Solution and Feedback to the second MATH43032/63032 Coursework, 2013-2014.

In the ⇐ direction we need to show that the \( \mathcal{A} \) axiom is valid, i.e. if \( \langle W, E, V \rangle \) is an \( \mathcal{A} \)-frame, \( i \in W \) and \( i \models \Diamond \Box \theta \) then \( i \models \theta \). So assume that in this \( \mathcal{A} \)-frame \( i \models \Diamond \Box \theta \). Then for some \( j \in W \) with \( \langle i, j \rangle \in E \), \( j \models \Box \theta \). Again for some \( k \in W \) with \( \langle j, k \rangle \in E \), \( k \models \Box \theta \). Since this is an \( \mathcal{A} \)-frame, \( \langle k, i \rangle \in E \), hence \( i \models \theta \), as required.

In the ⇒ direction form, as usual, the frame whose vertices are all maximal \( \mathcal{A} \)-consistent subsets \( \Gamma \) of SML with

\[
\langle \Gamma, \Delta \rangle \in E \iff \Delta \supseteq \{ \phi | \Box \phi \in \Gamma \},
\]
equivalently

\[
\langle \Gamma, \Delta \rangle \in E \iff \Gamma \supseteq \{ \Diamond \phi | \phi \in \Delta \} \quad \uparrow,
\]
and

\[
V_\Gamma(p) = 1 \iff p \in \Gamma.
\]

We must show that this is an \( \mathcal{A} \)-frame.

So suppose that \( \langle \Gamma, \Delta \rangle, \langle \Delta, \Omega \rangle \in E \). We need to show that \( \langle \Omega, \Gamma \rangle \in E \). Let \( \Box \phi \in \Omega \). Then from \( \uparrow \) with \( \langle \Delta, \Omega \rangle \in E \), \( \Diamond \Box \phi \in \Delta \). Again by \( \uparrow \) with \( \langle \Gamma, \Delta \rangle \in E \), \( \Diamond \Box \phi \in \Gamma \). Hence \( \Gamma \vdash \mathcal{A} \Diamond \Box \phi \) (\( \star \)). Since \( \Diamond \Box \phi \vdash \mathcal{A} \phi \), by IMR, \( \vdash \mathcal{A} \Diamond \Box \phi \rightarrow \phi \) and with \( \star \) and MP, \( \Gamma \vdash \mathcal{A} \phi \). Hence (by Lemma 9(1), for \( \mathcal{A} \)), \( \phi \in \Gamma \).

In conclusion we have shown that \( \Box \phi \in \Omega \Rightarrow \phi \in \Gamma \), so \( \langle \Omega, \Gamma \rangle \in E \), as required.

For the last part let \( \langle W, E, V \rangle \) be the \( \mathcal{A} \)-frame with \( W = \{1, 2, 3\} \), \( E = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle\} \) and \( V_1(p) = 0, V_2(p) = V_3(p) = 1 \). Then \( 2 \models \Box p \), \( 1 \models \Diamond \Box p \) but \( 1 \not\models p \) so \( \Diamond \Box p \not\vdash \mathcal{A} p \). By the first part then \( \Diamond \Box p \not\vdash \mathcal{A} p \).

**Feedback**

Generally well done though many students wrote much more than was required – see my version above.

One common mistake however was in showing that \( \Diamond \Box \theta \vdash \mathcal{A} \theta \). Students started off along the right lines by taking an \( \mathcal{A} \)-frame \( \langle W, E, V \rangle \) and \( i \in W \).
such that $i \models \Diamond \Diamond \Box \theta$. But then instead of picking $j, k \in W$ such that $\langle i, j \rangle, \langle j, k \rangle \in E$ with $j \models \Diamond \Box \theta$ and $k \models \Box \theta$ they just said that if $j, k \in W$ with $\langle i, j \rangle, \langle j, k \rangle \in E$ then $j \models \Diamond \Box \theta$ and $k \models \Box \theta$. In other words claiming this conclusion for any $j, k \in W$ with $\langle i, j \rangle, \langle j, k \rangle \in E$ whereas what you were given only allowed you to assert it for some such $j, k$. 