THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

14th January 2013
9.45 – 12.15

Answer ALL FIVE questions in Section A (56 marks in total).
Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

The use of calculators is not permitted
A1. Let the language $L$ have a binary relation symbol $R$ and a unary function symbol $f$.

Which of the following are terms of $L$? You should briefly justify your answers.
(i) $f(w_1)$
(ii) $f(x_1)$

Which of the following are formulae of $L$? You should briefly justify your answers.
(iii) $\exists w_2 (R(w_2, x_1) \rightarrow \forall w_1 R(w_1, x_1))$
(iv) $(-\exists w_1 R(x_1, x_1))$

Let $M$ be the structure for $L$ with $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $f^M(n) = n + 1$,

$$R^M = \{(n, m) \in |M|^2 \mid n \text{ divides } m\}.$$ 

Which of the following sentences of $L$ are true in $M$?
(v) $\forall w_1 R(w_1, f(w_1))$
(vi) $\forall w_1 \exists w_2 (R(w_1, w_2) \land \neg R(w_2, w_1))$
(vii) $\exists w_1 \forall w_2 \forall w_3 ((R(w_2, f(w_1)) \land R(w_3, f(w_1))) \rightarrow (R(w_2, w_3) \lor R(w_3, w_2)))$

Find formulae $\theta_1(x_1), \theta_2(x_1, x_2), \theta_3(x_1), \theta_4(x_1)$ of $L$ such that for $n, m \in |M|$,
(viii) $M \models \theta_1(n) \iff n = 1$
(ix) $M \models \theta_2(n, m) \iff n = m$
(x) $M \models \theta_3(n) \iff n \text{ is odd}$
(xi) $M \models \theta_4(n) \iff n \text{ is a power of } 2$

Let $K$ be the structure for $L$ with $|K| = |M| = \mathbb{N}^+$, $f^K(n) = f^M(n) = n + 1$,

$$R^K = \{(n, m) \in |K|^2 \mid n \leq m\}.$$ 

(xii) Find a sentence $\phi$ of $L$ such that $K \models \phi$ and $M \not\models \phi$. 

[23 marks]

A2. Write down two different sentences in Prenex Normal Form each logically equivalent to

$$(\forall w_1 \exists w_2 R(w_1, w_2) \rightarrow \exists w_1 R(w_1, w_1)),$$

where $R$ is a binary relation symbol. 

[5 marks]
A3. Define what is meant by a formal proof. Give a formal proof of

$$\exists w_1 P(w_1), \forall w_1 (P(w_1) \rightarrow Q(w_1)) \vdash \exists w_1 Q(w_1)$$

where $P$, $Q$ are unary relation symbols. [9 marks]

A4. State the Completeness Theorem for Relational Languages. Using this theorem or otherwise show that

(a) $$\exists w_1 \forall w_2 (R(w_1, w_2) \lor R(w_2, w_1)) \vdash \exists w_1 R(w_1, w_1)$$

(b) $$\forall w_1 \exists w_2 R(w_1, w_2) \not\vdash \exists w_1 R(w_1, w_1)$$

where $R$ is a binary relation symbol.

Is it the case that

$$R(x_1, x_1) \equiv R(x_2, x_2)$$

Briefly explain your answer. [10 marks]

A5. Let $L$ be the language with a unary relation symbol $P$ and a unary function symbol $f$. Show that no two of the following sentences of $L$ logically imply the third:

(i) $$\forall w_1 (P(w_1) \rightarrow P(f(w_1)))$$

(ii) $$\forall w_1 (P(w_1) \lor \neg P(f(w_1)))$$

(iii) $$\exists w_1 \neg P(f(w_1))$$

[9 marks]
B6. Let the language $L$ have just a single unary relation symbol $P$ and let $M$ be a structure for $L$ such that $|M| = \mathbb{N}$, $1 \in P^M$, $0 \notin P^M$. Define a function $q : |M| \to \{0, 1\}$ by
\[
q(n) = \begin{cases} 
1 & \text{if } n \in P^M, \\
0 & \text{if } n \notin P^M.
\end{cases}
\]
Show by induction on $|\theta|$ that for $\theta(x_1, x_2, \ldots, x_m) \in FL$ and $n_1, n_2, \ldots, n_m \in |M|$,
\[
M \models \theta(n_1, n_2, \ldots, n_m) \iff M \models \theta(q(n_1), q(n_2), \ldots, q(n_m)).
\]
[For the connectives and quantifiers it is enough to do just one of the cases.]

[12 marks]

B7. Let $L$ be the language with equality, a unary relation symbol $P$ and a constant symbol $c$. Give formal proofs of:
(i) $E_{QL}, \forall w_1 c = w_1 \vdash P(c) \rightarrow \forall w_1 P(w_1)$
(ii) $\exists w_1 P(w_1) \vdash \exists w_1 \exists w_2 (P(w_1) \land P(w_2))$

[12 marks]

B8. State the Compactness Theorem for Relational Languages.

Let $L$ be the language with just a ternary relation symbol $T$. A structure $M$ for $L$ is said to have a finite separation if there is a finite subset $A$ of $|M|$ such that for every $b, c \in |M|$, $M \models T(b, a, c)$ for some $a \in A$. Show that there can be no sentence $\theta \in SL$ such that for any structure $M$ for $L$,
\[
M \models \theta \iff M \text{ has a finite separation}.
\]

[12 marks]
The Rules of Proof and Axiom for the Predicate Calculus

And In (AND) \[ \Gamma | \theta, \Delta | \phi \quad \Gamma \cup \Delta | \theta \land \phi \]

And Out (AO) \[ \Gamma | \theta \land \phi \quad \Gamma | \theta \land \phi \]
\[ \Gamma | \theta \quad \Gamma | \phi \]

Or In (ORR) \[ \Gamma | \theta \quad \Gamma | \theta \]
\[ \Gamma | \theta \lor \phi \quad \Gamma | \theta \lor \phi \]

Disjunction (DIS) \[ \Gamma, \theta | \phi, \Delta, \phi | \psi \quad \Gamma \cup \Delta, \theta \lor \phi | \psi \]

Implies In (IMR) \[ \Gamma, \theta | \phi \quad \Gamma | \theta \rightarrow \phi \]

Modus Ponens (MP) \[ \Gamma | \theta, \Delta | \theta \rightarrow \phi \quad \Gamma \cup \Delta | \phi \]

Not In (NIN) \[ \Gamma, \theta | \phi, \Delta, \theta | \neg \phi \quad \Gamma \cup \Delta | \neg \theta \]

Not Not Out (NNO) \[ \Gamma | \neg \neg \theta \quad \Gamma | \theta \]

Monotonicity (MON) \[ \Gamma | \theta \quad \Gamma \cup \Delta | \theta \]

All In (\(\forall I\)) \[ \Gamma | \theta \quad \Gamma \cup \Delta | \theta \]

where \(x_i\) does not occur in any formula in \(\Gamma\) and \(w_j\) does not occur in \(\theta\)

All Out (\(\forall O\)) \[ \Gamma \cup \Delta | \theta(w_j/x_i) \]

for \(t(\vec{x}) \in TL\)

Exists In (\(\exists I\)) \[ \Gamma | \theta \quad \Gamma \cup \Delta | \exists w_j \theta' \]

where \(\theta'\) is the result of replacing any number of occurrences of the term \(t(\vec{x})\) in \(\theta\) by \(w_j\) and \(w_j\) does not occur in \(\theta\).

Exists Out (\(\exists O\)) \[ \Gamma, \phi | \theta \quad \Gamma \cup \Delta | \exists w_j \phi(w_j/x_i) \]

where \(x_i\) does not occur in \(\theta\) nor any formula in \(\Gamma\) and \(w_j\) does not occur in \(\phi\).

REF \[ \Gamma | \theta \quad \text{whenever } \theta \in \Gamma. \]
The Equality Axioms, $EqL$

Eq1 $\forall w_1 w_1 = w_1$

Eq2 $\forall w_1, w_2 (w_1 = w_2 \rightarrow w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3 ((w_1 = w_2 \land w_2 = w_3) \rightarrow w_1 = w_3)$

Eq4 $\forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$

for $R$ an $r$-ary relation symbol of $L$.

Eq5 $\forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$

for $f$ an $r$-ary function symbol of $L$.

Eq6 $\forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$

for $t(x_1, x_2, \ldots, x_r) \in TL$.

Eq7 $\forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$

for $\theta(x_1, x_2, \ldots, x_r) \in FL$. 

END OF EXAMINATION PAPER

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