MATH33001, January 2014 Exam, Solutions[1]

A1. This does not follow since even given the premises the truth of the conclusion depends on the meaning of 'divisible by 3', 'divisible by 5'. To see this set $B(x)$ to mean 'x is divisible by 3', $E(x)$ to mean 'x is divisible by 5'. Then when the variable ranges over, say, $\mathbb{N}$, the argument becomes

$$\begin{array}{l} \exists w_1\, B(w_1) \\ \exists w_1\, E(w_1) \\ \hline \therefore \exists w_1\, (B(w_1) \wedge E(w_1)) \end{array}$$

But this conclusion is not necessarily true for any interpretation of B, E, etc. which make the premises true (for example if we just change $E(x)$ here to mean 'x is a power of 2') so the 'argument' is not valid.

A2. (i) $f(x_1, f(x_1, x_1)) \in TL$ since $x_1 \in TL$, by Te1, so $f(x_1, x_1) \in TL$ by Te3, and in turn $f(x_1, f(x_1, x_1)) \in TL$ by Te3 again.

(ii) $f((f(x_1, x_2), x_1)) \notin TL$ since we can show by induction on $|t|$ that any $t \in TL$ contains the same number of left parentheses as right parentheses, whilst in the case of $f((f(x_1, x_2), x_1)))$ there are 3 left parentheses and 4 right parentheses.

(iii) $\forall w_1\, (R(x_1, x_1) \vee R(x_1, x_1)) \in FL$ since $R(x_1, x_1) \in FL$ by L1, so $(R(x_1, x_1) \vee R(x_1, x_1)) \in FL$ by L2. Since w_2 does not occur in this formula, by L3 the result of replacing all occurrences of x_2 in it by w_2 and prefixing with $\forall w_1$, yields $\forall w_1\, (R(x_1, x_1) \vee R(x_1, x_1)) \in FL$. [Notice that there's no requirement in L3 that x_2 actually occurs in $(R(x_1, x_1) \vee R(x_1, x_1))$.]

(iv) $\forall x_1\, (R(x_1, x_1) \vee \neg R(x_1, x_1)) \notin FL$ since we can prove by induction on $|\theta|$ for $\theta \in FL$ that an occurrence of $\forall$ in θ is never immediately followed by x_1 (or any *free* variable).

(v) False. It is enough to just say this (similarly in (vi),(vii)), but for the record,

$$\begin{aligned} M \models \forall w_1 \exists w_2\, R(w_2, w_1) \Leftrightarrow &\text{ for all } n \in |M|, \text{ there exists } m \in |M|, \langle m, n\rangle \in R^M \\ \Longleftrightarrow &\quad \text{for all } n \in |M|, \text{ there exists } m \in |M|,\ m < n \end{aligned}$$

which is *not true*, there is no such m when $n = 2$.

(vi) True.

$$\begin{aligned} M \models \forall w_1 \forall w_2\, (\exists w_3\, R(f(w_1, w_3), f(w_2, w_3)) \to R(w_1, w_2)) \\ \Longleftrightarrow &\ \text{for each } n, m \in |M|, \text{ if there is } k \in |M| \text{ such} \\ &\ \text{that } \langle f^M(n,k), f^M(m,k)\rangle \in R^M \text{ then } \langle n, m\rangle \in R^M \\ \Longleftrightarrow &\ \text{for each } n, m \in |M|, \text{ if there is } k \in |M| \text{ such} \\ &\ \text{that } \langle n \times k, m \times k\rangle \in R^M \text{ then } \langle n, m\rangle \in R^M \\ \Longleftrightarrow &\ \text{for each } n, m \in |M|, \text{ if there is } k \in |M| \text{ such} \\ &\ \text{that } n \times k < m \times k \text{ then } n < m, \end{aligned}$$

which is true since if $k \in |M|$ then $k > 0$ and so if $n \times k < m \times k$ then $n < m$.

(vii) True.

$M \models \exists w_1 \forall w_2\, (R(w_2, w_1) \to R(f(w_2, w_2), w_1))$

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

$\iff$ there is $n \in |M|$ such that for any $m \in |M|$ if $\langle m, n\rangle \in R^M$ then $\langle f^M(m,m)n\rangle \in R^M$,

$\iff$ there is $n \in |M|$ such that for any $m \in |M|$ if $m < n$ then $m^2 < n$.

which is *true* since for $n = 2$ there is no $m \in |M|$ such that $m < n$ so for every $m \in |M|$, $m < n \Rightarrow m^2 < n$ is true.

There are lots of possible choices here, for example:

$$\theta_1(x_1, x_2) = (\neg R(x_1, x_2) \wedge \neg R(x_2, x_1))$$

$$\theta_2(x_1) = \neg\exists w_1\, R(w_1, x_1)$$

$$\theta_3(x_1, x_2) = R(f(x_1, x_1), x_2)$$

$$\theta_4(x_1) = \exists w_1\, (\neg\exists w_2\, R(w_2, w_1) \wedge (R(w_1, x_1) \wedge R(x_1, f(w_1, w_1))))$$

Again lots of choices here but an easy one, since K has a multiplicative identity (i.e. 1) but M does not is

$$\phi = \exists w_1 \forall w_2\, \neg R(w_2, f(w_1, w_2)).$$

A3. A (formal) *proof* (in PC) is a sequence of sequents

$$\Gamma_1 \,|\, \phi_1,\ \Gamma_2 \,|\, \phi_2 \ldots, \Gamma_m \,|\, \phi_m$$

where the Γ_i are finite subsets of FL, the $\phi_i \in FL$ and for $i = 1, 2, \ldots, m$, either $\Gamma_i \,|\, \phi_i$ is an instance of REF or there are some $j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1} \,|\, \phi_{j_1},\ \Gamma_{j_2} \,|\, \phi_{j_2}, \ldots, \Gamma_{j_s} \,|\, \phi_{j_s}}{\Gamma_i \,|\, \phi_i}$$

is an instance of one of the rules of proof.

A formal proof of $\forall w_1\, (P(w_1) \to Q(w_1)), \exists w_1\, P(w_1) \vdash \exists w_1\, Q(w_1)$:

1	$\forall w_1\, (P(w_1) \to Q(w_1)), P(x_1) \,|\, \forall w_1\, (P(w_1) \to Q(w_1))$	REF
2	$\forall w_1\, (P(w_1) \to Q(w_1)), P(x_1) \,|\, P(x_1) \to Q(x_1)$	∀O, 1
3	$\forall w_1\, (P(w_1) \to Q(w_1)), P(x_1) \,|\, P(x_1)$	REF
4	$\forall w_1\, (P(w_1) \to Q(w_1)), P(x_1) \,|\, Q(x_1)$	MP, 2, 3
5	$\forall w_1\, (P(w_1) \to Q(w_1)), P(x_1) \,|\, \exists w_1\, Q(w_1)$	∃I, 4
6	$\forall w_1\, (P(w_1) \to Q(w_1)), \exists w_1\, P(w_1) \,|\, \exists w_1\, Q(w_1)$	∃O, 5

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.

(a) Let M be the structure for L with $|M| = \{0, 1\}$, $P^M = \{0\}$, $g^M(0) = g^M(1) = 0$. Then since $g^M(0), g^M(1) \in P^M$ and $|M| = \{0, 1\}$, $g^M(a) \in P^M$ for every $a \in |M|$, and $M \models \forall w_1\, P(g(w_1))$. However $1 \notin P^M$ so $M \not\models P(1)$ and $M \not\models \forall w_1\, P(w_1)$. Hence $\forall w_1\, P(g(w_1)) \not\models \forall w_1\, P(w_1)$.

(b) Let M be the structure for L such that

$$M \models \forall w_1\, (P(w_1) \to \neg P(g(w_1))) \quad \star$$

Let $a \in |M|$. If $M \models \neg P(a)$ then $M \models \exists w_1 \neg P(w_1)$. On the other hand if $M \models P(a)$ then from $\star$.

$$M \models P(a) \to \neg P(g(a))$$

and hence $M \models \neg P(g(a))$. By Lemma 16 (or directly) then $M \models \neg P(g^M(a))$ so $M \models \exists w_1 \neg P(w_1)$. Either way then $M \models \exists w_1 \neg P(w_1)$. This shows that

$$\forall w_1 (P(w_1) \to \neg P(g(w_1))) \models \exists w_1 \neg P(w_1)$$

so by the Completeness Theorem,

$$\forall w_1 (P(w_1) \to \neg P(g(w_1))) \vdash \exists w_1 \neg P(w_1).$$

It is not the case that $P(x_1) \to \neg P(g(x_1)), P(x_2) \vdash \neg P(g(x_2))$. To show this it is enough, by the Completeness Theorem, to show $P(x_1) \to P(g(x_1)), P(x_2) \not\models \neg P(g(x_2))$. To this end let M be the structure as in (a) above and consider the assignment $x_1 \mapsto 1$, $x_2 \mapsto 0$. Then with this interpretation $P(x_1) \to \neg P(g(x_1))$ is true (since $P(x_1)$ is false) and $P(x_2)$ is true but $\neg P(g(x_2))$ is false (since $M \models P(0)$).

B5. We prove this by induction on $|\theta|$.

If $\theta = R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})$ for some r-ary relation symbol of L then $\theta^* = \theta \in FL$.

Now assume that $\phi^* \in FL$ for ever $\phi \in FL$ with $|\phi| < |\theta|$. We need to consider various cases:

If $\theta = \neg\phi$ for some $\phi \in FL$ then by the IH $\phi^* \in FL$ (since $|\phi| < |\theta|$) and $\theta^* = \phi^*$ so $\theta^* \in FL$.

If $\theta = (\phi \wedge \psi)$, then by the IH $\phi^*, \psi^* \in FL$ (since $|\phi|, |\psi| < |\theta|$) and so

$$\theta^* = (\phi \wedge \psi)^* = (\phi^* \wedge \psi^*) \in FL.$$

The cases for $\vee$ and $\to$ are exactly similar.

If $\theta = \exists w_i \, \psi(w_i/x_j)$ then $|\psi| < |\theta|$ so $\psi^* \in FL$ by the IH and

$$\theta^* = (\exists w_i \, \psi(w_i/x_j))^* = \exists w_i \, \psi^*(w_i/x_j) \in FL.$$

The case for $\forall$ is exactly similar and this concludes the proof that $\theta^* \in FL$ for all $\theta \in FL$.

The conclusion does not hold when conjunctions $\wedge$ are removed. For example if R is as above then $(R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \wedge R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})) \in FL$ but

$$(R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) \wedge R(x_{i_1}, x_{i_2}, \ldots, x_{i_r}))^* = (R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})R(x_{i_1}, x_{i_2}, \ldots, x_{i_r})) \notin FL$$

since all formulae for a relational language have the same number of left parentheses as relation symbols plus binary connectives and this expression does not.

B6. A formal proof of

$$\forall w_1 P(w_1) \vee \forall w_1 \neg P(w_1) \vdash \neg \exists w_1 \exists w_2 (P(w_1) \wedge \neg P(w_2)) \ :$$

Putting $\psi = \exists w_1 \exists w_2 (P(w_1) \wedge \neg P(w_2))$ (in order to get it on the page!),

1	$\psi, \forall w_1\, P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \forall w_1\, P(w_1)$	REF
2	$\psi, \forall w_1\, P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, P(x_2)$	∀O, 1
3	$\psi, \forall w_1\, P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, P(x_1) \wedge \neg P(x_2)$	REF
4	$\psi, \forall w_1\, P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \neg P(x_2)$	AO, 3
5	$\forall w_1\, P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \neg\psi$	NIN, $2, 4$
6	$\psi, \forall w_1\, \neg P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \forall w_1\, \neg P(w_1)$	REF
7	$\psi, \forall w_1\, \neg P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \neg P(x_1)$	∀O, 1
8	$\psi, \forall w_1\, \neg P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, P(x_1) \wedge \neg P(x_2)$	REF
9	$\psi, \forall w_1\, \neg P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, P(x_1)$	AO, 3
10	$\forall w_1\, \neg P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \neg\psi$	NIN, $7, 9$
11	$\forall w_1\, P(w_1) \vee \forall w_1\, \neg P(w_1), P(x_1) \wedge \neg P(x_2) \,\|\, \neg\psi$	DIS, $5, 10$
12	$\forall w_1\, P(w_1) \vee \forall w_1\, \neg P(w_1), \exists w_2\, (P(x_1) \wedge \neg P(w_2)) \,\|\, \neg\psi$	∃O, 11
13	$\forall w_1\, P(w_1) \vee \forall w_1\, \neg P(w_1), \exists w_1 \exists w_2\, (P(w_1) \wedge \neg P(w_2)) \,\|\, \neg\psi$	∃O, 12
14	$\forall w_1\, P(w_1) \vee \forall w_1\, \neg P(w_1), \exists w_1 \exists w_2\, (P(w_1) \wedge \neg P(w_2)) \,\|\, \psi$	REF
15	$\forall w_1\, P(w_1) \vee \forall w_1\, \neg P(w_1) \,\|\, \neg\exists w_1 \exists w_2\, (P(w_1) \wedge \neg P(w_2))$	NIN, $13, 14$

B7. The Compactness Theorem for Normal Structures: For L a language with equality and $\Gamma \subseteq FL$, Γ is satisfiable in a normal structure iff every finite subset of Γ is satisfiable in a normal structure.

Assume on the contrary that such a sentence θ did exist and consider the set of sentences of L:

$$\Gamma = \{\theta\} \cup \{\exists w_1, \ldots, w_n \left(\bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \right) \mid n \in \mathbb{N}^+\}$$

Let $\Delta \subseteq \Gamma$ be finite. So there is an $m \in \mathbb{N}^+$ such that if

$$\exists w_1, \ldots, w_n \left(\bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \right) \qquad \star$$

appears in Δ then $n \leq m$. Let M_m be a finite normal structure for L with $|M_m|$ having m elements. Let $\phi \in \Delta$. If $\phi = \theta$ then $M_m \models \phi$ since $|M_m|$ is finite (using our assumption concerning θ). Otherwise ϕ is of the form $\star$ for some $n \leq m$ so again $M_m \models \phi$. Hence Δ is satisfied in a normal structure. Therefore by the above Compactness Theorem Γ is satisfied in some normal structure, M say. But for all $n \in \mathbb{N}^+$,

$$M \models \exists w_1, \ldots, w_n \left(\bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \right),$$

so $|M|$ must have cardinality at least n for every $n \in \mathbb{N}^+$, i.e. M must be infinite. But since $\theta \in \Gamma$, $M \models \theta$ so this contradicts the assumed property of θ.

Feedback

Generally the marks were good, students seem to have practiced well beforehand. Some comments on the individual queations:

A1. Most students got this correct and scored full marks.

A2. Parts (i),(ii),(iv) were done well. In part (iii) it was important not to drop outermost brackets since they are germane to the question. [This also applied to some other questions where the structure of the formulae was important, e.g. B5.]

Parts (v)-(vii) were mostly correct but a problem here was that students sometimes reduced the truth of the sentence in M to something 'understandable' but then never said whether it was true or not. In fact this was a problem elsewhere as well, students would go through the working but then never actually answer the question! Always check that you've answered the question that was asked.

The usual errors occurred when stating the required $\theta_1(x_1, x_2)$ etc.: What was written down wasn't actually a formula, typically brackets were wrong; The θ didn't actually mention any free variable, or had n, m where x_1, x_2 should be; The function f inexplicably became unary. Another error was to write invalid expressions like $R(\theta_2(w_1), x_2)$. Presumably the intention here was that $w_1 = 2$, which could instead be obtained via $\exists w_2\,(\theta_2(w_2) \wedge R(w_2, x_2))$.

Almost everyone solved the last part, separating M and K, correctly.

A3. Mostly well done. The commonest error, apart from still not being able to define what a formal proof is, was to invent new (incorrect) versions of rules, for example $\exists O$ as

$$\frac{\Gamma \mid \exists w_i\, \theta(w_i)}{\Gamma \mid \theta(x_j)}$$

and DIS as

$$\frac{\theta, \phi, \Gamma \mid \psi}{(\theta \vee \phi), \Gamma \mid \psi}$$

Obviously the advantage of inventing rules like this is that it can make it much easier to prove things. On the other hand it carries with it the disadvantage that you don't get any marks for doing so.

A further error was to incorrectly apply $\exists O$ when the same variable appeared on the left hand side. Usually this incorrect use can be avoided by applying the rules in a different order.

A4. Many students wrote things like $g(a) \in |M|$ when what they should have written was $g^M(a) \in |M|$. It was slightly disappointing that so many students didn't even realize the difference, which was important if you are to properly answer this question (see the model answers).

B5. This was not so well done. Even many of those who realized it should be proved by induction on $|\theta|$ (what other possibility was there?) said things like 'if $\theta = \neg\phi$ then $\theta^* = \phi$', overlooking that ϕ may itself contain negations and that correctly $\theta^* = \phi^*$.

B6. This was not an easy proof and only a few students got it out. However there was a clear strategy for producing the proof and I gave marks to students who recognized this. For example it seems likely that the last step will be NIN, inserting $\neg\exists w_1 \exists w_2\,(P(w_1) \wedge \neg P(w_2))$ on the right hand side. Also to ever make use of

$$\forall w_1\, P(w_1) \vee \forall w_1 \neg P(w_1)$$

you would need to prove $\neg\exists w_1 \exists w_2\, (P(w_1) \wedge \neg P(w_2))$ separately from each disjunct and then use DIS. Similarly to ever make use of $\exists w_1 \exists w_2\, (P(w_1) \wedge \neg P(w_2))$ on the left hand side you should start with $(P(x_1) \wedge \neg P(x_2))$ on the left hand side and later replace it by $\exists w_1 \exists w_2\, (P(w_1) \wedge \neg P(w_2))$ by using $\exists O$ twice (making sure that x_1, x_2 do not appear anywhere else at that point).

B7. A number of students seemed to have learnt up this material and so should have been in a good position to score well. Unfortunately this wasn't always combined with a full understanding so that any lapse of memory resulted in complete drivel, which was duely given the reward it deserved!