Two Hours and Thirty Minutes

THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

22nd April 2010

2.00–4.30

Answer **all** six questions in **section A** (58 marks in all)
and
two of the three questions in **section B** (11 marks each). If all three questions from Section B are attempted then credit will only be given for the first two answers appearing in the answer book.
The total number of marks on the paper is 80.
A further 20 marks are available from work during the semester making a total of 100.

A list of axioms and rules of proof is appended to this examination paper

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SECTION A

Answer **ALL** 6 questions

A1. State the *Representation Theorem for Monotone Consequence Relations.*

Let $\mathrel{|\!\sim}_0$ be a monotone consequence relation and $\theta \in SL$. Show that the relation $\mathrel{|\!\sim}$ defined by

$$\Gamma \mathrel{|\!\sim} \phi \iff \Gamma \mathrel{|\!\sim}_0 \theta \to \phi,$$

for $\phi \in SL$, is a monotone consequence relation.

[6 marks]

A2. Explain (without proof) how a finite sequence $s_1, s_2, ..., s_m$ of subsets of At^L determines a rational consequence relation $\mathrel{|\!\sim}_{\vec{s}}$.

In the case where $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{p \wedge q \wedge r,\ p \wedge q \wedge \neg r,\ p \wedge \neg q \wedge r\}, \\ s_2 &= \{\neg p \wedge q \wedge r\}, \\ s_3 &= \{\neg p \wedge q \wedge \neg r\}, \end{aligned}$$

which of the following are true? [You need not justify your answers.]

(i) $\neg p \wedge \neg r \mathrel{|\!\sim}_{\vec{s}} q$.

(ii) $(p \to q) \mathrel{|\!\sim}_{\vec{s}} r$.

(iii) $\neg p \wedge \neg q \mathrel{|\!\sim}_{\vec{s}} p$.

[8 marks]

A3. Use the Z-algorithm to find the rational closure of $K = \{\, (p \to \neg q) \mathrel{|\!\sim} p,\ \neg p \mathrel{|\!\sim} q \,\}$.

[7 marks]

A4. How is the relation $\models^D$ defined?

State the *Completeness Theorem* for D.

By using this theorem or otherwise show that:

(i) $\Box\Box\theta \models^D \Diamond\Diamond\theta$.

(ii) $\Box\Box p \not\models^D \Diamond p$.

Give a formal proof in D that $\vdash^D \Box\Box\theta \to \Box\Diamond\theta$.

[16 marks]

A5. State the desirable properties C1-C4 for a function $F_\wedge : [0,1]^2 \to [0,1]$.

Let $F_\wedge$ satisfy C1-C4. Show that if $b \in [0,1]$ is such that $F_\wedge(b,b) = b$ then for any $x \in [0,b]$,

$$F_\wedge(b,x) = x.$$

Suppose that, in addition to C1-C4 $F_\wedge$, also satisfies that for each $y \in (0,1)$ there is $x \in (0,y)$ such that $F_\wedge(y,x) = 0$. What can be said about the structure $\langle [0,1], F_\wedge, < \rangle$?

[11 marks]

A6. How is the relation $\models^{Ł}$ defined?

State the *Completeness Theorem* for Ł. Show that:

(i) $\not\models^{Ł} (p \to q)\underline{\vee}(\neg p \to q)$,

(ii) $\models^{Ł} (p \to q) \vee (\neg p \to q)$,

[10 marks]

SECTION B

Answer **2** of the 3 questions.

If all three questions from this section are attempted then credit will only be given for the first two answers appearing in the answer book.

B7. By using the Representation Theorem for Rational Consequence Relations, or otherwise, show that the rule

$$\frac{\theta \mid\!\sim \phi \qquad \phi \wedge \neg\psi \mid\!\sim \psi}{\theta \mid\!\sim \psi}$$

holds for all rational consequence relations, but that the rule

$$\frac{\theta \wedge \neg\phi \mid\!\sim \phi \qquad \phi \mid\!\sim \psi}{\theta \mid\!\sim \psi}$$

fails for some rational consequence relation and choice of θ, ϕ and ψ.

[11 marks]

B8. Let L^+ be the result of adding a new propositional variable q to the language L. For $\langle W^+, E^+, V^+\rangle$ a frame for the language L^+ let $\langle W, E, V\rangle$ be the frame for L such that $W = W^+$, $E = E^+$ and for $i \in W$, $p \in L$, $V_i(p) = V_i^+(p)$. Show that for $\theta \in SML$,

$$\forall i \in W, \quad \langle W, E, V\rangle, i \models \theta \iff \langle W^+, E^+, V^+\rangle, i \models \theta.$$

Hence show that if $\theta \in SML$ then

$$\models^K (q \to \theta) \implies \models^K \theta.$$

[11 marks]

B9. Let $L_1 = \{p\}$. State McNaughton's Theorem for L_1.

Show that if $\theta \in SL_1$, V is a $[0,1]$-valuation on L_1 and $V(\theta) = 1/2$ then $V(p) = n/m$ for some odd natural number n and even natural number m, where $0 < n < m$.

Conversely show that if V is a $[0,1]$-valuation on L_1 such that $V(p) = n/m$ for some odd natural number n and even natural number m, where $0 < n < m$, then there is a $\theta \in SL_1$ such that $V(\theta) = 1/2$.

[11 marks]

END OF EXAMINATION PAPER

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: REF $\theta \mid \theta$

Rules of Proof:

AND $\dfrac{\Gamma \mid \theta \quad \Gamma \mid \phi}{\Gamma \mid \theta \wedge \phi}$ ANL $\dfrac{\Gamma, \theta, \phi \mid \psi}{\Gamma, \theta \wedge \phi \mid \psi}$

ORR $\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \quad \dfrac{\Gamma \mid \phi}{\Gamma \mid \theta \vee \phi}$ DIS $\dfrac{\Gamma, \theta \mid \psi \quad \Gamma, \phi \mid \psi}{\Gamma, \theta \vee \phi \mid \psi}$

IMR $\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$ MP $\dfrac{\Gamma \mid \theta \quad \Gamma \mid \theta \to \phi}{\Gamma \mid \phi}$

NIN $\dfrac{\Gamma, \theta \mid \phi \quad \Gamma, \theta \mid \neg\phi}{\Gamma \mid \neg\theta}$ NNO $\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$

MON $\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$ AO $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \quad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$

Modal Logics

Axioms:

K := REF + $\Box(\theta \to \phi) \mid \Box\theta \to \Box\phi$
T := $K + \Box\theta \mid \theta$
D := $K + \Box\theta \mid \Diamond\theta$
B := $K + \theta \mid \Box\Diamond\theta$
S_4 := $T + \Box\theta \mid \Box\Box\theta$
S_5 := $T + \Diamond\theta \mid \Box\Diamond\theta$

Rules of Proof: All the rules of SC plus NEC $\dfrac{\mid \theta}{\mid \Box\theta}$

Łukasiewicz Logic, Ł

Axioms: REF together with

Ł1 : $\mid \theta \to (\phi \to \theta)$
Ł2 : $\mid (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$
Ł3 : $\mid (\neg\theta \to \neg\phi) \to (\phi \to \theta)$
Ł4 : $\mid ((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $\mid (\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$
Ł5 : $\mid (\theta \to \phi) \underline{\vee} (\phi \to \theta)$

Rules of Proof: Only MP

GM Rules and Axiom

REF, $\theta \mid\!\sim \theta$, together with:

left logical equivalence	$$\frac{\theta \equiv \phi, \quad \theta \mid\!\sim \psi}{\phi \mid\!\sim \psi}$$	LLE
right weakening	$$\frac{\theta \mid\!\sim \phi, \quad \phi \models \psi}{\theta \mid\!\sim \psi}$$	RWE
cautious monotonicity	$$\frac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \wedge \phi \mid\!\sim \psi}$$	CMO
and on right	$$\frac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \mid\!\sim \phi \wedge \psi}$$	AND
disjunction, or left	$$\frac{\theta \mid\!\sim \psi, \quad \phi \mid\!\sim \psi}{\theta \vee \phi \mid\!\sim \psi}$$	DIS
rational monotonicity	$$\frac{\theta \mid\!\sim \psi, \quad \theta \not\mid\!\sim \neg\phi}{\theta \wedge \phi \mid\!\sim \psi}$$	RMO