The Problems

1. Give formal proofs of
   (i) \( \forall w_1 \neg (P(w_1) \lor Q(w_1)) \vdash \forall w_1 \neg P(w_1) \)
   (ii) \( \forall w_1 \exists w_2 P(w_2) \vdash \exists w_2 \forall w_1 P(w_1) \)
where \( P, Q \) are unary relation symbols.

2. Show that if \( f \) is a unary function symbol of \( L \) which does not occur in \( \theta(x_1) \in FL \) and \( \models \forall w_1 \theta(f(w_1)) \) then \( \models \forall w_1 \theta(w_1) \).

[To simplify the notation (and in line with our convention) you may assume that \( x_1 \) is the only free variable appearing in \( \theta(x_1) \).]

The Solutions

1.(i) A formal proof of \( \forall w_1 \neg (P(w_1) \lor Q(w_1)) \vdash \forall w_1 \neg P(w_1) \)

   1 \( P(x_1), \forall w_1 \neg (P(w_1) \lor Q(w_1)) \mid \forall w_1 \neg (P(w_1) \lor Q(w_1)) \) \text{ REF}
   2 \( P(x_1), \forall w_1 \neg (P(w_1) \lor Q(w_1)) \mid \neg (P(x_1) \lor Q(x_1)) \) \text{ \forall O, 1}
   3 \( P(x_1), \forall w_1 \neg (P(w_1) \lor Q(w_1)) \mid P(x_1) \) \text{ REF}
   4 \( P(x_1), \forall w_1 \neg (P(w_1) \lor Q(w_1)) \mid P(x_1) \lor Q(x_1) \) \text{ ORR, 3}
   5 \( \forall w_1 \neg (P(w_1) \lor Q(w_1)) \mid \neg P(x_1) \) \text{ NIN, 2, 4}
   6 \( \forall w_1 \neg (P(w_1) \lor Q(w_1)) \mid \forall w_1 \neg P(w_1) \) \text{ \forall I, 5}

(ii) A formal proof of \( \forall w_1 \exists w_2 P(w_2) \vdash \exists w_2 \forall w_1 P(w_2) \)

   1 \( \forall w_1 \exists w_2 P(w_2) \mid \forall w_1 \exists w_2 P(w_2) \) \text{ REF}
   2 \( \forall w_1 \exists w_2 P(w_2) \mid \exists w_2 P(w_2) \) \text{ \forall O, 1}
   3 \( P(x_1) \mid P(x_1) \) \text{ REF}
   4 \( P(x_1) \mid \forall w_1 P(x_1) \) \text{ \forall I, 3}
   5 \( P(x_1) \mid \exists w_2 \forall w_1 P(w_2) \) \text{ \exists I, 4}
   6 \( \exists w_2 P(w_2) \mid \exists w_2 \forall w_1 P(w_2) \) \text{ \exists O, 5}
   7 \( \mid \exists w_2 P(w_2) \rightarrow \exists w_2 \forall w_1 P(w_2) \) \text{ IMR, 6}
   8 \( \forall w_1 \exists w_2 P(w_2) \mid \exists w_2 \forall w_1 P(w_2) \) \text{ MP, 2, 7}

2. Let \( M \) be a structure for \( L \) and \( a \in |M| \). Let \( K \) be the structure for \( L \) which completely agrees with \( M \) except that \( f^K(a) = a \). Then since \( \theta(x_1) \) does not mention \( f \), \( M \) and \( K \) must be exactly the same on the relation, constant, function symbols appearing in \( \theta(x_1) \), so

\[ K \models \theta(a) \iff M \models \theta(a). \]  \hspace{1cm} (1)

\(^1\)Only for levels 4&6 students.
[This is clear but in any case it is easily proved by induction on the length of formulae, see Example 28 on the Example Sheet.]
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\[ \therefore \text{given that } | \forall w_1 \theta(f(w_1)), K \models \theta(f(a)), \text{ so by Lemma 16}^* \text{ (or take it as obvious) } K \models \theta(f^K(a)), \text{ i.e. } K \models \theta(a). \therefore \text{by (1), } M \models \theta(a), \text{ so since } a \text{ was an arbitrary element of } |M|, M \models \forall w_1 \theta(w_1). \text{ Finally then since } M \text{ was an arbitrary structure for } L, \models \forall w_1 \theta(w_1). \]

The Feedback

Generally the MATH33001 students did very well, the average mark was close to 9. Strangely the MATH43001/63001 students on average didn’t perform well at all, even on question 1 which was common to both tests, their average mark being around 5.

An error on Question 1(i) was to invent new rules which weren’t on the list – no way is this permitted. For example going from \( | \neg (P(x_1) \lor Q(x_1)) \) to \( | \neg P(x_1) \land \neg Q(x_1) \) by the ‘rule of logical equivalence’!

Amongst those students who did not get Question 2 correct a common error was to incorrectly apply the rule MP as in:

\[
\begin{align*}
n &. \quad \Gamma, \forall w_1 P(w_1) | \forall w_1 P(w_1) \\
n + 1 &. \quad \Gamma, \forall w_1 P(w_1) | P(x_1) \quad \forall O \quad n \\
n + 2 &. \quad \Gamma | \forall w_1 P(w_1) \rightarrow P(x_1) \quad \text{IMR} \quad n + 1 \\
n + 3 &. \quad \Gamma | P(x_1) \quad \text{MP; } n, n + 2
\end{align*}
\]

The error was that on line \( n + 3 \) the \( \forall w_1 P(w_1) \) on the left hand side of line \( n \) should now reappear on the left of line \( n + 3 \). It should have been obvious that there was something wrong here since for \( \Gamma = \emptyset \) here we could in this way prove any \( P(x_1) \) from no assumptions at all. Indeed it is always a good idea as you’re producing a proof to check at each stage that you think it reasonable that the rhs does follow from the lhs. [Similarly you should always be suspicious at the end of a proof if you never actually used one of the given lhs assumptions!]

As expected a mistake some students made on this question was to apply the \( \exists O \) rule using a variable which also appeared on the rhs. This can often be avoided by arranging the order in which you use the quantifier rules so that at the stage when \( \exists O \) is applied the free variable in question has already been removed from the rhs (usually by an application of \( \exists I \), see the model answer).

Question 2 for the level 4 & 6 students was not well done, almost nobody saw why it held. I can only suggest you look at the model answer.