MATH33001, January 2012 Exam, Solutions[1]

A1. This does follow since it does not depend on the meaning of 'frog', 'sings' etc. To see this set $P(x)$ to mean 'x plays the piano' and $S(x)$ to mean 'x sings'. Then when the variable ranges over frogs the argument becomes

$$\frac{\exists w\,(P(w) \wedge S(w))}{\therefore\ \exists w\, P(w)}$$

But this conclusion follows from the premise for *any* interpretation of P, S and range of the variable so the argument is 'valid'.

A2. (i) $f(f(w_2)) \notin TL$ since this word contains a bound variable (w_2) and we can prove by induction on $|t|$ that no term t of L can contain a bound variable. [Not necessary to give the proof but for the record: Clearly true if t is a constant or free variable x_i and if $t = g(t_1, \dots, t_n)$ and no bound variables occur in $t_1, \dots, t_n$ then none will occur in t either.]

(ii) $f(f(f(f(x_1))) \notin TL$ since this word has different numbers of right ')' and left '(' round brackets and we can prove by induction on $|t|$ that any $t \in TL$ has the same number of each. [Not necessary to give the proof but for the record: Clearly true if t is a free variable x_i (when there are zero of either) and if $t = f(t_1)$ then the number of '(' in t equals 1 plus the number in t_1, equals 1 plus the number of ')' in t_1, by inductive hypothesis, which equals the number of ')' in t.]

(iii) $\forall w_1 \neg R(w_1, x_1, w_1) \in FL$ since $R(x_2, x_1, x_2) \in FL$ by L1, so $\neg R(x_2, x_1, x_2) \in FL$ by L2, and finally then $\forall w_1 \neg R(w_1, x_1, w_1) \in FL$ by L3.

(iv) $\forall w_1\, R(w_1, x_1, w_1) \vee R(w_1, w_1, x_1) \notin FL$ since we can prove by induction on $|\theta|$ for $\theta \in FL$ that the number of left round brackets '(' in θ equals the number of relation, function and binary connective (i.e. $\wedge, \vee, \rightarrow$) symbols occurring in θ and this is not the case for $\forall w_1\, R(w_1, x_1, w_1) \vee R(w_1, w_1, x_1)$. [Again it is not necessary to prove this but, for the record, such a proof could go as follows: We first prove it for terms $t \in TL$ (where of course there are are no relation symbols nor connectives) by induction on $|t|$. Moving on to formulae it is clearly true for $R(t_1, t_2, t_3)$ since it is true for t_1, t_2, t_3 and along with R we introduce one new '('. Finally, by inspection we can see that if it holds for $\phi, \psi \in FL$ then it holds for $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $\exists w_j\, \psi(w_j/x_i)$ and $\forall w_j\, \psi(w_j/x_i)$ (assuming here of course that w_j does not already occur in ψ).]

(v) $M \models \forall w_1\, R(w_1, f(w_1), f(f(w_1)))$
$$\iff \text{for all } n \in \mathbb{N},\ \langle n, f^M(n), f^M(f^M(n))\rangle \in R^M$$
$$\iff \text{for all } n \in \mathbb{N},\ n < f^M(n) < f^M(f^M(n))$$
$$\iff \text{for all } n \in \mathbb{N},\ n < n+1 < (n+1)+1,$$

which is *true*. [In your exam script it is enough to simply give an answer 'true'/'false', similarly with parts (vi),(vii).]

(vi) $M \models \exists w_1 \exists w_2\, R(w_1, w_2, f(w_1))$
$$\iff \text{there are } n, m \in \mathbb{N} \text{ such that } \langle n, m, f^M(n)\rangle \in R^M$$
$$\iff \text{there are } n, m \in \mathbb{N} \text{ such that } n < m < n+1,$$

which is *false* since there can be no natural number m between the natural numbers n and $n+1$.

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

(vii) $M \models \forall w_1 \forall w_2 \exists w_3\, (R(w_1, w_3, f(f(w_2))) \vee R(w_2, w_3, f(f(w_1))))$

$\iff$ for all $n, m \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that either $\langle n, k, f^M(f^M(m)) \rangle \in R^M$ or $\langle m, k, f^M(f^M(n)) \rangle \in R^M$

$\iff$ for all $n, m \in \mathbb{N}$, there is a $k \in \mathbb{N}$ such that either $n < k < m+2$ or $m < k < n+2$. This is *true* since if $n \leq m$ we can take $k = n+1$ and if $m < n$ we can take $k = m+1$.

$\theta_1(x_1, x_2) = R(x_1, x_2, f(x_2))$

$\theta_2(x_1, x_2) = \neg(\theta_1(x_1, x_2) \vee \theta_1(x_2, x_1)) = \neg(R(x_1, x_2, f(x_2)) \vee R(x_2, x_1, f(x_1)))$

$\theta_3(x_1, x_2) = \exists w_1\, (R(x_1, w_1, x_2) \vee R(x_2, w_1, x_1))$

$\theta_4(x_1) = \neg \exists w_1 \exists w_2\, R(w_1, x_1, w_2)$

$\phi = \exists w_1\, \neg R(w_1, f(w_1), f(f(w_1)))$ (since this holds in K when $w_1 = 0$ but by (v) does not hold in M).

A3. A (formal) *proof* (in PC) is a sequence of sequents

$$\Gamma_1 \,|\, \phi_1,\ \Gamma_2 \,|\, \phi_2 \ldots, \Gamma_m \,|\, \phi_m$$

where the Γ_i are finite subsets of FL, the $\phi_i \in FL$ and for $i = 1, 2, \ldots, m$, either $\Gamma_i \,|\, \phi_i$ is an instance of REF or there are some $j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1} \,|\, \phi_{j_1},\ \Gamma_{j_2} \,|\, \phi_{j_2}, \ldots, \Gamma_{j_s} \,|\, \phi_{j_s}}{\Gamma_i \,|\, \phi_i}$$

is an instance of one of the rules of proof.

A formal proof of $\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \vdash \forall w_1\, (P(w_1) \wedge Q(w_1))$:

1	$\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \,\vert\, \forall w_1\, P(w_1)$	REF
2	$\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \,\vert\, P(x_1)$	$\forall$O, 1
3	$\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \,\vert\, \forall w_1\, Q(w_1)$	REF
4	$\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \,\vert\, Q(x_1)$	$\forall$O, 3
5	$\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \,\vert\, (P(x_1) \wedge Q(x_1))$	AND, 2, 4
6	$\forall w_1\, P(w_1),\ \forall w_1\, Q(w_1) \,\vert\, \forall w_1\, (P(w_1) \wedge Q(w_1))$	$\forall$I, 5
7	$\forall w_1\, P(w_1) \,\vert\, \forall w_1\, Q(w_1) \to \forall w_1\, (P(w_1) \wedge Q(w_1))$	IMR, 6

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.
(a) Let M be the structure for L such that $|M| = \mathbb{N}$, $P^M = \{\, n \in \mathbb{N} \,|\, n \text{ is even } \}$, $Q^M = \{\, n \in \mathbb{N} \,|\, n \text{ is odd } \}$. Then

$$M \models \exists w_1\, P(w_1) \tag{1}$$

since $M \models P(0)$. Similarly

$$M \models \exists w_1\, Q(w_1) \tag{2}$$

since $M \models Q(1)$. However for any $n \in \mathbb{N}$ $M \not\models P(n) \wedge Q(n)$ since n cannot be both even and odd. Hence $M \not\models \exists w_1\, (P(w_1) \wedge Q(w_1))$ so with (2)

$$M \not\models \exists w_1\, Q(w_1) \to \exists w_1\, (P(w_1) \wedge Q(w_1))$$

and with (1) we obtain

$$\exists w_1\, P(w_1) \nvDash \exists w_1\, Q(w_1) \to \exists w_1\, (P(w_1) \wedge Q(w_1))$$

and by the Completeness Theorem

$$\exists w_1\, P(w_1) \nvdash \exists w_1\, Q(w_1) \to \exists w_1\, (P(w_1) \wedge Q(w_1)).$$

(b) Let M be a structure for L and suppose that

$$M \models \forall w_1\, (P(w_1) \vee Q(w_1)) \qquad \star$$

but

$$M \nvDash \forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1) \qquad \dagger$$

Then

$$M \nvDash \forall w_1\, P(w_1) \quad \text{and} \quad M \nvDash \exists w_2\, Q(w_2).$$

Hence for some $a \in |M|$, $M \nvDash P(a)$ and also $M \nvDash Q(a)$ since $M \nvDash \exists w_1\, Q(w_1)$. Hence $M \nvDash P(a) \vee Q(a)$. But this contradicts $\star$. Hence given $\star$ $\dagger$ must fail, so

$$\forall w_1\, (P(w_1) \vee Q(w_1)) \models \forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1)$$

and by the Completeness Theorem

$$\forall w_1\, (P(w_1) \vee Q(w_1)) \vdash \forall w_1\, P(w_1) \vee \exists w_1\, Q(w_1).$$

B5. (i)+(ii) $\nvDash$ (iii): Let M be the structure for L such that $|M| = \{0,1\}$ and $R^M = \{\langle 0,1\rangle, \langle 1,1\rangle\}$. Then (i) is true in M since $M \nvDash R(0,0)$ and $M \nvDash R(1,0)$ so for each $n \in |M|$ there is an $m \in |M|$ such that $M \vDash \neg R(n,m)$. Also (ii) is true in M since $M \vDash R(1,1), R(0,1)$ so $M \vDash \forall w_2\, R(w_2,1)$. However (iii) fails to hold in M since $M \nvDash R(0,0) \vee R(0,0)$ so $M \nvDash \forall w_1 \forall w_2\, (R(w_1,w_2) \vee R(w_2,w_1))$.

(i)+(iii) $\nvDash$ (ii): Let M be the structure for L with $|M| = \{0,1,2\}$ and

$$R^M = \{\langle 0,1\rangle, \langle 0,0\rangle, \langle 1,2\rangle, \langle 1,1,\rangle, \langle 2,0\rangle, \langle 2,2\rangle\}.$$

Then (i) is true in M since $R(0,2), R(1,0), R(2,1)$ all fail in M and (iii) is true in M since for each of the pairs $\langle n,m\rangle$, $\langle m,n\rangle$ with $n,m \in |M|$ at least one of them is in R^M. However (ii) fails since $\langle 0,2\rangle \notin R^M$, $\langle 1,0\rangle \notin R^M$,$\langle 2,1\rangle \notin R^M$.

(ii)+(iii) $\nvDash$ (i): Let M be the structure for L with $|M| = \{0\}$ and $R^M = \{\langle 0,0\rangle\}$. Then for any $n \in |M|$, $\langle n,0\rangle \in R^M$ so (ii) holds in M. Also (iii) holds since $M \vDash R(0,0)$ and 0 is the sole element of $|M|$. However there is no $n \in |M|$ such that $M \nvDash R(0,n)$ so (i) fails in M.

B6. A formal proof of $\forall w_1\, P(w_1) \vdash \neg\exists w_1\, (\neg P(w_1) \wedge Q(w_1))$:

1	$\forall w_1\, P(w_1),\ \neg P(x_1) \wedge Q(x_1) \mid \neg P(x_1) \wedge Q(x_1)$	REF
2	$\forall w_1\, P(w_1),\ \neg P(x_1) \wedge Q(x_1) \mid \neg P(x_1)$	AO, 1
3	$\forall w_1\, P(w_1),\ \neg P(x_1) \wedge Q(x_1) \mid \forall w_1\, P(w_1)$	REF,
4	$\forall w_1\, P(w_1),\ \neg P(x_1) \wedge Q(x_1) \mid P(x_1)$	$\forall$O, 3
5	$\neg P(x_1) \wedge Q(x_1) \mid \neg\forall w_1\, P(w_1)$	NIN, 2, 4
6	$\exists w_1\, (\neg P(w_1) \wedge Q(w_1)) \mid \neg\forall w_1\, P(w_1)$	$\exists$O, 5
7	$\forall w_1\, P(w_1),\ \exists w_1\, (\neg P(w_1) \wedge Q(w_1)) \mid \forall w_1\, P(w_1)$	REF,
8	$\forall w_1\, P(w_1) \mid \neg\exists w_1\, (\neg P(w_1) \wedge Q(w_1))$	NIN, 6, 7

B7. The Compactness Theorem for Normal Structures: For L a language with equality and $\Gamma \subseteq FL$, Γ is satisfiable in a normal structure iff every finite subset of Γ is satisfiable in a normal structure.

Assume on the contrary that such a sentence θ did exist and consider the set of sentences of L:

$$\Gamma = \{\theta\} \cup \{\exists w_1, \ldots, w_n \bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \mid n \in \mathbb{N}^+\}$$

Let $\Delta \subseteq \Gamma$ be finite. So there is an $m \in \mathbb{N}^+$ such that if

$$\exists w_1, \ldots, w_n \bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j \qquad \star$$

appears in Δ then $n \leq m$. Let M_m be a finite normal structure for L with $|M_m|$ having m elements. Let $\phi \in \Delta$. If $\phi = \theta$ then $M_m \models \phi$ since $|M_m|$ is finite (using our assumption concerning θ). Otherwise ϕ is of the form $\star$ for some $n \leq m$ so again $M_m \models \phi$. Hence Δ is satisfied in a normal structure. Therefore by the above Compactness Theorem Γ is satisfied in some normal structure, M say. But for all $n \in \mathbb{N}^+$,

$$M \models \exists w_1, \ldots, w_n \bigwedge_{1 \leq i < j \leq n} \neg w_i = w_j$$

, so $|M|$ must have cardinality at least n for every $n \in \mathbb{N}^+$, i.e. M must be infinite. But since $\theta \in \Gamma$, $M \models \theta$ so this contradicts the assumed property of θ.