Two and a half hours

UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

17th January 2011

9.45 – 12.15

Answer **ALL** questions in Section A and **TWO** questions in Section B.

A list of axioms and rules of proof is appended to this examination paper

---

Calculators may be used but only if they cannot store text.

SECTION A

Answer **ALL** five questions

A1. Let the language L have a binary relation symbol R and binary function symbol f. Which of the following are terms of L? You should justify your answers.

(i) $f(x_1, f(x_1, x_2))$

(ii) $f((f(x_1, x_2), x_1)$

Which of the following are formulae of L? You should justify your answers.

(iii) $\forall w_1 \neg R(w_1, x_1)$

(iv) $\forall w_1 \neg R(w_2, x_1)$

Let M be the structure for L with $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $f^M(n, m) = nm$,

$$R^M = \{ \langle n, m \rangle \in |M|^2 \mid n < m \}.$$

Which of the following sentences of L are true in M?

(v) $\forall w_1 \forall w_2 (R(w_1, w_2) \to R(w_2, w_1))$

(vi) $\exists w_1 \forall w_2 \neg R(w_2, f(w_1, w_2))$

(vii) $\forall w_1 (R(w_1, f(w_1, w_1)) \to \forall w_2 R(w_2, f(w_1, w_2)))$

Find formulae $\theta_1(x_1, x_2)$, $\theta_2(x_1, x_2)$, $\theta_3(x_1, x_2)$, $\theta_4(x_1, x_2)$ of L such that for $n, m \in |M|$,

$$\begin{aligned}
M \models \theta_1(n, m) &\iff n^2 < m \\
M \models \theta_2(n, m) &\iff n = m \\
M \models \theta_3(n, m) &\iff n + 1 = m \\
M \models \theta_4(n, m) &\iff n \text{ divides } m
\end{aligned}$$

Let K be the structure for L with $|K| = \mathbb{N} = \{ 0, 1, 2, 3, \ldots\}$, $f^K(n, m) = nm$,

$$R^K = \{ \langle n, m \rangle \in |K|^2 \mid n < m \}.$$

Find a sentence ϕ of L such that $M \models \phi$ and $K \not\models \phi$.

[24 marks]

A2. Write down a sentence in Prenex Normal Form logically equivalent to

$$(\exists w_1 P(w_1) \to \neg \exists w_1 R(w_1)).$$

[4 marks]

A3. Give a formal proof of

$$\exists w_1\, \theta(w_1) \to \phi \vdash \forall w_1\, (\theta(w_1) \to \phi)$$

where w_1 does not occur in ϕ.

[8 marks]

A4. State the Completeness Theorem. Using this theorem or otherwise show that

(a) $\forall w_1\, P(w_1) \to \forall w_1\, Q(w_1) \nvdash \forall w_1\, (P(w_1) \to Q(w_1))$

(b) $\forall w_1 \forall w_2\, (P(w_1) \vee Q(w_2)) \vdash \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2)$

where P Q are unary relation symbols.

[10 marks]

A5. Let L be the language with a single binary relation symbol R. Show that no two of the following sentences of L logically imply the third:

(i) $\forall w_1 \exists w_2\, R(w_1, w_2)$

(ii) $\exists w_1 \forall w_2\, \neg R(w_2, w_1)$

(iii) $\forall w_1 \forall w_2 (R(w_1, w_2) \to \exists w_3\, (R(w_1, w_3) \wedge R(w_3, w_2)))$

[10 marks]

SECTION B

Answer **TWO** of the three questions

B6. Let L be a relational language and let P and Q be relation symbols of L of the same arity. For any $\phi(\vec{x}) \in FL$ let $\phi^*(\vec{x})$ denote the formula of L which results by replacing P everywhere in $\phi(\vec{x})$ by Q. For M a structure for L let M^* be the structure for L such that $|M^*| = |M|$, $R^{M^*} = R^M$ for R a relation symbol of L different from P whilst $P^{M^*} = Q^M$. Show that for any $\vec{a} \in |M|$,

$$M \models \phi^*(\vec{a}) \iff M^* \models \phi(\vec{a}).$$

Hence show that if $\theta(\vec{x}) \in FL$ and $\models \theta(\vec{x})$ then $\models \theta^*(\vec{x})$. Is the converse true? You should justify your answer.

[12 marks]

B7. Give a formal proof that

$$\mathrm{EqL}(=),\ \forall w_1\, R(w_1, w_1) \vdash x_1 = x_2 \to R(x_1, x_2).$$

[12 marks]

B8. State the Compactness Theorem.

Let L be the language with the single binary relation symbol R. For M a structure for L we say M has a *finite cover* if there is a finite set $A \subseteq |M|$ such that for each $b \in |M|$ there is an $a \in A$ such that $M \models R(a, b)$. Show that there can be no sentence θ of L such that for any structure M for L,

$$M \models \theta \iff M \text{ has a finite cover.}$$

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$	
And Out (AO)	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta}$ $\qquad$ $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$	
Or In (ORR)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi}$ $\qquad$ $\dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$	
Disjunction (DIS)	$\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$	
Implies In (IMR)	$\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$	
Modus Ponens (MP)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$	
Not In (NIN)	$\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$	
Not Not Out (NNO)	$\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$	
Monotonicity (MON)	$\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$	
All In (∀I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \forall w_j\, \theta(w_j/x_i)}$	where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out (∀O)	$\dfrac{\Gamma \mid \forall w_j\, \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In (∃I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists\, w_j\, \theta'}$	where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ.
Exists Out (∃O)	$\dfrac{\Gamma,\, \phi \mid \theta}{\Gamma, \exists w_j \phi(w_j/x_i) \mid \theta}$	where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.
REF	$\Gamma \mid \theta$ whenever $\theta \in \Gamma$.	

The Equality Axioms, Eq

Eq1 $\forall w_1 \, w_1 = w_1$

Eq2 $\forall w_1, w_2 \, (w_1 = w_2 \to w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3 \, ((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3)$

Eq4

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for R an r-ary relation symbol of L.

Eq5

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{n+i} \right) \to f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for f an r-ary function symbol of L.

Eq6

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for $t(x_1, x_2, \ldots, x_r) \in TL$.

Eq7

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for $\theta(x_1, x_2, \ldots, x_r) \in FL$.

END OF EXAMINATION PAPER