Solution and Feedback to the second MATH43032/63032 Coursework, 2013-2014.

In the $\Leftarrow$ direction we need to show that the $\mathcal{A}$ axiom is valid, i.e. if $\langle W, E, V\rangle$ is an $\mathcal{A}$-frame, $i \in W$ and $i \models \Diamond\Diamond\Box\theta$ then $i \models \theta$. So assume that in this $\mathcal{A}$-frame $i \models \Diamond\Diamond\Box\theta$. Then for some $j \in W$ with $\langle i, j\rangle \in E$, $j \models \Diamond\Box\theta$. Again for some $k \in W$ with $\langle j, k\rangle \in E$, $k \models \Box\theta$. Since this is an $\mathcal{A}$-frame, $\langle k, i\rangle \in E$, hence $i \models \theta$, as required.

In the $\Rightarrow$ direction form, as usual, the frame whose vertices are all maximal $\mathcal{A}$-consistent subsets Γ of SML with

$$\langle \Gamma, \Delta\rangle \in E \iff \Delta \supseteq \{\phi \,|\, \Box\phi \in \Gamma\},$$

equivalently

$$\langle \Gamma, \Delta\rangle \in E \iff \Gamma \supseteq \{\Diamond\phi \,|\, \phi \in \Delta\} \qquad \ddagger,$$

and

$$V_\Gamma(p) = 1 \iff p \in \Gamma.$$

We must show that this is an $\mathcal{A}$-frame.

So suppose that $\langle \Gamma, \Delta\rangle, \langle \Delta, \Omega\rangle \in E$. We need to show that $\langle \Omega, \Gamma\rangle \in E$. Let $\Box\phi \in \Omega$. Then from $\ddagger$ with $\langle \Delta, \Omega\rangle \in E$, $\Diamond\Box\phi \in \Delta$. Again by $\ddagger$ with $\langle \Gamma, \Delta\rangle \in E$, $\Diamond\Diamond\Box\phi \in \Gamma$. Hence $\Gamma \vdash^{\mathcal{A}} \Diamond\Diamond\Box\phi$ $(\star)$. Since $\Diamond\Diamond\Box\phi \vdash^{\mathcal{A}} \phi$, by IMR, $\vdash^{\mathcal{A}} \Diamond\Diamond\Box\phi \to \phi$ and with $\star$ and MP, $\Gamma \vdash^{\mathcal{A}} \phi$. Hence (by Lemma 9(1), for $\mathcal{A}$), $\phi \in \Gamma$.

In conclusion we have shown that $\Box\phi \in \Omega \Rightarrow \phi \in \Gamma$, so $\langle \Omega, \Gamma\rangle \in E$, as required.

For the last part let $\langle W, E, V\rangle$ be the $\mathcal{A}$-frame with $W = \{1, 2, 3\}$, $E = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$ and $V_1(p) = 0, V_2(p) = V_3(p) = 1$. Then $2 \models \Box p$, $1 \models \Diamond\Box p$ but $1 \not\models p$ so $\Diamond\Box p \not\models^{\mathcal{A}} p$. By the first part then $\Diamond\Box p \nvdash^{\mathcal{A}} p$.

Feedback

Generally well done though many students wrote much more than was required – see my version above.

One common mistake however was in showing that $\Diamond\Diamond\Box\theta \models^{\mathcal{A}} \theta$. Students started off along the right lines by taking an $\mathcal{A}$-frame $\langle W, E, V\rangle$ and $i \in W$

such that $i \models \Diamond\Diamond\Box\theta$. But then instead of *picking* $j, k \in W$ such that $\langle i, j\rangle, \langle j, k\rangle \in E$ with $j \models \Diamond\Box\theta$ and $k \models \Box\theta$ they just said that *if* $j, k \in W$ with $\langle i, j\rangle, \langle j, k\rangle \in E$ *then* $j \models \Diamond\Box\theta$ and $k \models \Box\theta$. In other words claiming this conclusion for *any* $j, k \in W$ with $\langle i, j\rangle, \langle j, k\rangle \in E$ whereas what you were given only allowed you to assert it for *some* such j, k.