R^M = \{\langle 0, 0\rangle\}$. Then for any $n \in |M|$, $\langle n, 0\rangle \in R^M$ so (ii) holds in M. Also (iii) holds since $M \models R(0,0)$ and 0 is the sole element of $|M|$. However there is no $n \in |M|$ such that $M \not\models R(0,n)$ so (i) fails in M.

B6 A suitable a formal proof of

$$EqL,\ \forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2) \vdash (R(x_1,x_2) \to R(x_1,x_1)).$$

is:

1	$\mid \forall w_1, w_2, w_3, w_4\,((w_1 = w_2 \wedge w_3 = w_4) \to (R(w_1,w_2) \leftrightarrow R(w_3,w_4)))$	Eq4,
2	$\mid \forall w_2, w_3, w_4\,((x_1 = w_2 \wedge w_3 = w_4) \to (R(x_1,w_2) \leftrightarrow R(w_3,w_4)))$	$\forall$O, 1
3	$\mid \forall w_3, w_4\,((x_1 = x_2 \wedge w_3 = w_4) \to (R(x_1,x_2) \leftrightarrow R(w_3,w_4)))$	$\forall$O, 2
4	$\mid \forall w_4\,((x_1 = x_2 \wedge x_1 = w_4) \to (R(x_1,x_2) \leftrightarrow R(x_1,w_4)))$	$\forall$O, 3
5	$\mid ((x_1 = x_2 \wedge x_1 = x_1) \to (R(x_1,x_2) \leftrightarrow R(x_1,x_1)))$	$\forall$O, 3
6	$\mid \forall w_1\, w_1 = w_1$	Eq1,
7	$\mid x_1 = x_1$	$\forall$O, 6
8	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2) \mid \forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2)$	REF
9	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2) \mid \forall w_2\,(R(x_1,w_2) \to x_1 = w_2)$	$\forall$O,8
9	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2) \mid (R(x_1,x_2) \to x_1 = x_2)$	$\forall$O,9
10	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2),\ R(x_1,x_2) \mid R(x_1,x_2)$	REF,
11	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2),\ R(x_1,x_2) \mid x_1 = x_2$	MP, 9, 10
12	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2),\ R(x_1,x_2) \mid (x_1 = x_2 \wedge x_1 = x_1)$	AND, 7, 11
13	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2),\ R(x_1,x_2) \mid (R(x_1,x_2) \leftrightarrow R(x_1,x_1))$	MP, 5, 12
14	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2),\ R(x_1,x_2) \mid (R(x_1,x_2) \to R(x_1,x_1))$	AO, 13
15	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2),\ R(x_1,x_2) \mid R(x_1,x_1)$	MP, 10, 14
16	$\forall w_1, w_2\,(R(w_1,w_2) \to w_1 = w_2) \mid (R(x_1,x_2) \to R(x_1,x_1))$	IMR, 15

B7 Suppose that $\Gamma \vdash \theta$, say $\Gamma_1 \mid \theta_1, \ldots, \Gamma_m \mid \theta_m$ is a proof of this, so $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$. We shall show that $\Gamma_1^* \mid \theta_1^*, \ldots, \Gamma_m^* \mid \theta_m^*$ is also a proof, which suffices to prove the result since clearly $\Gamma_m^* \subseteq \Gamma^*$ and $\theta_m^* = \theta^*$.

Suppose that $i < m$ and we have confirmed that $\Gamma_1^* \mid \theta_1^*, \ldots, \Gamma_{i_1}^* \mid \theta_{i-1}^*$ (vacuously true if $i = 1$). To show that $\Gamma_1^* \mid \theta_1^*, \ldots, \Gamma_i^* \mid \theta_i^*$ is a proof it is enough to show that the final sequent $\Gamma_i^* \mid \theta_i^*$ is justified. There are various cases depending on the justification for $\Gamma_i \mid \theta_i$ in the proof $\Gamma_1 \mid \theta_1, \ldots, \Gamma_m \mid \theta_m$.

Case 1: $\Gamma_i \mid \theta_i$ is justified by REF.

In this case $\theta_i \in \Gamma_i$ so $\theta_i^* \in \Gamma_i^*$ and $\Gamma_i^* \mid \theta_i^*$ is justified by REF too.

Case 2: $\Gamma_i \mid \theta_i$ is justified by the rule MP.

In this case there are $j, k < i$ such that $\theta_j = (\theta_k \to \theta_i)$ and $\Gamma_i = \Gamma_j \cup \Gamma_k$. But then $\Gamma_i^* = (\Gamma_j \cup \Gamma_k)^* = \Gamma_j^* \cup \Gamma_k^*$ and

$$\theta_i^* = (\theta_k \to \theta_i)^* = (\theta_k^* \to \theta_i^*).$$

Thus $\Gamma_i^* \mid \theta_i^*$ is also justified by MP from the corresponding $\Gamma_j^* \mid \theta_j^*$, $\Gamma_k^* \mid \theta_k^*$.

The cases for the remaining rules are exactly analogous, in each case it is the same justification from the exactly corresponding earlier sequents.

The converse is not necessarily true. For example $\nvdash P(x_1) \vee \neg Q(x_1)$ but $\vdash (P(x_1) \vee \neg Q(x_1))^*$, i.e. $\nvdash Q(x_1) \vee \neg Q(x_1)$, does hold.

B8. The Compactness Theorem: For L a language and $\Gamma \subseteq FL$, Γ is satisfiable iff every finite subset of Γ is satisfiable.

Suppose on the contrary that there was such a formula $\psi(x_1, x_2)$. Let Γ be the set of sentences $\{\psi(x_1, x_2)\} \cup \{\neg\phi_n(x_1, x_2) \mid n \in \mathbb{N}\}$ of L where $\phi_n(x_1, x_2)$ is the formula

$$\exists w_1, w_2, \ldots, w_n \left(R(x_1, w_1) \wedge R(w_n, x_2) \wedge \bigwedge_{i=1}^{n-1} R(w_i, w_{i+1}) \right).$$

Let Δ be a finite subset of Γ. Since there must be a bound on the n such that $\phi_n \in \Delta$ (otherwise Δ would be infinite) it must be that

$$\Delta \subseteq \{\psi(x_1, x_2)\} \cup \{\neg\phi_n(x_1, x_2) \mid n < m\} = \Upsilon \text{ say}$$

for some $m \in \mathbb{N}$. This Υ, and hence also Δ is satisfiable, indeed it is satisfied by $0, m+1$ in the structure M for L with universe $\{0, 1, 2, \ldots, m, m+1\}$ and

$$R^M = \{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \ldots, \langle m-1, m\rangle, \langle m, m+1\rangle,$$
$$\langle m+1, m\rangle, \langle m, m-1\rangle, \ldots, \langle 2,1\rangle, \langle 1,0\rangle, \}$$

since 0 is connected to $m+1$ in M but not by any 'path' of with less than m intermediary points.

By the Compactness Theorem then Γ is satisfiable, say in a structure K for L by b, c. Then since $K \models \psi(b, c)$, by the assumed property of $\psi(x_1, x_2)$, there is some path

$$K \models R(b, a_1) \wedge R(a_1, a_2) \wedge R(a_2, a_3) \wedge \ldots \wedge R(a_{n-1}, a_n) \wedge R(a_n, c)$$

from b to c in K. But then

$$K \models \exists w_1, w_2, \ldots, w_n \left(R(b, w_1) \wedge R(w_n, c) \wedge \bigwedge_{i=1}^{n-1} R(w_i, w_{i+1}) \right),$$

in other words, $K \models \phi_n(b, c)$ which is a contradiction since $\neg\phi_n(x_1, x_2) \in \Gamma$ and b, c are supposed to satisfy Γ in K. We conclude that no such formula $\psi(x_1, x_2)$ can exist.