MATH43001/63001, January 2011 Exam, Solutions

A1.  (i) \( f(x_1, f(x_1, x_2)) \in TL \) since \( x_1, x_2 \in TL \) by Te1, so \( f(x_1, x_2) \in TL \) by Te2 and \( f(x_1, f(x_1, x_2)) \in TL \) by Te2 again.

(ii) \( f((x_1, x_2), x_1) \notin TL \) since this word has different numbers of right and left round brackets and we can prove by induction on \(|t|\) that any \( t \in TL \) has the same number. [Not necessary to give the proof but for the record: Clearly true if \( t \) is a constant or free variable \( x_i \) (when there are zero of either) and if \( t = f(t_1, \ldots, t_n) \) then the number of ‘(‘ in \( t \) equals 1 plus the number in \( t_1, \ldots, t_n, \) equals 1 plus the number of ‘)’ in \( t_1, \ldots, t_n, \) by inductive hypothesis, equals the number of ‘)’ in \( t. \)]

(iii) \( \forall w_1 \neg R(w_1, x_1) \in FL \) since \( R(x_2, x_1) \in FL \) by L1, so \( \neg R(x_2, x_1) \in FL \) by L2, and finally then \( \forall w_1 \neg R(w_1, x_1) \in FL \) by L3.

(iv) \( \forall w_1 \neg R(w_2, x_1) \notin FL \) since we can prove by induction on \(|\theta|\) for \( \theta \in FL \) that if \( w_2 \) occurs in \( \theta \) then so does either \( \exists w_2 \) or \( \forall w_2, \) which rules out \( \forall w_1 \neg R(w_2, x_1) \) being in \( FL. \) [Again no need to prove this but for the record: Clearly true, vacuously, for \( R(t_1, t_2), \) and if it holds for \( \phi, \psi \) then it holds for \( \neg \phi, (\phi \land \psi), (\phi \lor \psi), (\phi \rightarrow \psi). \) Also if it holds for \( \eta \) and \( \eta \) does not mention \( w_j, \) if \( j \neq 2 \) then it holds for \( \exists w_j \eta(w_j/x_i) \) and \( \forall w_j \eta(w_j/x_i), \) whilst if \( j = 2 \) then the condition holds trivially for \( \exists w_2 \eta(w_2/x_2) \) and \( \forall w_2 \eta(w_2/x_2). ]

(v) \( M \models \forall w_1 \forall w_2 (R(w_1, w_2) \rightarrow R(w_2, w_1)) \iff \quad \text{for all } n, m \in N^+, \text{ if } n < m \text{ then } m < n,

which is false since, e.g. \( 1 < 2 \) but \( 2 \notin 1. \)

(vi) \( M \models \exists w_1 \forall w_2 \neg R(w_2, f(w_1, w_2)) \iff \quad \text{there is an } n \in N^+ \text{ such that for all } m \in N^+, m \notin nm, \)

which is true when we take \( n = 1 \) since \( m \neq 1 \times m \) for any \( m \in N^+. \)

(vii) \( M \models \forall w_1 (R(w_1, f(w_1, w_1)) \rightarrow \forall w_2 R(w_2, f(w_1, w_2))) \iff \quad \text{for all } n \in N^+, \text{ if } n < n^2 \text{ then for all } m \in N^+, m < nm. \)

This is true since if \( n \in N^+ \) and \( n < n^2 \) then \( n > 1 \) so \( m < nm \) for \( m \in N^+. \)

\[
\begin{align*}
\theta_1(x_1, x_2) &= R(f(x_1, x_1), x_2) \\
\theta_2(x_1, x_2) &= (\neg R(x_1, x_2) \land \neg R(x_2, x_1)) \\
\theta_3(x_1, x_2) &= (R(x_1, x_2) \land \neg \exists w_1 (R(x_1, w_1) \land R(w_1, x_2))) \\
\theta_4(x_1, x_2) &= \exists w_1 \theta_2(f(x_1, w_1), w_2) = \exists w_1 (\neg R(f(x_1, w_1), x_2) \land \neg R(x_2, f(x_1, w_1))) \\
\phi &= \forall w_1 \exists w_2 R(w_1, f(w_1, w_2)) \quad \text{(since this fails in } K \text{ when } w_1 = 0). \\
\end{align*}
\]

\(^1\)These solutions are more detailed than I would expect in the exam. That’s because I want them to also serve an educational purpose when given with ‘last year’s paper’ next year(!)
A2. A suitable logical equivalent (there are many possibilities here) in PNF is

$$\forall w_2 \forall w_1 (P(w_2) \to \neg R(w_1)).$$

It is enough to just write this down for the marks but for the record we could argue:

$$\neg \exists w_1 R(w_1) \equiv \forall w_1 \neg R(w_1) \quad \text{and} \quad \exists w_1 P(w_1) \equiv \exists w_2 P(w_2)$$

by the ‘Useful Equivalents’ (UEs for short).

$$\therefore (\exists w_1 P(w_1) \to \neg \exists w_1 R(w_1)) \equiv (\exists w_2 P(w_2) \to \forall w_1 \neg R(w_1)) \quad \text{by Lemma 1},$$

$$\therefore (\exists w_1 P(w_1) \to \neg \exists w_1 R(w_1)) \equiv \forall w_2 (P(w_2) \to \forall w_1 \neg R(w_1))$$

by UEs and transitivity of $\equiv$. Also by UEs,

$$(P(x_2) \to \forall w_1 \neg R(w_1)) \equiv \forall w_1 (P(x_2) \to \neg R(w_1))$$

so by Lemma 1,

$$\forall w_2 (P(w_2) \to \forall w_1 \neg R(w_1)) \equiv \forall w_2 \forall w_1 (P(w_2) \to \neg R(w_1))$$

and the result follows by transitivity of $\equiv$.

A3. A formal proof of $\exists w_1 \theta(w_1) \to \phi \vdash \forall w_1 (\theta(w_1) \to \phi)$ where $w_1$ does not occur in $\phi$:

1. $\theta(x_1), \exists w_1 \theta(w_1) \to \phi \mid \exists w_1 \theta(w_1) \to \phi$  
   REF
2. $\theta(x_1), \exists w_1 \theta(w_1) \to \phi \mid \theta(x_1)$  
   REF
3. $\theta(x_1), \exists w_1 \theta(w_1) \to \phi \mid \exists w_1 \theta(w_1)$  
   $\exists I, 2$
4. $\theta(x_1), \exists w_1 \theta(w_1) \to \phi \mid \phi$  
   MP, 1, 3
5. $\exists w_1 \theta(w_1) \to \phi \mid (\theta(x_1) \to \phi)$  
   IMR, 4
6. $\exists w_1 \theta(w_1) \to \phi \mid \forall w_1 (\theta(w_1) \to \phi)$  
   $\forall I, 5$

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.

(a) Let $M$ be the structure for $L$ such that $|M| = \mathbb{N}$, $P^M = \{ n \in \mathbb{N} | n \text{ is even} \}$, $Q^M = \{ n \in \mathbb{N} | n \text{ is odd} \}$. Then $M \models \forall w_1 P(w_1) \to \forall w_1 Q(w_1)$ since $M \not\models \forall w_1 P(w_1)$. However $M \not\models \forall w_1 (P(w_1) \to Q(w_1))$ since $0 \in \mathbb{N}$ is even but not odd. Hence

$$\forall w_1 P(w_1) \to \forall w_1 Q(w_1) \not\models \forall w_1 (P(w_1) \to Q(w_1))$$

and by the Completeness Theorem

$$\forall w_1 P(w_1) \to \forall w_1 Q(w_1) \not\vdash \forall w_1 (P(w_1) \to Q(w_1)).$$
(b) Let $M$ be a structure for $L$ and suppose that

$$M \models \forall w_1 \forall w_2 (P(w_1) \lor Q(w_2)) \quad \ast$$

but

$$M \not\models \forall w_1 P(w_1) \lor \exists w_2 Q(w_2) \quad \dagger$$

Then

$$M \not\models \forall w_1 P(w_1) \quad \text{and} \quad M \not\models \exists w_2 Q(w_2).$$

Hence for some $a \in |M|$, $M \not\models P(a)$ and also $M \not\models Q(a)$ since $M \not\models \exists w_2 Q(w_2)$. Hence $M \not\models P(a) \lor Q(a)$. But this contradicts $\ast$. Hence given $\ast \dagger$ must fail, so

$$\forall w_1 \forall w_2 (P(w_1) \lor Q(w_2)) \models \forall w_1 P(w_1) \lor \exists w_2 Q(w_2)$$

and by the Completeness Theorem

$$\forall w_1 \forall w_2 (P(w_1) \lor Q(w_2)) \models \forall w_1 P(w_1) \lor \exists w_2 Q(w_2).$$

A5. (i)+(ii) \(\not\equiv\) (iii): Let $M$ be the structure for $L$ such that $|M| = \mathbb{N}$ and $R^M = \{\langle n, m \rangle \in \mathbb{N}^2 \mid n < m\}$. Then (i) is true in $M$ since for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $n < m$ and (ii) is true in $M$ since $0 \in \mathbb{N}$ and $m \not\in 0$ for every $m \in \mathbb{N}$. However $M \models R(0,1)$ since $0 < 1$ but there is no $n \in \mathbb{N}$ such that $M \models R(0,n) \land R(n,1)$, i.e. $0 < n < 1$ so (iii) fails in $M$.

(i)+(ii) \(\not\equiv\) (ii): Let $M$ be the structure for $L$ with $|M| = \mathbb{R}$ and $R^M = \{\langle n, m \rangle \in \mathbb{R}^2 \mid n < m\}$. Then (i) is true in $M$ since for every $r \in \mathbb{R}$ there is an $s \in \mathbb{R}$ such that $r < s$ and (iii) is true in $M$ since if $r, s \in \mathbb{R}$ and $r < s$ then there is a $t \in \mathbb{R}$ (for example $(r + s)/2$) such that $r < t < s$. However (ii) fails in $M$ since otherwise there would have to be some $r \in \mathbb{R}$ such that for all $s \in \mathbb{R}$, $s \not< r$, which is false (take $s = r - 1$).

(ii)+(iii) \(\not\equiv\) (i): Let $M$ be the structure for $L$ with $|M| = \{0\}$ and $R^M = \emptyset$. Then for any $s \in |M|$, $\langle 0, s \rangle \not\in R^M$ so (ii) holds in $M$. Also since $\langle s, r \rangle \not\in R^M$ for any $r, s \in |M|$, $M \not\models R(s,r)$ and $M \models R(s,r) \rightarrow \exists w_3 (R(w_1, w_3) \land R(w_3, w_2))$. Hence (iii) holds in $M$. However (i) fails in $M$ since for the only element of $|M|$, 0, there is no $s \in |M|$ such that $M \models R(0, s)$, i.e. $\langle 0, s \rangle \in R^M$.

B6. **Claim** For any $\phi(\vec{x}) \in FL$ and any $\vec{a} \in |M|$, $M^* \models \phi(\vec{a}) \iff M \models \phi^*(\vec{a})$

where (as expected) $\phi^*(\vec{x})$ is the result of replacing the relation symbol $P$ everywhere in $\phi(\vec{x})$ by $Q$.

The claim is proved by induction on $|\phi|$ (for all $\vec{a}$ simultaneously). If $\phi(\vec{x}) = R(x_1, \ldots, x_m)$ and $R \not\equiv P$ then $\phi^*(\vec{x}) = \phi(\vec{x})$ and

$$M \models \phi^*(\vec{a}) \iff M \models \phi(\vec{a}) \iff \langle a_{i_1}, \ldots, a_{i_m} \rangle \in R^M$$
\[\iff (a_1, \ldots, a_m) \in R^{M^*} \iff M^* \models \phi^*(\vec{a}) \iff M^* \models \phi(\vec{a}).\]

If \( R = P \) then
\[M \models \phi^*(\vec{a}) \iff M \models Q(a_1, \ldots, a_m) \iff (a_1, \ldots, a_m) \in Q^M\]
\[\iff (a_1, \ldots, a_m) \in P^{M^*} \iff M^* \models P(a_1, \ldots, a_m) \iff M^* \models \phi(\vec{a}).\]

Assuming the result for \( \psi(\vec{x}), \eta(\vec{x}), \chi(x, \vec{x}) \) (and noticing that \(((\psi(\vec{x}) \land \eta(\vec{x})))^* = (\psi^*(\vec{x}) \land \eta^*(\vec{x}))\)) we have
\[M \models \psi^*(\vec{a}) \land \eta^*(\vec{a}) \iff M \models \psi^*(\vec{a}) \text{ and } M \models \eta^*(\vec{a}) \iff M^* \models \psi(\vec{a}) \land \eta(\vec{a})\]
and similarly for the other connectives. Also (noticing that \((\exists w_j \chi(w_j, \vec{x}))^* = \exists w_j \chi^*(w_j, \vec{a})\))
\[M \models \exists w_j \chi^*(w_j, \vec{a}) \iff \text{ for some } b \in |M|, M \models \chi^*(b, \vec{a}) \iff\]
\[\iff \text{ for some } b \in |M|(| = |M^*|), M^* \models \chi(b, \vec{a}) \text{ by Ind.Hyp.} \iff M^* \models \exists w_j \chi(w_j, \vec{a}),\]
and similarly for \( \chi(x, \vec{x}) \), completing the induction.

Now suppose that \( \models \theta(\vec{x}) \). Then for any structure \( M \) for \( L \) and assignment \( \vec{x} \mapsto \vec{a}, M^* \models \theta(\vec{a}) \) so by the claim \( M \models \theta^*(\vec{a}) \). Hence \( \models \theta^*(\vec{x}) \), as required.

The converse is not true, for example \( \models Q(x_1) \lor \neg Q(x_1) \) but \( \not\models Q(x_1) \lor \neg P(x_1) \).

**B7. A proof of EqL(=), \forall w_1 R(w_1, w_1) \models x_1 = x_2 \rightarrow R(x_1, x_2).**

1. \(x_1 = x_2, \forall w_1 R(w_1, w_1) \mid x_1 = x_2\) \hspace{1cm} \text{REF}
2. \(x_1 = x_1\) \hspace{1cm} \text{Eq1}
3. \(x_1 = x_2, \forall w_1 R(w_1, w_1) \mid (x_1 = x_1 \land x_1 = x_2)\) \hspace{1cm} \text{AND, 1, 2}
4. \(\forall w_1, w_2, w_3, w_4 ((w_1 = w_3 \land w_2 = w_4) \rightarrow (R(w_1, w_2) \rightarrow R(w_3, w_4)))\) \hspace{1cm} \text{Eq4}
5. \(\forall w_2, w_3, w_4 ((x_1 = x_1 \land x_1 = x_1) \rightarrow (R(x_1, w_2) \rightarrow R(x_1, w_4)))\) \hspace{1cm} \text{\forall O, 4}
6. \(\forall w_3, w_4 ((x_1 = x_1 \land w_1 = w_4) \rightarrow (R(x_1, w_2) \rightarrow R(x_1, w_4)))\) \hspace{1cm} \text{\forall O, 5}
7. \(\forall w_1 ((x_1 = x_1 \land x_1 = w_1) \rightarrow (R(x_1, x_1) \rightarrow R(x_1, x_1)))\) \hspace{1cm} \text{\forall O, 6}
8. \(\forall w_1 (\{x_1 = x_1 \land x_1 = x_2 \rightarrow (R(x_1, x_1) \rightarrow R(x_1, x_2)))\) \hspace{1cm} \text{\forall O, 7}
9. \(x_1 = x_2, \forall w_1 R(w_1, w_1) \mid (R(x_1, x_1) \rightarrow R(x_1, x_2))\) \hspace{1cm} \text{MP, 3, 8}
10. \(x_1 = x_2, \forall w_1 R(w_1, w_1) \mid \forall w_1 R(w_1, w_1)\) \hspace{1cm} \text{REF}
11. \(x_1 = x_2, \forall w_1 R(w_1, w_1) \mid R(x_1, x_1)\) \hspace{1cm} \text{\forall O, 10}
12. \(x_1 = x_2, \forall w_1 R(w_1, w_1) \mid R(x_1, x_2)\) \hspace{1cm} \text{MP, 9, 11}
13. \(\forall w_1 R(w_1, w_1) \mid x_1 = x_2 \rightarrow R(x_1, x_2)\) \hspace{1cm} \text{IMR, 12}
The Compactness Theorem: For a language and $\Gamma \subseteq FL$, $\Gamma$ is satisfiable iff every finite subset of $\Gamma$ is satisfiable.

Suppose on the contrary that there was such a sentence $\theta$. Let $\Gamma$ be the set of sentences $\{\theta\} \cup \{\neg \phi_n \mid n \in \mathbb{N}^+\}$ of $L$ where $\phi_n$ is the sentence

$$\exists w_1, w_2, \ldots, w_n \forall w_{n+1} \bigvee_{i=1}^{n} R(w_i, w_{n+1}).$$

Let $\Delta$ be a finite subset of $\Gamma$, so there is an $m \in \mathbb{N}^+$ such that if $\neg \phi_i \in \Delta$ then $i \leq m$. So $\Delta \subseteq \{\theta\} \cup \{\neg \phi_i \mid 1 \leq i \leq m\}$. Let $M$ be the structure for $L$ such that $|M| = \{1, 2, 3, \ldots, m + 1\}$ and

$$R^M = \{(i, i) \mid 1 \leq i \leq m + 1\}.$$

Then $M \models \theta$ since $M$ has a finite cover, namely $\{1, 2, \ldots, m + 1\}$. Also $\phi_n$ fails in $M$ for $n \leq m$ since for any $j_1, j_2, \ldots, j_n \in |M|$, $M \not\models \bigvee_{i=1}^{n} R(i, k)$ for any $k$ from the non-empty set

$$|M| - \{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, m + 1\} - \{j_1, j_2, \ldots, j_n\},$$

non-empty because

$$m + 1 = |\{1, 2, \ldots, m + 1\}| > m \geq n \geq |\{j_1, j_2, \ldots, j_n\}|.$$

Hence $M$ is a model of $\Delta$.

$. \therefore$ By the above Compactness Theorem $\Gamma$ is satisfied in some structure $K$ for $L$. Hence $K \models \theta$ so by assumption $K$ has a finite cover, $\{a_1, a_2, \ldots, a_n\}$ say. Therefore

$$K \models \forall w_{n+1} \bigvee_{i=1}^{n} R(a_i, w_{n+1})$$

and hence

$$K \models \exists w_1, w_2, \ldots, w_n \forall w_{n+1} \bigvee_{i=1}^{n} R(w_i, w_{n+1}),$$

i.e. $K \models \phi_n$. But this is a contradiction since $K \models \Gamma$ and $\neg \phi_n \in \Gamma$. We conclude that no such $\theta$ can exist, as required.