

Two and a half hours

THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

17th January 2012

9.45 – 12.15

Answer **ALL** five questions in Section A (56 marks in all) and
TWO questions in Section B (24 marks in all).

If you answer more than two questions from Section B only the first two
appearing in your answer book will be marked.

A list of axioms and rules of proof is appended to this examination paper

The use of calculators is not permitted

SECTION AAnswer **ALL** five questions

A1. Let the language L have a ternary (i.e. 3-ary) relation symbol R and a unary function symbol f . Which of the following are terms of L ? You should briefly justify your answers.

- (i) $f(f(w_2))$
- (ii) $f(f(f(f(x_1))))$

Which of the following are formulae of L ? You should briefly justify your answers.

- (iii) $\forall w_1 \neg R(w_1, x_1, w_1)$
- (iv) $\forall w_1 R(w_1, x_1, w_1) \vee R(w_1, w_1, x_1)$

Let M be the structure for L with $|M| = \mathbb{N} = \{0, 1, 2, 3, \dots\}$, $f^M(n) = n + 1$,

$$R^M = \{ \langle n, m, k \rangle \in \mathbb{N}^3 \mid n < m < k \}.$$

Which of the following sentences of L are true in M ?

- (v) $\forall w_1 R(w_1, f(w_1), f(f(w_1)))$
- (vi) $\exists w_1 \exists w_2 R(w_1, w_2, f(w_1))$
- (vii) $\forall w_1 \forall w_2 \exists w_3 (R(w_1, w_3, f(f(w_2))) \vee R(w_2, w_3, f(f(w_1))))$

Find formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1, x_2)$, $\theta_3(x_1, x_2)$, $\theta_4(x_1)$ of L such that for $n, m \in |M|$,

$$\begin{aligned} M \models \theta_1(n, m) &\iff n < m \\ M \models \theta_2(n, m) &\iff n = m \\ M \models \theta_3(n, m) &\iff |n - m| \geq 2 \\ M \models \theta_4(n) &\iff n = 0 \end{aligned}$$

Let K be the structure for L with $|K| = |M| = \mathbb{N} = \{0, 1, 2, 3, \dots\}$, $f^K(n) = n^2$,

$$R^K = R^M = \{ \langle n, m, k \rangle \in \mathbb{N}^3 \mid n < m < k \}.$$

Find a sentence ϕ of L such that $K \models \phi$ and $M \not\models \phi$.

[23 marks]

A2. Write down a sentence in Prenex Normal Form logically equivalent to

$$\neg(\exists w_1 P(w_1) \wedge \neg \exists w_1 R(w_1))$$

where P, R are unary relation symbols.

[4 marks]

A3. Give a formal proof of

$$\exists w_1 \theta(w_1) \rightarrow \phi \vdash \forall w_1 (\theta(w_1) \rightarrow \phi)$$

where w_1 does not occur in ϕ .

[8 marks]

A4. State the Completeness Theorem. Using this theorem or otherwise show that

$$(a) \quad \exists w_1 P(w_1) \not\vdash \exists w_1 Q(w_1) \rightarrow \exists w_1 (P(w_1) \wedge Q(w_1))$$

$$(b) \quad \forall w_1 (P(w_1) \vee Q(w_1)) \vdash \forall w_1 P(w_1) \vee \exists w_1 Q(w_1)$$

where P Q are unary relation symbols.

[10 marks]

A5. Let L be the language with a single binary relation symbol R . Show that no two of the following sentences of L logically imply the third:

$$(i) \quad \forall w_1 \exists w_2 R(w_1, w_2)$$

$$(ii) \quad \exists w_1 \forall w_2 \neg R(w_2, w_1)$$

$$(iii) \quad \forall w_1 \forall w_2 (R(w_1, w_2) \rightarrow \exists w_3 (R(w_1, w_3) \wedge R(w_3, w_2)))$$

[10 marks]

SECTION BAnswer **TWO** of the three questions**B6.** Give a formal proof of

$$EqL, \forall w_1, w_2 (R(w_1, w_2) \rightarrow w_1 = w_2) \vdash (R(x_1, x_2) \rightarrow R(x_1, x_1)).$$

[12 marks]

B7. Suppose that the relational language L has relation symbols P, Q of the same arity. For θ a formula of L let θ^* be the formula of L resulting from replacing each occurrence of P in θ by Q and for $\Gamma \subseteq FL$ let $\Gamma^* = \{\theta^* \mid \theta \in \Gamma\}$.

Outline a proof by induction on the length of (formal) proof that if $\Gamma \vdash \theta$ then $\Gamma^* \vdash \theta^*$.

Is the converse necessarily true?

[12 marks]

B8. State the Compactness Theorem.

Let L be the language with the single binary relation symbol R . For M a structure for L and $b, c \in |M|$ we say that b is *connected* to c in M if there is some finite sequence a_1, a_2, \dots, a_n of elements of $|M|$ such that

$$M \models R(b, a_1) \wedge R(a_1, a_2) \wedge R(a_2, a_3) \wedge \dots \wedge R(a_{n-1}, a_n) \wedge R(a_n, c).$$

Show that there is no formula $\psi(x_1, x_2)$ of L such that for M a structure for L and $b, c \in |M|$,

$$M \models \psi(b, c) \iff b \text{ is connected to } c \text{ in } M.$$

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\frac{\Gamma \theta, \Delta \phi}{\Gamma \cup \Delta \theta \wedge \phi}$	
And Out (AO)	$\frac{\Gamma \theta \wedge \phi}{\Gamma \theta} \quad \frac{\Gamma \theta \wedge \phi}{\Gamma \phi}$	
Or In (ORR)	$\frac{\Gamma \theta}{\Gamma \theta \vee \phi} \quad \frac{\Gamma \theta}{\Gamma \phi \vee \theta}$	
Disjunction (DIS)	$\frac{\Gamma, \theta \psi, \Delta, \phi \psi}{\Gamma \cup \Delta, \theta \vee \phi \psi}$	
Implies In (IMR)	$\frac{\Gamma, \theta \phi}{\Gamma \theta \rightarrow \phi}$	
Modus Ponens (MP)	$\frac{\Gamma \theta, \Delta \theta \rightarrow \phi}{\Gamma \cup \Delta \phi}$	
Not In (NIN)	$\frac{\Gamma, \theta \phi, \Delta, \theta \neg \phi}{\Gamma \cup \Delta \neg \theta}$	
Not Not Out (NNO)	$\frac{\Gamma \neg \neg \theta}{\Gamma \theta}$	
Monotonicity (MON)	$\frac{\Gamma \theta}{\Gamma \cup \Delta \theta}$	
All In (\forall I)	$\frac{\Gamma \theta}{\Gamma \forall w_j \theta(w_j/x_i)}$	where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out (\forall O)	$\frac{\Gamma \forall w_j \theta(w_j, \vec{x})}{\Gamma \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In (\exists I)	$\frac{\Gamma \theta}{\Gamma \exists w_j \theta'}$	where θ' is the result of replacing any number of occurrences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ .
Exists Out (\exists O)	$\frac{\Gamma, \phi \theta}{\Gamma, \exists w_j \phi(w_j/x_i) \theta}$	where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ .
REF	$\Gamma \theta$ whenever $\theta \in \Gamma$.	

The Equality Axioms, Eq

Eq1 $\forall w_1 w_1 = w_1$

Eq2 $\forall w_1, w_2 (w_1 = w_2 \rightarrow w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3 ((w_1 = w_2 \wedge w_2 = w_3) \rightarrow w_1 = w_3)$

Eq4

$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow (R(w_1, w_2, \dots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \dots, w_{2r})) \right)$$

for R an r -ary relation symbol of L .

Eq5

$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow f(w_1, w_2, \dots, w_r) = f(w_{r+1}, w_{r+2}, \dots, w_{2r}) \right)$$

for f an r -ary function symbol of L .

Eq6

$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow t(w_1, w_2, \dots, w_r) = t(w_{r+1}, w_{r+2}, \dots, w_{2r}) \right)$$

for $t(x_1, x_2, \dots, x_r) \in TL$.

Eq7

$$\forall w_1, \dots, w_{2r} \left(\left(\bigwedge_{i=1}^r w_i = w_{r+i} \right) \rightarrow (\theta(w_1, w_2, \dots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \dots, w_{2r})) \right)$$

for $\theta(x_1, x_2, \dots, x_r) \in FL$.