MATH43001/63001, January 2011 Exam, Solutions[1]

A1. (i) $f(x_1, f(x_1, x_2)) \in TL$ since $x_1, x_2 \in TL$ by Te1, so $f(x_1, x_2) \in TL$ by Te2 and $f(x_1, f(x_1, x_2)) \in TL$ by Te2 again.

(ii) $f((f(x_1, x_2), x_1) \notin TL$ since this word has different numbers of right and left round brackets and we can prove by induction on $|t|$ that any $t \in TL$ has the same number. [Not necessary to give the proof but for the record: Clearly true if t is a constant or free variable x_i (when there are zero of either) and if $t = f(t_1, \dots, t_n)$ then the number of '(' in t equals 1 plus the number in $t_1, \dots, t_n$, equals 1 plus the number of ')' in $t_1, \dots, t_n$, by inductive hypothesis, equals the number of ')' in t.]

(iii) $\forall w_1 \neg R(w_1, x_1) \in FL$ since $R(x_2, x_1) \in FL$ by L1, so $\neg R(x_2, x_1) \in FL$ by L2, and finally then $\forall w_1 \neg R(w_1, x_1) \in FL$ by L3.

(iv) $\forall w_1 \neg R(w_2, x_1) \notin FL$ since we can prove by induction on $|\theta|$ for $\theta \in FL$ that if w_2 occurs in θ then so does either $\exists w_2$ or $\forall w_2$, which rules out $\forall w_1 \neg R(w_2, x_1)$ being in FL. [Again no need to prove this but for the record: Clearly true, vacuously, for $R(t_1, t_2)$, and if it holds for ϕ, ψ then it holds for $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi)$. Also if it holds for η and η does not mention w_j, if $j \neq 2$ then it holds for $\exists w_j\, \eta(w_j/x_i)$ and $\forall w_j\, \eta(w_j/x_i)$, whilst if $j = 2$ then the condition holds trivially for $\exists w_2\, \eta(w_2/x_i)$ and $\forall w_2\, \eta(w_2/x_i)$.]

(v) $M \models \forall w_1 \forall w_2\, (R(w_1, w_2) \to R(w_2, w_1)) \iff$
for all $n, m \in \mathbb{N}^+$, if $n < m$ then $m < n$,

which is false since, e.g. $1 < 2$ but $2 \not< 1$.

(vi) $M \models \exists w_1 \forall w_2\, \neg R(w_2, f(w_1, w_2)) \iff$
there is an $n \in \mathbb{N}^+$ such that for all $m \in \mathbb{N}^+$, $m \not< nm$,

which is true when we take $n = 1$ since $m \not< 1 \times m$ for any $m \in \mathbb{N}^+$.

(vii) $M \models \forall w_1\, (R(w_1, f(w_1, w_1)) \to \forall w_2\, R(w_2, f(w_1, w_2))) \iff$
for all $n \in \mathbb{N}^+$, if $n < n^2$ then for all $m \in \mathbb{N}^+$, $m < nm$.

This is true since if $n \in \mathbb{N}^+$ and $n < n^2$ then $n > 1$ so $m < nm$ for $m \in \mathbb{N}^+$.

$\theta_1(x_1, x_2) = R(f(x_1, x_1), x_2)$

$\theta_2(x_1, x_2) = (\neg R(x_1, x_2) \wedge \neg R(x_2, x_1))$

$\theta_3(x_1, x_2) = (R(x_1, x_2) \wedge \neg \exists w_1\, (R(x_1, w_1) \wedge R(w_1, x_2)))$

$\theta_4(x_1, x_2) = \exists w_1\, \theta_2(f(x_1, w_1), w_2) = \exists w_1\, (\neg R(f(x_1, w_1), x_2) \wedge \neg R(x_2, f(x_1, w_1)))$

$\phi = \forall w_1 \exists w_2\, R(w_1, f(w_1, w_2))$ (since this fails in K when $w_1 = 0$).

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

A2. A suitable logical equivalent (there are many possibilities here) in PNF is

$$\forall w_2 \, \forall w_1 \, (P(w_2) \to \neg R(w_1)).$$

It is enough to just write this down for the marks but for the record we could argue:

$$\neg \exists w_1 \, R(w_1) \equiv \forall w_1 \, \neg R(w_1) \;\text{ and }\; \exists w_1 \, P(w_1) \equiv \exists w_2 \, P(w_2)$$

by the 'Useful Equivalents' (UEs for short).

$$\therefore \;\; (\exists w_1 \, P(w_1) \to \neg \exists w_1 \, R(w_1)) \equiv (\exists w_2 \, P(w_2) \to \forall w_1 \, \neg R(w_1)) \;\text{ by Lemma 1,}$$

$$\therefore \;\; (\exists w_1 \, P(w_1) \to \neg \exists w_1 \, R(w_1)) \equiv \forall w_2 \, (P(w_2) \to \forall w_1 \, \neg R(w_1))$$

by UEs and transitivity of $\equiv$. Also by UEs,

$$(P(x_2) \to \forall w_1 \, \neg R(w_1)) \equiv \forall w_1 \, (P(x_2) \to \neg R(w_1))$$

so by Lemma 1,

$$\forall w_2 \, (P(w_2) \to \forall w_1 \, \neg R(w_1)) \equiv \forall w_2 \forall w_1 \, (P(w_2) \to \neg R(w_1))$$

and the result follows by transitivity of $\equiv$.

A3. A formal proof of $\exists w_1 \, \theta(w_1) \to \phi \vdash \forall w_1 \, (\theta(w_1) \to \phi)$ where w_1 does not occur in ϕ:

1	$\theta(x_1),\ \exists w_1 \theta(w_1) \to \phi \mid \exists w_1 \theta(w_1) \to \phi$	REF
2	$\theta(x_1),\ \exists w_1 \theta(w_1) \to \phi \mid \theta(x_1)$	REF
3	$\theta(x_1),\ \exists w_1 \theta(w_1) \to \phi \mid \exists w_1 \theta(w_1)$	$\exists$I, 2
4	$\theta(x_1),\ \exists w_1 \theta(w_1) \to \phi \mid \phi$	MP, 1, 3
5	$\exists w_1 \theta(w_1) \to \phi \mid (\theta(x_1) \to \phi)$	IMR, 4
6	$\exists w_1 \theta(w_1) \to \phi \mid \forall w_1 \, (\theta(w_1) \to \phi)$	$\forall$I, 5

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.
(a) Let M be the structure for L such that $|M| = \mathbb{N}$, $P^M = \{ n \in \mathbb{N} \mid n \text{ is even} \}$, $Q^M = \{ n \in \mathbb{N} \mid n \text{ is odd} \}$. Then $M \models \forall w_1 \, P(w_1) \to \forall w_1 \, Q(w_1)$ since $M \not\models \forall w_1 \, P(w_1)$. However $M \not\models \forall w_1 \, (P(w_1) \to Q(w_1))$ since $0 \in \mathbb{N}$ is even but not odd. Hence

$$\forall w_1 \, P(w_1) \to \forall w_1 \, Q(w_1) \not\models \forall w_1 \, (P(w_1) \to Q(w_1))$$

and by the Completeness Theorem

$$\forall w_1 \, P(w_1) \to \forall w_1 \, Q(w_1) \nvdash \forall w_1 \, (P(w_1) \to Q(w_1)).$$

(b) Let M be a structure for L and suppose that

$$M \models \forall w_1 \forall w_2\,(P(w_1) \vee Q(w_2)) \qquad \star$$

but

$$M \not\models \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2) \qquad \dagger$$

Then

$$M \not\models \forall w_1\, P(w_1) \quad \text{and} \quad M \not\models \exists w_2\, Q(w_2).$$

Hence for some $a \in |M|$, $M \not\models P(a)$ and also $M \not\models Q(a)$ since $M \not\models \exists w_2\, Q(w_2)$. Hence $M \not\models P(a) \vee Q(a)$. But this contradicts $\star$. Hence given $\star$ $\dagger$ must fail, so

$$\forall w_1 \forall w_2\,(P(w_1) \vee Q(w_2)) \models \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2)$$

and by the Completeness Theorem

$$\forall w_1 \forall w_2\,(P(w_1) \vee Q(w_2)) \vdash \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2).$$

A5. (i)+(ii) $\not\models$ (iii): Let M be the structure for L such that $|M| = \mathbb{N}$ and $R^M = \{\langle n, m\rangle \in \mathbb{N}^2 \,|\, n < m\}$. Then (i) is true in M since for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $n < m$ and (ii) is true in M since $0 \in \mathbb{N}$ and $m \not< 0$ for every $m \in \mathbb{N}$. However $M \models R(0,1)$ since $0 < 1$ but there is no $n \in \mathbb{N}$ such that $M \models R(0,n) \wedge R(n,1)$, i.e. $0 < n < 1$ so (iii) fails in M.

(i)+(iii) $\not\models$ (ii): Let M be the structure for L with $|M| = \mathbb{R}$ and $R^M = \{\langle n, m\rangle \in \mathbb{R}^2 \,|\, n < m\}$. Then (i) is true in M since for every $r \in \mathbb{R}$ there is an $s \in \mathbb{R}$ such that $r < s$ and (iii) is true in M since if $r, s \in \mathbb{R}$ and $r < s$ then there is a $t \in \mathbb{R}$ (for example $(r+s)/2$) such that $r < t < s$. However (ii) fails in M since otherwise there would have to be some $r \in \mathbb{R}$ such that for all $s \in \mathbb{R}$, $s \not< r$, which is false (take $s = r - 1$).

(ii)+(iii) $\not\models$ (i): Let M be the structure for L with $|M| = \{0\}$ and $R^M = \emptyset$. Then for any $s \in |M|$, $\langle 0, s\rangle \notin R^M$ so (ii) holds in M. Also since $\langle s, r\rangle \notin R^M$ for any $r, s \in |M|$, $M \not\models R(s,r)$ and $M \models R(s,r) \to \exists w_3\,(R(w_1,w_3) \wedge R(w_3,w_2))$. Hence (iii) holds in M. However (i) fails in M since for the only element of $|M|$, 0, there is no $s \in |M|$ such that $M \models R(0,s)$, i.e. $\langle 0, s\rangle \in R^M$.

B6. <u>Claim</u> For any $\phi(\vec{x}) \in FL$ and any $\vec{a} \in |M|$,

$$M^* \models \phi(\vec{a}) \iff M \models \phi^*(\vec{a})$$

where (as expected) $\phi^*(\vec{x})$ is the result of replacing the relation symbol P everywhere in $\phi(\vec{x})$ by Q.

The claim is proved by induction on $|\phi|$ (for all $\vec{a}$ simultaneously). If $\phi(\vec{x}) = R(x_{i_1}, \ldots, x_{i_m}))$ and $R \neq P$ then $\phi^*(\vec{x}) = \phi(\vec{x})$ and

$$M \models \phi^*(\vec{a}) \iff M \models \phi(\vec{a}) \iff \langle a_{i_1}, \ldots, a_{i_m}\rangle \in R^M$$

$$\iff \langle a_{i_1},\ldots,a_{i_m}\rangle \in R^{M^*} \iff M^* \models \phi^*(\vec{a}) \iff M^* \models \phi(\vec{a}).$$

If $R = P$ then

$$M \models \phi^*(\vec{a}) \iff M \models Q(a_{i_1},\ldots,a_{i_m}) \iff \langle a_{i_1},\ldots,a_{i_m}\rangle \in Q^M$$

$$\iff \langle a_{i_1},\ldots,a_{i_m}\rangle \in P^{M^*} \iff M^* \models P(a_{i_1},\ldots,a_{i_m}) \iff M^* \models \phi(\vec{a}).$$

Assuming the result for $\psi(\vec{x}), \eta(\vec{x}), \chi(x_i,\vec{x})$ (and noticing that $((\psi(\vec{x}) \wedge \eta(\vec{x})))^* = (\psi^*(\vec{x}) \wedge \eta^*(\vec{x}))$ we have

$$M \models \psi^*(\vec{a}) \wedge \eta^*(\vec{a}) \iff M \models \psi^*(\vec{a}) \text{ and } M \models \eta^*(\vec{a}) \iff$$

$$\iff M^* \models \psi(\vec{a}) \text{ and } M^* \models \eta(\vec{a}) \text{ (by Ind.Hyp.)} \iff M^* \models \psi(\vec{a}) \wedge \eta(\vec{a})$$

and similarly for the other connectives. Also (noticing that $(\exists w_j\, \chi(w_j,\vec{x}))^* = \exists w_j\, \chi^*(w_j,\vec{x}))$

$$M \models \exists w_j\, \chi^*(w_j,\vec{a}) \iff \text{for some } b \in |M|,\ M \models \chi^*(b,\vec{a}) \iff$$

$$\iff \text{for some } b \in |M|(=|M^*|),\ M^* \models \chi(b,\vec{a}) \text{ (by Ind.Hyp.} \iff M^* \models \exists w_j\, \chi(w_j,\vec{a}),$$

and similarly for $\chi(x_i,\vec{x})$, completing the induction.

Now suppose that $\models \theta(\vec{x})$. Then for any structure M for L and assignment $\vec{x} \mapsto \vec{a}$, $M^* \models \theta(\vec{a})$ so by the claim $M \models \theta^*(\vec{a})$. Hence $\models \theta^*(\vec{x})$, as required.

The converse is not true, for example $\models Q(x_1) \vee \neg Q(x_1)$ but $\not\models Q(x_1) \vee \neg P(x_1)$.

B7. A proof of EqL$(=), \forall w_1\, R(w_1,w_1) \vdash x_1 = x_2 \to R(x_1,x_2)$.

1	$x_1 = x_2,\ \forall w_1\, R(w_1,w_1) \mid x_1 = x_2$	REF
2	$\mid x_1 = x_1$	Eq1
3	$x_1 = x_2,\ \forall w_1\, R(w_1,w_1) \mid (x_1 = x_1 \wedge x_1 = x_2)$	AND, 1, 2
4	$\mid \forall w_1,w_2,w_3,w_4\, ((w_1 = w_3 \wedge w_2 = w_4) \to (R(w_1,w_2) \to R(w_3,w_4)))$	Eq4
5	$\mid \forall w_2,w_3,w_4\, ((x_1 = w_3 \wedge w_2 = w_4) \to (R(x_1,w_2) \to R(w_3,w_4)))$	$\forall$O, 4
6	$\mid \forall w_3,w_4\, ((x_1 = w_3 \wedge x_1 = w_4) \to (R(x_1,w_2) \to R(x_1,w_4)))$	$\forall$O, 5
7	$\mid \forall w_4\, ((x_1 = x_1 \wedge x_1 = w_4) \to (R(x_1,x_1) \to R(x_1,w_4)))$	$\forall$O, 6
8	$\mid ((x_1 = x_1 \wedge x_1 = x_2) \to (R(x_1,x_1) \to R(x_1,x_2)))$	$\forall$O, 7
9	$x_1 = x_2,\ \forall w_1\, R(w_1,w_1) \mid (R(x_1,x_1) \to R(x_1,x_2))$	MP, 3, 8
10	$x_1 = x_2,\ \forall w_1\, R(w_1,w_1) \mid \forall w_1\, R(w_1,w_1)$	REF
11	$x_1 = x_2,\ \forall w_1\, R(w_1,w_1) \mid R(x_1,x_1)$	$\forall$O, 10
12	$x_1 = x_2,\ \forall w_1\, R(w_1,w_1) \mid R(x_1,x_2)$	MP, 9, 11
13	$\forall w_1\, R(w_1,w_1) \mid x_1 = x_2 \to R(x_1,x_2)$	IMR, 12

B8. The Compactness Theorem: For L a language and $\Gamma \subseteq FL$, Γ is satisfiable iff every finite subset of Γ is satisfiable.

Suppose on the contrary that there was such a sentence θ. Let Γ be the set of sentences $\{\theta\} \cup \{\neg\phi_n \,|\, n \in \mathbb{N}^+\}$ of L where ϕ_n is the sentence

$$\exists w_1, w_2, \ldots, w_n \forall w_{n+1} \bigvee_{i=1}^{n} R(w_i, w_{n+1}).$$

Let Δ be a finite subset of Γ, so there is an $m \in \mathbb{N}+$ such that if $\neg\phi_i \in \Delta$ then $i \leq m$. So $\Delta \subseteq \{\theta\} \cup \{\neg\phi_i \,|\, 1 \leq i \leq m\}$. Let M be the structure for L such that $|M| = \{1, 2, 3, \ldots, m+1\}$ and

$$R^M = \{\langle i, i\rangle \,|\, 1 \leq i \leq m+1\}.$$

Then $M \models \theta$ since M has a finite cover, namely $\{1, 2, \ldots, m+1\}$. Also ϕ_n fails in M for $n \leq m$ since for any $j_1, j_2, \ldots, j_n \in |M|$,

$$M \not\models \bigvee_{i=1}^{n} R(i_j, k)$$

for any k from the non-empty set

$$|M| - \{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, m+1\} - \{j_1, j_2, \ldots, j_n\},$$

non-empty because

$$m + 1 = |\{1, 2, \ldots, m+1\}| > m \geq n \geq |\{j_1, j_2, \ldots, j_n\}|.$$

Hence M is a model of Δ.

$\therefore$ By the above Compactness Theorem Γ is satisfied in some structure K for L. Hence $K \models \theta$ so by assumption K has a finite cover, $\{a_1, a_2, \ldots, a_n\}$ say. Therefore

$$K \models \forall w_{n+1} \bigvee_{i=1}^{n} R(a_i, w_{n+1})$$

and hence

$$K \models \exists w_1, w_2, \ldots, w_n \forall w_{n+1} \bigvee_{i=1}^{n} R(w_i, w_{n+1}),$$

i.e. $K \models \phi_n$. But this is a contradiction since $K \models \Gamma$ and $\neg\phi_n \in \Gamma$. We conclude that no such θ can exist, as required.