Two Hours and Thirty Minutes

UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

24th April 2009

2.00–4.30

Answer **all** 8 questions (80 marks in all).

A list of axioms and rules of proof is appended to this examination paper

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SECTION A

Answer **ALL** 8 questions

A1. How is $\mid\!\sim_{\vec{s}}$ defined for $\vec{s} = s_1, \ldots, s_m \subseteq At^L$?

State the *Representation Theorem for Rational Consequence Relations.*

In the case when $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{\neg p \wedge \neg q \wedge \neg r\} \\ s_2 &= \{p \wedge \neg q \wedge \neg r, \neg p \wedge q \wedge r\} \\ s_3 &= \{p \wedge q \wedge r\} \end{aligned}$$

which of the following are true? [You need not justify your answers.]

$$\begin{array}{rc} \text{(i)} & \neg p \mid\!\sim_{\vec{s}} \neg q \wedge \neg r \\ \text{(ii)} & p \vee q \mid\!\sim_{\vec{s}} r \\ \text{(iii)} & q \wedge \neg r \mid\!\sim_{\vec{s}} p \end{array}$$

[12 marks]

A2. By giving a direct derivation from the GM rules show that the rule

$$\frac{\neg\phi \mid\!\sim \psi \qquad \neg\phi \mid\!\not\sim \neg\theta}{\theta \mid\!\sim \phi \vee \psi}$$

is satisfied by all rational consequence relations. [You may use the derived rule Con but in that case you should give a derivation of it from the GM rules.]

[6 marks]

A3. By using the Representation Theorem for rational consequence relations show that the rule

$$\frac{\theta \vee \phi \mid\!\sim \neg\theta}{\psi \vee \phi \mid\!\sim \neg\theta}$$

holds for all rational consequence relations, but that the rule

$$\frac{\theta \vee \phi \mid\!\sim \neg\theta}{\psi \wedge \phi \mid\!\sim \neg\theta}$$

fails for some rational consequence relation (and choice of θ, ϕ, ψ).

[12 marks]

A4. How is the relation $\models^{S_4}$ defined?

State the Completeness Theorem for S_4.

Show that:

(i) $\models^{S_4} \Diamond\theta \vee \Box\Box\neg\theta$.

(ii) $\not\models^{S_4} p \vee \Box\Box\neg p$.

[12 marks]

A5. What is meant by a *proof* in B?

What does it mean to say $\Gamma \vdash^{B} \theta$?

Give a formal proof to show that

$$\vdash^{B} \Box\theta \to \Box\Box\Diamond\theta.$$

[8 marks]

A6. Let H be K together with the axiom schema

$$\Diamond\theta \mid \Diamond\Diamond\theta$$

By considering a suitable family of frames, or otherwise, show that

$$\Diamond p \not\vdash^{H} p.$$

[10 marks]

A7. List the desiderata (C1)-(C4) for a function $F_\wedge : [0,1]^2 \to [0,1]$.

What can be said about $F_\wedge$ according to the classification given by the Mostert-Shields Theorem when, in addition to (C1)-(C4), $F_\wedge$ also satisfies that $0 < F_\wedge(x,x) < x$ for all $x \in (0,1)$?

[9 marks]

A8. How is the relation $\models^{\text{Ł}}$ defined?

Which of the following are true (for all $\theta, \phi \in SL$)? In each case you should justify your answer.

(i) If $\theta \models^{\text{Ł}} \phi$ then $\models^{\text{Ł}} \theta \to \phi$.

(ii) $\models^{\text{Ł}} ((\phi \to \theta) \to \phi) \to (\theta \to \phi)$.

[11 marks]

END OF EXAMINATION PAPER

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: REF $\theta \mid \theta$

Rules of Proof:

AND $\dfrac{\Gamma \mid \theta \quad \Gamma \mid \phi}{\Gamma \mid \theta \wedge \phi}$ ANL $\dfrac{\Gamma, \theta, \phi \mid \psi}{\Gamma, \theta \wedge \phi \mid \psi}$

ORR $\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \quad \dfrac{\Gamma \mid \phi}{\Gamma \mid \theta \vee \phi}$ DIS $\dfrac{\Gamma, \theta \mid \psi \quad \Gamma, \phi \mid \psi}{\Gamma, \theta \vee \phi \mid \psi}$

IMR $\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$ MP $\dfrac{\Gamma \mid \theta \quad \Gamma \mid \theta \to \phi}{\Gamma \mid \phi}$

NIN $\dfrac{\Gamma, \theta \mid \phi \quad \Gamma, \theta \mid \neg\phi}{\Gamma \mid \neg\theta}$ NNO $\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$

MON $\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$ AO $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \quad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$

Modal Logics

Axioms:

$$
\begin{aligned}
\mathrm{K} &:= \quad \mathrm{REF} + \Box(\theta \to \phi) \mid \Box\theta \to \Box\phi \\
\mathrm{T} &:= \quad K + \Box\theta \mid \theta \\
\mathrm{D} &:= \quad K + \Box\theta \mid \Diamond\theta \\
\mathrm{B} &:= \quad K + \theta \mid \Box\Diamond\theta \\
\mathrm{S}_4 &:= \quad T + \Box\theta \mid \Box\Box\theta \\
\mathrm{S}_5 &:= \quad T + \Diamond\theta \mid \Box\Diamond\theta
\end{aligned}
$$

Rules of Proof: All the rules of SC plus NEC $\dfrac{\mid \theta}{\mid \Box\theta}$

Łukasiewicz Logic, Ł

Axioms: REF together with

Ł1 : $\mid \theta \to (\phi \to \theta)$
Ł2 : $\mid (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$
Ł3 : $\mid (\neg\theta \to \neg\phi) \to (\phi \to \theta)$
Ł4 : $\mid ((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $\mid (\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$
Ł5 : $\mid (\theta \to \phi) \underline{\vee} (\phi \to \theta)$

Rules of Proof: Only MP

GM Rules and Axiom

REF, $\theta \mathrel{|}\!\sim \theta$, together with:

left logical equivalence	$\dfrac{\theta \equiv \phi, \quad \theta \mathrel{\mid}\!\sim \psi}{\phi \mathrel{\mid}\!\sim \psi}$	LLE
right weakening	$\dfrac{\theta \mathrel{\mid}\!\sim \phi, \quad \phi \models \psi}{\theta \mathrel{\mid}\!\sim \psi}$	RWE
cautious monotonicity	$\dfrac{\theta \mathrel{\mid}\!\sim \phi, \quad \theta \mathrel{\mid}\!\sim \psi}{\theta \wedge \phi \mathrel{\mid}\!\sim \psi}$	CMO
and on right	$\dfrac{\theta \mathrel{\mid}\!\sim \phi, \quad \theta \mathrel{\mid}\!\sim \psi}{\theta \mathrel{\mid}\!\sim \phi \wedge \psi}$	AND
disjunction, or left	$\dfrac{\theta \mathrel{\mid}\!\sim \psi, \quad \phi \mathrel{\mid}\!\sim \psi}{\theta \vee \phi \mathrel{\mid}\!\sim \psi}$	DIS
rational monotonicity	$\dfrac{\theta \mathrel{\mid}\!\sim \psi, \quad \theta \mathrel{\mid}\!\not\sim \neg\phi}{\theta \wedge \phi \mathrel{\mid}\!\sim \psi}$	RMO