MATH33001/43001/63001 Coursework 2 Solutions and Feedback, 2011-12

The Problems

1. Give formal proofs of

(i) $\forall w_1 \neg(P(w_1) \vee Q(w_1)) \vdash \forall w_1 \neg P(w_1)$

(ii) $\forall w_1 \exists w_2 \, P(w_2) \vdash \exists w_2 \forall w_1 \, P(w_2)$

where P, Q are unary relation symbols.

2.[1] Show that if f is a unary function symbol of L which does not occur in $\theta(x_1) \in FL$ and $\models \forall w_1 \, \theta(f(w_1))$ then $\models \forall w_1 \, \theta(w_1)$.

[To simplify the notation (and in line with our convention) you may assume that x_1 is the only free variable appearing in $\theta(x_1)$.]

The Solutions

1.(i) A formal proof of $\forall w_1 \neg(P(w_1) \vee Q(w_1)) \vdash \forall w_1 \neg P(w_1)$

1	$P(x_1), \forall w_1 \neg(P(w_1) \vee Q(w_1)) \mid \forall w_1 \neg(P(w_1) \vee Q(w_1))$	REF
2	$P(x_1), \forall w_1 \neg(P(w_1) \vee Q(w_1)) \mid \neg(P(x_1) \vee Q(x_1))$	$\forall$O, 1
3	$P(x_1), \forall w_1 \neg(P(w_1) \vee Q(w_1)) \mid P(x_1)$	REF
4	$P(x_1), \forall w_1 \neg(P(w_1) \vee Q(w_1)) \mid P(x_1) \vee Q(x_1)$	ORR, 3
5	$\forall w_1 \neg(P(w_1) \vee Q(w_1)) \mid \neg P(x_1)$	NIN, 2, 4
6	$\forall w_1 \neg(P(w_1) \vee Q(w_1)) \mid \forall w_1 \neg P(w_1)$	$\forall$I, 5

(ii) A formal proof of $\forall w_1 \exists w_2 \, P(w_2) \vdash \exists w_2 \forall w_1 \, P(w_2)$

1	$\forall w_1 \exists w_2 \, P(w_2) \mid \forall w_1 \exists w_2 \, P(w_2)$	REF
2	$\forall w_1 \exists w_2 \, P(w_2) \mid \exists w_2 \, P(w_2)$	$\forall$O, 1
3	$P(x_1) \mid P(x_1)$	REF
4	$P(x_1) \mid \forall w_1 \, P(x_1)$	$\forall$I, 3
5	$P(x_1) \mid \exists w_2 \forall w_1 \, P(w_2)$	$\exists$I, 4
6	$\exists w_2 \, P(w_2) \mid \exists w_2 \forall w_1 \, P(w_2)$	$\exists$O, 5
7	$\mid \exists w_2 \, P(w_2) \to \exists w_2 \forall w_1 \, P(w_2)$	IMR, 6
8	$\forall w_1 \exists w_2 \, P(w_2) \mid \exists w_2 \forall w_1 \, P(w_2)$	MP, 2, 7

2. Let M be a structure for L and $a \in |M|$. Let K be the structure for L which completely agrees with M except that $f^K(a) = a$. Then since $\theta(x_1)$ does not mention f, M and K must be exactly the same on the relation, constant, function symbols appearing in $\theta(x_1)$, so

$$K \models \theta(a) \iff M \models \theta(a). \qquad (1)$$

[1] Only for levels 4&6 students.

[This is clear but in any case it is easily proved by induction on the length of formulae, see Example 28 on the Example Sheet.]
$\therefore$ given that $\models \forall w_1\, \theta(f(w_1))$, $K \models \theta(f(a))$, so by Lemma 16* (or take it as obvious) $K \models \theta(f^K(a))$, i.e. $K \models \theta(a)$. $\therefore$ by (1), $M \models \theta(a)$, so since a was an arbitrary element of $|M|$, $M \models \forall w_1\, \theta(w_1)$. Finally then since M was an arbitrary structure for L, $\models \forall w_1\, \theta(w_1)$.

The Feedback

Generally the MATH33001 students did very well, the average mark was close to 9. Strangely the MATH43001/63001 students on average didn't perform well at all, even on question 1 which was common to both tests, their average mark being around 5.

An error on Question 1(i) was to invent new rules which weren't on the list – no way is this permitted. For example going from $\Gamma \mid \neg(P(x_1) \vee Q(x_1))$ to $\Gamma \mid \neg P(x_1) \wedge \neg Q(x_1)$ by the 'rule of logical equivalence'!

Amongst those students who did not get Question 2 correct a common error was to incorrectly apply the rule MP as in:

$n.$	$\Gamma, \forall w_1 P(w_1) \mid \forall w_1 P(w_1)$		
$n+1.$	$\Gamma, \forall w_1 P(w_1) \mid P(x_1)$	$\forall O$	n
$n+2.$	$\Gamma \mid \forall w_1 P(w_1) \to P(x_1)$	IMR	$n+1$
$n+3.$	$\Gamma \mid P(x_1)$	$MP,$	$n, n+2$

The error was that on line $n+3$ the $\forall w_1 P(w_1)$ on the left hand side of line n should now reappear on the left of line $n+3$. It should have been obvious that there was something wrong here since for $\Gamma = \emptyset$ here we could in this way prove *any* $P(x_1)$ from no assumptions at all. Indeed it is always a good idea as you're producing a proof to check at each stage that you think it reasonable that the rhs does follow from the lhs. [Similarly you should always be suspicious at the end of a proof if you never actually used one of the given lhs assumptions!]

As expected a mistake some students made on this question was to apply the $\exists O$ rule using a variable which also appeared on the rhs. This can often be avoided by arranging the order in which you use the quantifier rules so that at the stage when $\exists O$ is applied the free variable in question has already been removed from the rhs (usually by an application of $\exists I$, see the model answer).

Question 2 for the level 4 & 6 students was not well done, almost nobody saw why it held. I can only suggest you look at the model answer.