UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

17th January 2011
9.45 – 12.15

Answer **ALL** questions in Section A and **TWO** questions in Section B.

A list of axioms and rules of proof is appended to this examination paper

Calculators may be used but only if they cannot store text.
A1. Let the language $L$ have a binary relation symbol $R$ and binary function symbol $f$. Which of the following are terms of $L$? You should justify your answers.

(i) $f(x_1, f(x_1, x_2))$

(ii) $f((f(x_1, x_2), x_1))$

Which of the following are formulae of $L$? You should justify your answers.

(iii) $\forall w_1 \neg R(w_1, x_1)$

(iv) $\forall w_1 \neg R(w_2, x_1)$

Let $M$ be the structure for $L$ with $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $f^M(n, m) = nm$,

$$R^M = \{ \langle n, m \rangle \in |M|^2 \mid n < m \}.$$ 

Which of the following sentences of $L$ are true in $M$?

(v) $\forall w_1 \forall w_2 (R(w_1, w_2) \rightarrow R(w_2, w_1))$

(vi) $\exists w_1 \forall w_2 \neg R(w_2, f(w_1, w_2))$

(vii) $\forall w_1 (R(w_1, f(w_1, w_1)) \rightarrow \forall w_2 R(w_2, f(w_1, w_2)))$

Find formulae $\theta_1(x_1, x_2), \theta_2(x_1, x_2), \theta_3(x_1, x_2), \theta_4(x_1, x_2)$ of $L$ such that for $n, m \in |M|$, $M \models \theta_1(n, m) \iff n^2 < m$, $M \models \theta_2(n, m) \iff n = m$, $M \models \theta_3(n, m) \iff n + 1 = m$, $M \models \theta_4(n, m) \iff n \text{ divides } m$.

Let $K$ be the structure for $L$ with $|K| = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $f^K(n, m) = nm$,

$$R^K = \{ \langle n, m \rangle \in |K|^2 \mid n < m \}.$$ 

Find a sentence $\phi$ of $L$ such that $M \models \phi$ and $K \not\models \phi$. [24 marks]

A2. Write down a sentence in Prenex Normal Form logically equivalent to

$$(\exists w_1 P(w_1) \rightarrow \neg \exists w_1 R(w_1)).$$ 

[4 marks]
A3. Give a formal proof of
\[ \exists w_1 \theta(w_1) \rightarrow \phi \vdash \forall w_1 (\theta(w_1) \rightarrow \phi) \]
where \( w_1 \) does not occur in \( \phi \). [8 marks]

A4. State the Completeness Theorem. Using this theorem or otherwise show that

(a) \( \forall w_1 P(w_1) \rightarrow \forall w_1 Q(w_1) \not\vdash \forall w_1 (P(w_1) \rightarrow Q(w_1)) \)
(b) \( \forall w_1 \forall w_2 (P(w_1) \lor Q(w_2)) \vdash \forall w_1 P(w_1) \lor \exists w_2 Q(w_2) \)

where \( P Q \) are unary relation symbols. [10 marks]

A5. Let \( L \) be the language with a single binary relation symbol \( R \). Show that no two of the following sentences of \( L \) logically imply the third:

(i) \( \forall w_1 \exists w_2 R(w_1, w_2) \)
(ii) \( \exists w_1 \forall w_2 \neg R(w_2, w_1) \)
(iii) \( \forall w_1 \forall w_2 (R(w_1, w_2) \rightarrow \exists w_3 (R(w_1, w_3) \land R(w_3, w_2))) \)

[10 marks]
B6. Let $L$ be a relational language and let $P$ and $Q$ be relation symbols of $L$ of the same arity. For any $\phi(\vec{x}) \in FL$ let $\phi^*(\vec{x})$ denote the formula of $L$ which results by replacing $P$ everywhere in $\phi(\vec{x})$ by $Q$. For $M$ a structure for $L$ let $M^*$ be the structure for $L$ such that $|M^*| = |M|$, $R^M = R^M$ for $R$ a relation symbol of $L$ different from $P$ whilst $P^M = Q^M$. Show that for any $\vec{a} \in |M|,$

$$M \models \phi^*(\vec{a}) \iff M^* \models \phi(\vec{a}).$$

Hence show that if $\theta(\vec{x}) \in FL$ and $\models \theta(\vec{x})$ then $\models \theta^*(\vec{x})$. Is the converse true? You should justify your answer.

[12 marks]

B7. Give a formal proof that

$$\text{EqL}(=), \forall w_1 R(w_1, w_1) \models x_1 = x_2 \rightarrow R(x_1, x_2).$$

[12 marks]

B8. State the Compactness Theorem.

Let $L$ be the language with the single binary relation symbol $R$. For $M$ a structure for $L$ we say $M$ has a finite cover if there is a finite set $A \subseteq |M|$ such that for each $b \in |M|$ there is an $a \in A$ such that $M \models R(a, b)$. Show that there can be no sentence $\theta$ of $L$ such that for any structure $M$ for $L$,

$$M \models \theta \iff M \text{ has a finite cover}.$$

[12 marks]
The Rules of Proof and Axiom for the Predicate Calculus

And In (AND) \[
\frac{\Gamma \mid \theta, \ \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \land \phi}
\]

And Out (AO) \[
\frac{\Gamma \mid \theta \land \phi}{\Gamma \mid \theta}, \quad \frac{\Gamma \mid \theta \land \phi}{\Gamma \mid \phi}
\]

Or In (ORR) \[
\frac{\Gamma \mid \theta}{\Gamma \mid \theta \lor \phi}, \quad \frac{\Gamma \mid \theta}{\Gamma \mid \phi \lor \theta}
\]

Disjunction (DIS) \[
\frac{\Gamma, \theta \mid \psi, \ \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \lor \phi \mid \psi}
\]

Implies In (IMR) \[
\frac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \rightarrow \phi}
\]

Modus Ponens (MP) \[
\frac{\Gamma \mid \theta, \ \Delta \mid \theta \rightarrow \phi}{\Gamma \cup \Delta \mid \phi}
\]

Not In (NIN) \[
\frac{\Gamma, \theta \mid \phi, \ \Delta, \theta \mid \neg \phi}{\Gamma \cup \Delta \mid \neg \theta}
\]

Not Not Out (NNO) \[
\frac{\Gamma \mid \neg \neg \theta}{\Gamma \mid \theta}
\]

Monotonicity (MON) \[
\frac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}
\]

All In (\(\forall I\)) \[
\frac{\Gamma \mid \theta}{\Gamma \mid \forall w_j \theta(w_j/x_i)}
\] where \(x_i\) does not occur in any formula in \(\Gamma\) and \(w_j\) does not occur in \(\theta\)

All Out (\(\forall O\)) \[
\frac{\Gamma \mid \forall w_j \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}
\] for \(t(\vec{x}) \in TL\)

Exists In (\(\exists I\)) \[
\frac{\Gamma \mid \theta}{\Gamma \mid \exists w_j \theta'}
\] where \(\theta'\) is the result of replacing any number of occurrences of the term \(t(\vec{x})\) in \(\theta\) by \(w_j\) and \(w_j\) does not occur in \(\theta\).

Exists Out (\(\exists O\)) \[
\frac{\Gamma, \phi \mid \theta}{\Gamma, \exists w_j \phi(w_j/x_i) \mid \theta}
\] where \(x_i\) does not occur in \(\theta\) nor any formula in \(\Gamma\) and \(w_j\) does not occur in \(\phi\).

REF \[
\Gamma \mid \theta \quad \text{whenever } \theta \in \Gamma.
\]
The Equality Axioms, Eq

Eq1 \( \forall w_1 \, w_1 = w_1 \)

Eq2 \( \forall w_1, w_2 \,( w_1 = w_2 \rightarrow w_2 = w_1 ) \)

Eq3 \( \forall w_1, w_2, w_3 \,( ( w_1 = w_2 \land w_2 = w_3 ) \rightarrow w_1 = w_3 ) \)

Eq4
\[ \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow ( R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r}) ) \right) \]

for \( R \) an \( r \)-ary relation symbol of \( L \).

Eq5
\[ \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{n+i} \right) \rightarrow f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right) \]

for \( f \) an \( r \)-ary function symbol of \( L \).

Eq6
\[ \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right) \]

for \( t(x_1, x_2, \ldots, x_r) \in TL \).

Eq7
\[ \forall w_1, \ldots, w_{2r} \left( \left( \bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \rightarrow ( \theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r}) ) \right) \]

for \( \theta(x_1, x_2, \ldots, x_r) \in FL \).