MATH33001, January 2011 Exam, Solutions[1]

A1. This does not follow. For set $F(x)$ to mean 'x loves football' and $S(x)$ to mean 'x loves Spain'. Then when the variable range over people the argument becomes

$$\begin{array}{l} \forall w\, (F(w) \to S(w)) \\ \quad \forall w\, S(w) \\ \hline \therefore \forall w F(w) \end{array}$$

But this conclusion does not follow for *any* interpretation of F, S and the range of the variables, for example if the variables range over $\mathbb{N}^+$, $F(x)$ means 'x is prime' and $S(x)$ means just $x = x$ then the premises $\forall w\, (F(w) \to S(w))$, $\forall w\, S(w)$ are true (since every prime does equal itself, as does every number, but $\forall w F(w)$, i.e. every number is prime, is false.

A2. (i) $f(x_1, f(x_1, x_2)) \in TL$ since $x_1, x_2 \in TL$ by Te1, so $f(x_1, x_2) \in TL$ by Te3 and $f(x_1, f(x_1, x_2)) \in TL$ by Te3 again.

(ii) $f((f(x_1, x_2), x_1) \notin TL$ since this word has different numbers of right and left round brackets and we can prove by induction on $|t|$ that any $t \in TL$ has the same number. [Not necessary to give the proof but for the record: Clearly true if t is a constant or free variable x_i (when there are zero of either) and if $t = f(t_1, \ldots, t_n)$ then the number of '(' in t equals 1 plus the number in $t_1, \ldots, t_n$, equals 1 plus the number of ')' in $t_1, \ldots, t_n$, by inductive hypothesis, equals the number of ')' in t.]

(iii) $\forall w_1\, \neg R(w_1, x_1) \in FL$ since $R(x_2, x_1) \in FL$ by L1, so $\neg R(x_2, x_1) \in FL$ by L2, and finally then $\forall w_1\, \neg R(w_1, x_1) \in FL$ by L3.

(iv) $\forall w_1\, \neg R(w_2, x_1) \notin FL$ since we can prove by induction on $|\theta|$ for $\theta \in FL$ that if w_2 occurs in θ then so does either $\exists w_2$ or $\forall w_2$, which rules out $\forall w_1\, \neg R(w_2, x_1)$ being in FL. [Again no need to prove this but for the record: Clearly true, vacuously, for $R(t_1, t_2)$, and if it holds for ϕ, ψ then it holds for $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi)$. Also if it holds for η and η does not mention w_j, if $j \neq 2$ then it holds for $\exists w_j\, \eta(w_j/x_i)$ and $\forall w_j\, \eta(w_j/x_i)$, whilst if $j = 2$ then the condition holds trivially for $\exists w_2\, \eta(w_2/x_i)$ and $\forall w_2\, \eta(w_2/x_i)$.]

(v) $M \models \forall w_1 \forall w_2\, (R(w_1, w_2) \to R(w_2, w_1)) \iff$
for all $n, m \in \mathbb{N}^+$, if $n < m$ then $m < n$,

which is false since, e.g. $1 < 2$ but $2 \not< 1$.

(vi) $M \models \exists w_1 \forall w_2\, \neg R(w_2, f(w_1, w_2)) \iff$
there is an $n \in \mathbb{N}^+$ such that for all $m \in \mathbb{N}^+$, $m \not< nm$,

which is true when we take $n = 1$ since $m \not< 1 \times m$ for any $m \in \mathbb{N}^+$.

(vii) $M \models \forall w_1\, (R(w_1, f(w_1, w_1)) \to \forall w_2\, R(w_2, f(w_1, w_2))) \iff$
for all $n \in \mathbb{N}^+$, if $n < n^2$ then for all $m \in \mathbb{N}^+$, $m < nm$.

This is true since if $n \in \mathbb{N}^+$ and $n < n^2$ then $n > 1$ so $m < nm$ for $m \in \mathbb{N}^+$.

$\theta_1(x_1, x_2) = R(f(x_1, x_1), x_2)$

$\theta_2(x_1, x_2) = (\neg R(x_1, x_2) \wedge \neg R(x_2, x_1))$

$\theta_3(x_1, x_2) = (R(x_1, x_2) \wedge \neg \exists w_1\, (R(x_1, w_1) \wedge R(w_1, x_2)))$

$\theta_4(x_1, x_2) = \exists w_1\, \theta_2(f(x_1, w_1), w_2) = \exists w_1\, (\neg R(f(x_1, w_1), x_2) \wedge \neg R(x_2, f(x_1, w_1)))$

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

$\phi = \forall w_1 \exists w_2\, R(w_1, f(w_1, w_2))$ (since this fails in K when $w_1 = 0$).

A3. A (formal) *proof* (in PC) is a sequence of sequents

$$\Gamma_1 \,|\, \phi_1,\ \Gamma_2 \,|\, \phi_2 \ldots, \Gamma_m \,|\, \phi_m$$

where the Γ_i are finite subsets of FL, the $\phi_i \in FL$ and for $i = 1, 2, \ldots, m$, either $\Gamma_i \,|\, \phi_i$ is an instance of REF or there are some $j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1} \,|\, \phi_{j_1},\ \Gamma_{j_2} \,|\, \phi_{j_2}, \ldots, \Gamma_{j_s} \,|\, \phi_{j_s}}{\Gamma_i \,|\, \phi_i}$$

is an instance of one of the rules of proof.

A proof of $\forall w_1\, (P(w_1) \to Q(w_1)) \vdash \forall w_1\, P(w_1) \to \forall w_1 Q(w_1)$:

1	$\forall w_1\, (P(w_1) \to Q(w_1)),\ \forall w_1\, P(w_1) \,|\, \forall w_1\, (P(w_1) \to Q(w_1))$	REF
2	$\forall w_1\, (P(w_1) \to Q(w_1)),\ \forall w_1\, P(w_1) \,|\, \forall w_1\, P(w_1)$	REF
3	$\forall w_1\, (P(w_1) \to Q(w_1)),\ \forall w_1\, P(w_1) \,|\, (P(x_1) \to Q(x_1))$	$\forall$O, 1
4	$\forall w_1\, (P(w_1) \to Q(w_1)),\ \forall w_1\, P(w_1) \,|\, P(x_1)$	$\forall$O, 2
5	$\forall w_1\, (P(w_1) \to Q(w_1)),\ \forall w_1\, P(w_1) \,|\, Q(x_1)$	MP, 3, 4
6	$\forall w_1\, (P(w_1) \to Q(w_1)),\ \forall w_1\, P(w_1) \,|\, \forall w_1\, Q(w_1)$	$\forall$I, 5
7	$\forall w_1\, (P(w_1) \to Q(w_1)) \,|\, \forall w_1\, P(w_1) \to \forall w_1\, Q(w_1)$	IMR, 6

A4. Completeness Theorem: For $\Gamma \subseteq FL$ and $\theta \in FL$, $\Gamma \vdash \theta \iff \Gamma \models \theta$.
(a) Let M be the structure for L such that $|M| = \mathbb{N}$, $P^M = \{ n \in \mathbb{N} \,|\, n \text{ is even} \}$, $Q^M = \{ n \in \mathbb{N} \,|\, n \text{ is odd} \}$. Then $M \models \forall w_1\, P(w_1) \to \forall w_1\, Q(w_1)$ since $M \not\models \forall w_1\, P(w_1)$. However $M \not\models \forall w_1\, (P(w_1) \to Q(w_1))$ since $0 \in \mathbb{N}$ is even but not odd. Hence

$$\forall w_1\, P(w_1) \to \forall w_1\, Q(w_1) \not\models \forall w_1\, (P(w_1) \to Q(w_1))$$

and by the Completeness Theorem

$$\forall w_1\, P(w_1) \to \forall w_1\, Q(w_1) \nvdash \forall w_1\, (P(w_1) \to Q(w_1)).$$

(b) Let M be a structure for L and suppose that

$$M \models \forall w_1 \forall w_2\, (P(w_1) \vee Q(w_2)) \qquad \star$$

but

$$M \not\models \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2) \qquad \dagger$$

Then

$$M \not\models \forall w_1\, P(w_1) \quad \text{and} \quad M \not\models \exists w_2\, Q(w_2).$$

Hence for some $a \in |M|$, $M \not\models P(a)$ and also $M \not\models Q(a)$ since $M \not\models \exists w_2\, Q(w_2)$. Hence $M \not\models P(a) \vee Q(a)$. But this contradicts $\star$. Hence given $\star$ $\dagger$ must fail, so

$$\forall w_1 \forall w_2\, (P(w_1) \vee Q(w_2)) \models \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2)$$

and by the Completeness Theorem

$$\forall w_1 \forall w_2\, (P(w_1) \vee Q(w_2)) \vdash \forall w_1\, P(w_1) \vee \exists w_2\, Q(w_2).$$

B5. (i)+(ii) $\not\models$ (iii): Let M be the structure for L such that $|M| = \mathbb{N}$ and $R^M = \{\langle n, m\rangle \in \mathbb{N}^2 \mid n < m\}$. Then (i) is true in M since for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $n < m$ and (ii) is true in M since $0 \in \mathbb{N}$ and $m \not< 0$ for every $m \in \mathbb{N}$. However $M \models R(0,1)$ since $0 < 1$ but there is no $n \in \mathbb{N}$ such that $M \models R(0,n) \wedge R(n,1)$, i.e. $0 < n < 1$ so (iii) fails in M.

(i)+(iii) $\not\models$ (ii): Let M be the structure for L with $|M| = \mathbb{R}$ and $R^M = \{\langle n, m\rangle \in \mathbb{R}^2 \mid n < m\}$. Then (i) is true in M since for every $r \in \mathbb{R}$ there is an $s \in \mathbb{R}$ such that $r < s$ and (iii) is true in M since if $r, s \in \mathbb{R}$ and $r < s$ then there is a $t \in \mathbb{R}$ (for example $(r+s)/2$) such that $r < t < s$. However (ii) fails in M since otherwise there would have to be some $r \in \mathbb{R}$ such that for all $s \in \mathbb{R}$, $s \not< r$, which is false (take $s = r - 1$).

(ii)+(iii) $\not\models$ (i): Let M be the structure for L with $|M| = \{0\}$ and $R^M = \emptyset$. Then for any $s \in |M|$, $\langle 0, s\rangle \notin R^M$ so (ii) holds in M. Also since $\langle s, r\rangle \notin R^M$ for any $r, s \in |M|$, $M \not\models R(s,r)$ and $M \models R(s,r) \to \exists w_3\,(R(w_1,w_3) \wedge R(w_3,w_2))$. Hence (iii) holds in M. However (i) fails in M since for the only element of $|M|$, 0, there is no $s \in |M|$ such that $M \models R(0,s)$, i.e. $\langle 0, s\rangle \in R^M$.

B6. A proof of $\exists w_1\,(P(w_1) \vee Q(w_1)) \vdash \forall w_1 \neg P(w_1) \to \exists w_1\, Q(w_1)$.

1	$P(x_1),\ \neg\exists w_1\, Q(w_1),\ \forall w_1\, \neg P(w_1) \mid \forall w_1\, \neg P(w_1)$	REF
2	$P(x_1),\ \neg\exists w_1\, Q(w_1),\ \forall w_1\, \neg P(w_1) \mid \neg P(x_1)$	∀O, 1
3	$P(x_1),\ \neg\exists w_1\, Q(w_1),\ \forall w_1\, \neg P(w_1) \mid P(x_1)$	REF
4	$P(x_1),\ \forall w_1\, \neg P(w_1) \mid \neg\neg\exists w_1\, Q(w_1)$	NIN, $2, 3$
5	$P(x_1),\ \forall w_1\, \neg P(w_1) \mid \exists w_1\, Q(w_1)$	NNO, 4
6	$Q(x_1),\ \forall w_1\, \neg P(w_1) \mid Q(x_1)$	REF,
7	$Q(x_1),\ \forall w_1\, \neg P(w_1) \mid \exists w_1\, Q(w_1)$	∃I, 6
8	$P(x_1) \vee Q(x_1),\ \forall w_1\, \neg P(w_1) \mid \exists w_1\, Q(w_1)$	DIS, $5, 7$
9	$\exists w_1\,(P(w_1) \vee Q(w_1)),\ \forall w_1\, \neg P(w_1) \mid \exists w_1\, Q(w_1)$	∃O, 8

B7. The Compactness Theorem for Normal Structures: For L a language with equality and $\Gamma \subseteq FL$, Γ is satisfiable in a normal structure iff every finite subset of Γ is satisfiable in a normal structure.

Assume on the contrary that such a sentence θ did exist and consider the set of sentences of L:

$$\Gamma = \{\theta\} \cup \{\exists w_1, \ldots, w_n \bigwedge_{1 \le i < j \le n} \neg w_i = w_j \mid n \in \mathbb{N}^+\}$$

Let $\Delta \subseteq \Gamma$ be finite. So there is an $m \in \mathbb{N}^+$ such that if

$$\exists w_1, \ldots, w_n \bigwedge_{1 \le i < j \le n} \neg w_i = w_j \qquad \star$$

appears in Δ then $n \le m$. Let M_m be a finite normal structure for L with $|M_m|$ having m elements. Let $\phi \in \Delta$. If $\phi = \theta$ then $M_m \models \phi$ since $|M_m|$ is finite (using our assumption concerning θ). Otherwise ϕ is of the form $\star$ for some $n \le m$ so again $M_m \models \phi$. Hence Δ is satisfied in a normal structure. Therefore by the above Compactness Theorem Γ is satisfied in some normal structure, M say. But for all $n \in \mathbb{N}^+$,

$$M \models \exists w_1, \ldots, w_n \bigwedge_{1 \le i < j \le n} \neg w_i = w_j$$

, so $|M|$ must have cardinality at least n for every $n \in \mathbb{N}^+$, i.e. M must be infinite. But since $\theta \in \Gamma$, $M \models \theta$ so this contradicts the assumed property of θ.