Three Hours

THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

23 January 2014

9.45 – 12.45

Answer ALL FIVE questions in Section A (56 marks in all).
Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted, then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

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Electronic calculators are not permitted

SECTION A

Answer ALL FIVE questions

A1. Let the language L have a binary relation symbol R and a binary function symbol f. Which of the following are terms of L? You should justify your answers.

(i) $f(x_1, f(x_1, x_1))$

(ii) $f((f(x_1, x_2), x_1))$

Which of the following are formulae of L? You should briefly justify your answers.

(iii) $\forall w_1 \, (R(x_1, x_1) \vee R(x_1, x_1))$

(iv) $\forall x_1 \, (R(x_1, x_1) \vee \neg R(x_1, x_1))$

Let M be the structure for L with $|M| = \{2, 3, 4, \ldots\}$, $f^M(n, m) = n \times m$, and

$$R^M = \{ \langle n, m \rangle \in |M|^2 \mid n < m \}.$$

Which of the following sentences of L are true in M?

(v) $\forall w_1 \exists w_2 \, R(w_2, w_1)$,

(vi) $\forall w_1 \forall w_2 \, (\exists w_3 \, R(f(w_1, w_3), f(w_2, w_3)) \to R(w_1, w_2))$,

(vii) $\exists w_1 \forall w_2 \, (R(w_2, w_1) \to R(f(w_2, w_2), w_1))$.

Find formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1)$, $\theta_3(x_1, x_2)$, $\theta_4(x_1)$ of L such that for $n \in |M|$,

$$\begin{aligned} M \models \theta_1(n, m) &\iff n = m, \\ M \models \theta_2(n) &\iff n = 2, \\ M \models \theta_3(n, m) &\iff n^2 \leq m, \\ M \models \theta_4(n) &\iff n = 3. \end{aligned}$$

Let K be the structure for L with $|K| = \mathbb{N} = \{ 1, 2, 3, \ldots \}$, $f^K(n, m) = n \times m$,

$$R^K = \{ \langle n, m \rangle \in |K|^2 \mid n < m \}.$$

Find a sentence ϕ of L such that $K \vDash \phi$ and $M \nvDash \phi$.

[25 marks]

A2. Write down a sentence in Prenex Normal Form logically equivalent to

$$(\exists w_1 \, P(w_1) \to \neg(\forall w_1 \, P(w_1) \wedge \exists w_1 \, Q(w_1)))$$

where P, Q are unary relation symbols.

[4 marks]

A3. Let L be a relational language and for $\theta \in FL$ let θ^* be the expression resulting from removing every occurrence of $\neg$ in θ. Outline a proof that for $\theta \in FL$, $\theta^* \in FL$.

[7 marks]

A4. Define what is meant by a *formal proof*. Give a formal proof of

$$\exists w_1\,(\theta(w_1) \to \phi) \vdash \forall w_1\,\theta(w_1) \to \phi$$

where w_1 does not occur in ϕ.

[8 marks]

A5. State the *Completeness Theorem*. Using this theorem or otherwise show that

(a) $\forall w_1\,P(g(w_1)) \nvDash \forall w_1\,P(w_1)$

(b) $\forall w_1\,(P(w_1) \to \neg P(g(w_1))) \vdash \exists w_1\,\neg P(w_1)$

where P is a unary relation symbol and g a unary function symbol.

Does

$$P(x_1) \to \neg P(g(x_1)), P(x_2) \vdash \neg P(g(x_2)) \quad ?$$

Justify your answer.

[12 marks]

SECTION B

Answer TWO of the THREE questions

If more than TWO questions are attempted then
credit will be given for the FIRST TWO answers.

B6. (a) Show that

$$EqL,\ \forall w_1 \forall w_2\, (\theta(w_1, w_2) \to w_1 = w_2) \models \forall w_1, w_2\, (\theta(w_1, w_2) \to \theta(w_2, w_1)).$$

(b) Give a formal proof of

$$EqL,\ \phi(x_1), \neg\phi(x_2) \vdash \neg\, x_1 = x_2.$$

[12 marks]

B7. Suppose that the language L has constant symbols c, d and for θ a formula of L let $\underline{\theta}$ be the formula of L resulting from replacing each occurrence of c in θ by d. *Outline* a proof that if $\vdash \theta$ then $\vdash \underline{\theta}$.

Does the result still hold if only some occurrences of c are replaced by d? You should justify your answer.

[12 marks]

B8. Describe an infinite set Γ of sentences of the language L with just equality such that for M a normal structure for L,

$$M \models \Gamma \iff |M| \text{ is infinite.}$$

Suppose that $\Gamma' \subseteq SL$ is another set of sentences with this same property. Show that for every $\theta \in \Gamma'$ there is a finite subset Δ of Γ such that

$$EqL,\, \Delta \vdash \theta.$$

Hence show that any such Γ' cannot be finite.

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$	
And Out (AO)	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \qquad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$	
Or In (ORR)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \qquad \dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$	
Disjunction (DIS)	$\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$	
Implies In (IMR)	$\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$	
Modus Ponens (MP)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$	
Not In (NIN)	$\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$	
Not Not Out (NNO)	$\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$	
Monotonicity (MON)	$\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$	
All In ($\forall$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \forall w_j \, \theta(w_j/x_i)}$	where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out ($\forall$O)	$\dfrac{\Gamma \mid \forall w_j \, \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In ($\exists$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists \, w_j \, \theta'}$	where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ.
Exists Out ($\exists$O)	$\dfrac{\Gamma, \, \phi \mid \theta}{\Gamma, \exists w_j \phi(w_j/x_i) \mid \theta}$	where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.
REF	$\Gamma \mid \theta$ whenever $\theta \in \Gamma$.	

The Equality Axioms, EqL

Eq1 $\forall w_1\, w_1 = w_1$

Eq2 $\forall w_1, w_2\, (w_1 = w_2 \to w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3\, ((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3)$

Eq4
$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (R(w_1, w_2, \dots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \dots, w_{2r})) \right)$$
for R an r-ary relation symbol of L.

Eq5
$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{n+i} \right) \to f(w_1, w_2, \dots, w_r) = f(w_{r+1}, w_{r+2}, \dots, w_{2r}) \right)$$
for f an r-ary function symbol of L.

Eq6
$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to t(w_1, w_2, \dots, w_r) = t(w_{r+1}, w_{r+2}, \dots, w_{2r}) \right)$$
for $t(x_1, x_2, \dots, x_r) \in TL$.

Eq7
$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (\theta(w_1, w_2, \dots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \dots, w_{2r})) \right)$$
for $\theta(x_1, x_2, \dots, x_r) \in FL$.

END OF EXAMINATION PAPER