A1. \[ \theta \models_{\sim} \phi \iff \begin{cases} \forall i \ s_i \cap S_{\theta} = \emptyset \quad \text{or} \\ \exists i \ s_i \cap S_{\theta} \neq \emptyset \quad \text{and for the least such} \ i, \ s_i \cap S_{\theta} \subseteq S_{\phi}. \end{cases} \]

The Representation Theorem for Rational Consequence Relations: Every rational consequence relation on SL is of the form \( \models_{\sim} \) for some \( \bar{s} = s_1, s_2, \ldots, s_m \subseteq A^L \), and conversely every \( \models_{\sim} \) is a rational consequence relation.

(i) True, \quad (ii) Not true \quad (iii) True.

A2. (a) Assume that \( \theta \land \neg \phi \models \phi \). Since \( \theta \land \phi \models \phi \) by Supraclassicality, SCL, \( \theta \land \phi \models \phi \). This with \( \theta \land \neg \phi \models \phi \) gives by DIS \( (\theta \land \phi) \lor (\theta \land \neg \phi) \models \phi \). Finally, since

\[ (\theta \land \phi) \lor (\theta \land \neg \phi) \equiv \theta, \]

by LLE, \( \theta \models \phi \). [Alternatively use CON on \( \theta \land \neg \phi \models \phi \) to get \( \theta \models (\neg \phi \rightarrow \phi) \) and then RWE with \( \neg \phi \rightarrow \phi \models \phi \).]

(b) Letting \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) be as usual \( (p \land q), (p \land \neg q), (\neg p \land q), (\neg p \land \neg q) \) we have for

\( K = \{ p \models \neg q, \neg q \models \neg p \} \)

\[ S_{\neg p \lor \neg q} = \{ \alpha_2, \alpha_3, \alpha_4 \}, \quad S_{p \land \neg q} = \{ \alpha_2 \}, \]

\[ S_{\neg q \lor \neg p} = S_{q \lor p} = \{ \alpha_1, \alpha_3, \alpha_4 \}, \quad S_{q \land \neg p} = \{ \alpha_4 \}. \]

Running the Z-algorithm we get:

\[ A_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}, \quad K_0 = K = \{ p \models \neg q, \neg q \models \neg p \}, \]

\[ u_1 = A_0 \cap S_{\neg p \lor \neg q} \cap S_{\neg q \lor \neg p} \quad = \{ \alpha_2, \alpha_3, \alpha_4 \} \cap \{ \alpha_1, \alpha_3, \alpha_4 \} = \{ \alpha_3, \alpha_4 \}. \]

\[ A_1 = A_0 - u_1 = \{ \alpha_1, \alpha_2 \}, \quad K_1 = \{ p \models \neg q \}, \text{since} \ S_{\neg q \lor \neg p} \cap u_1 = \{ \alpha_4 \} \neq \emptyset, \]

\[ S_{p \land \neg q} \cap u_1 = \{ \alpha_2 \} \cap \{ \alpha_3, \alpha_4 \} = \emptyset, \]

\[ u_2 = A_1 \cap S_{\neg p \lor \neg q} = \{ \alpha_1, \alpha_2 \} \cap \{ \alpha_2, \alpha_3, \alpha_4 \} = \{ \alpha_2 \}. \]

\[ A_2 = A_1 - u_2 = \{ \alpha_1 \}, \quad K_2 = \emptyset \text{ since } u_2 \cap S_{p \land q} = \{ \alpha_2 \} \neq \emptyset. \]

\[ u_3 = A_2 = \{ \alpha_1 \}. \]

Since all the atoms have now been used up we must have \( u_4 = \emptyset \) so the rational closure of \( K \) is \( \models_{\sim} \) where

\[ \bar{u} = u_1, u_2, u_3 = \{ \alpha_3, \alpha_4 \}, \{ \alpha_2 \}, \{ \alpha_1 \}. \]

A3. Completeness Theorem for B: For \( \Gamma \subseteq SML, \theta \in SML, \)

\[ \Gamma \models^B \theta \iff \Gamma \models^B \theta. \]

\(^1\)As usual these are more detailed than I would necessarily require from the students since they are also intended to serve an instructional purpose.
[No need to define $|=B$.]

(i) Let $\langle W, E, V \rangle$ be a $B$-frame, so $E$ is symmetric, and let $i \in W$, $i \models \theta$. Suppose that $\langle i, j \rangle \in E$, so $\langle j, i \rangle \in E$ by symmetry and $j \models \diamond \theta$. Now suppose that $\langle j, k \rangle \in E$, so by symmetry $\langle k, j \rangle \in E$ and $k \models \diamond \diamond \theta$. Since $k$ was an arbitrary element of $W$ such that $\langle j, k \rangle \in E$, $j \models \Box \diamond \diamond \theta$. Similarly since $j$ was an arbitrary element of $W$ such that $\langle i, j \rangle \in E$, $i \models \Box \Box \diamond \diamond \theta$, as required.

(ii) Let $\langle W, E, V \rangle$ be the symmetric frame

```
0
\[\hspace{1cm} \Rightarrow \hspace{1cm} \]
\[\hspace{1cm} \Leftarrow \hspace{1cm} \]
1
\leftarrow
\neg p
\rightarrow
p
```

Then $1 \models \diamond p$ so $0 \models \diamond \diamond p$ and $1 \models \diamond \diamond p$. Hence $1 \models \Box \diamond \diamond p$ so $0 \models \Box \Box \diamond p$. But $0 \not\models p$ so $\Box \Box \diamond p \not\models B p$. By the Completeness Theorem for $B$ then, $\Box \Box \diamond p \not\models B p$.

A4. A proof in $T$ is a sequence of sequents $\Gamma_1|\theta_1, \Gamma_2|\theta_2, \ldots, \Gamma_m|\theta_m$, where the $\Gamma_i$ are finite subsets of SML and the $\theta_i \in SML$, such that for each $i = 1, 2, \ldots, m$, either $\Gamma_i|\theta_i$ is an instance of an axiom of $T$ or $\exists j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1}|\theta_{j_1}, \Gamma_{j_2}|\theta_{j_2}, \ldots, \Gamma_{j_s}|\theta_{j_s}}{\Gamma_i|\theta_i}$$

is an instance of a rule of proof of $T$.

$\Gamma \vdash^T \theta$ if $\exists$ a proof in $T$, $\Gamma_1|\theta_1, \Gamma_2|\theta_2, \ldots, \Gamma_m|\theta_m$, such that $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$.

Proof in $T$ of $\vdash^T \Box(\Box \theta \to \theta)$:

1. $\Box \theta \to \theta$ $\hspace{1cm}$ T axiom
2. $\equiv (\Box \theta \to \theta)$ IMR, 1
3. $\equiv (\Box \theta \to \theta)$ NEC, 2

A5. (i) Suppose that $|=L \theta \to \phi$, $\mathbb{S}$, and let $w$ be a $[0,1]$-valuation such that $w(\theta) = 1$. Then from $\mathbb{S}$, $w(\theta \to \phi) = 1$ so $w(\theta) \leq w(\phi)$, giving that also $w(\phi) = 1$. Hence $\theta \models L \phi$.

(ii) Let $w$ be a $[0,1]$-valuation. If $w(\theta) \leq w(\phi)$ then $w(\theta \to \phi) = 1$. If $w(\theta) > w(\phi)$ then

$$w(\neg \theta) = 1 - w(\theta) < 1 - w(\phi) = w(\neg \phi)$$

so $w(\neg \theta \to \neg \phi) = 1$. Either way then

$$1 = \max\{w(\theta \to \phi), w(\neg \theta \to \neg \phi)\} = w((\theta \to \phi) \lor (\neg \theta \to \neg \phi)).$$

Hence $|=L (\theta \to \phi) \lor (\neg \theta \to \neg \phi)$.

(iii) Assume that

$$|=L \theta \to \phi \hspace{1cm} |=L \phi \to \psi$$

From L2 (and Proposition 8, which as usual we assume throughout) we have

$$|=L (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi)).$$
Modus Ponens with \((\ast)\) now gives

\[ \vdash^L (\phi \rightarrow \psi) \rightarrow (\theta \rightarrow \psi) \]

and now Modus Ponens with \((\dagger)\) gives as required that \(\vdash^L \theta \rightarrow \psi\).

**A6.** Writing, as usual, just \(x \land y\) for \(F_\land(x, y)\):

(C1) \(0 \land 1 = 1 \land 0 = 0, 1 \land 1 = 1\),
(C2) \(\land\) is continuous,
(C3) \(\land\) is increasing (not necessarily strictly) in each coordinate,
(C4) \(\land\) is associative.

**Mostert-Shields Theorem (for \(F_\land\)):** Let \(\land\) satisfy (C1-4) and let \(A = \{ x \in [0, 1] | x \land x = x \}\). Then for \(x \in A\) and \(0 \leq z \leq x \leq y \leq 1\),

\[ z \land y = y \land z = z = \min\{y, z\}, \]

and if \(a < b\), \(a, b \in A\) and \((a, b) \cap A = \emptyset\) then on \([a, b]\) either

\[ \langle[a, b], \land, <\rangle \cong \langle[0, 1], x, <\rangle \] or \[ \langle[a, b], \land, <\rangle \cong \langle[0, 1], \max\{0, x + y - 1\}, <\rangle. \]

Let \(0 \leq x \leq 1\). Since \(0 = 0 \land 1 \leq x \leq 1 \land 1 = 1\) and the function \(t \mapsto t \land 1\) is continuous by (C2) the IVT gives that there is some \(0 \leq t \leq 1\) such that \(x = t \land 1\). Hence

\[ x = t \land 1 = t \land (1 \land 1) \text{ (by (C1))} = (t \land 1) \land 1 \text{ (by (C4))} = x \land 1. \]

For the last part, since \(x \land x < x\) for \(0 < x < 1\) the set \(A\) in the Mostert-Shields Theorem is just \([0, 1]\). Hence \(\langle[0, 1], \land, <\rangle\) is isomorphic to one of \(\langle[0, 1], \max\{0, x + y - 1\}, <\rangle, \langle[0, 1], x, <\rangle\). It cannot be the former since \(x \land x > 0\) for \(x > 0\), so it must be isomorphic to \(\langle[0, 1], x, <\rangle\).

**B7.** By the Representation Theorem it is enough to show \(\theta \vdash^L \phi \lor \psi\) that under the assumption that

\[ \theta \land \neg\phi \vdash^L \phi \lor \psi \quad (\ast) \]

If \(s_i \cap S_\theta = \emptyset\) for all \(i\) then \(\theta \vdash^L \phi \lor \psi\) holds directly from the definition. So suppose that \(s_i \cap S_\theta \neq \emptyset\) for some \(i\) and let \(k\) be the least such. Let \(\alpha \in s_k \cap S_\theta\).

If \(\alpha \notin S_{\neg\phi}\) then

\[ \alpha \in (At^L - S_{\neg\phi}) = S_{\neg\neg\phi} = S_\phi \subseteq S_\phi \cup S_\psi = S_{\phi \lor \psi} \quad (\dagger). \]

If \(\alpha \in S_{\neg\phi}\) then \(\alpha \in S_\theta \cap S_{\neg\phi} = S_{\theta \land \neg\phi}\) so \(s_k \cap S_{\theta \land \neg\phi} \neq \emptyset\). Indeed \(k\) must be the least such since if \(j < k\) and \(s_j \cap S_{\theta \land \neg\phi} \neq \emptyset\) then \(s_j \cap S_\theta \neq \emptyset\) (since \(S_{\theta \land \neg\phi} = S_\theta \cap S_{\neg\phi} \subseteq S_\theta\)) contradicting the choice of \(k\).

From \((\ast)\) then \(\alpha \in S_{\phi \lor \psi}\), again. With \((\dagger)\) this gives that \(s_k \cap S_\theta \subseteq S_{\phi \lor \psi}\) so, as required, \(\theta \vdash^L \phi \lor \psi\).

Letting \(L = \{p, q, r\}\), \(\theta = p, \phi = q, \psi = r\) and

\[ \bar{s} = s_1, s_2 = \{p \land q \land r\}, \{p \land \neg q \land \neg r\}. \]
we have that \( \theta \vdash_{\mathcal{S}} \phi \lor \psi \) (i.e. \( p \vdash_{\mathcal{S}} q \lor r \)) but not \( \theta \land \neg \phi \vdash_{\mathcal{S}} \phi \lor \psi \) (i.e. not \( p \land \neg q \vdash_{\mathcal{S}} q \lor r \)) so this ‘rule’ fails for \( \mathcal{S} \), and by the Representation Theorem \( \vdash_{\mathcal{S}} \) is a rational consequence relation.

**B8.** To show the required Completeness Theorem for \( \mathcal{R} \) there are two ‘new’ features that we have to check. Firstly we need to show that this additional axiom

\[ | \Diamond (\theta \lor \neg \theta) \quad (R) \]

is valid in serial frames, meaning that if \( \langle W, E, V \rangle \) is a serial frame and \( i \in W \) then \( i \models \Diamond (\theta \lor \neg \theta) \). But since this frame is serial there is a \( j \in W \) such that \( (i, j) \in E \). Clearly \( j \models \theta \lor \neg \theta \) (since either \( j \models \theta \) or \( j \nmid \theta \) (– in which case \( j \models \neg \theta \)), so \( i \models \Diamond (\theta \lor \neg \theta) \), as required.

Secondly we need to show that if \( \Gamma \) is a maximally \( \mathcal{R} \)-consistent frame \( \Delta \) such that

\[ \{ \phi \mid \Box \phi \in \Gamma \} \subseteq \Delta, \]

equivalently that \( \{ \phi \mid \Box \phi \in \Gamma \} \) is \( \mathcal{R} \)-consistent. Suppose not, say \( \Box \phi_1, \Box \phi_2, \ldots, \Box \phi_m \in \Gamma \) and for some \( \theta \),

\[ \phi_1, \phi_2, \ldots, \phi_m \vdash \mathcal{R} \theta \land \neg \theta. \]

Then since \( K \subseteq \mathcal{R} \),

\[ \Box \phi_1, \Box \phi_2, \ldots, \Box \phi_m \vdash \mathcal{R} \Box (\theta \land \neg \theta), \]

so

\[ \Gamma \vdash \mathcal{R} \Box (\theta \land \neg \theta) \quad (\dagger) \]

since the \( \Box \phi_i \in \Gamma \). But directly from \( (R) \),

\[ \Gamma \vdash \mathcal{R} \Diamond (\theta \lor \neg \theta) \quad (\ast). \]

Since

\[ \vdash \Box \neg \Box (p \lor \neg p) \leftrightarrow (p \land \neg p), \]

so

\[ \vdash \Box \neg \Box (\theta \lor \neg \theta) \leftrightarrow (\theta \land \neg \theta), \]

and

\[ \vdash \Box \neg \Box (\theta \lor \neg \theta) \leftrightarrow \neg \Box (\theta \lor \neg \theta). \]

Hence with \( (\ast) \),

\[ \Gamma \vdash \mathcal{R} \neg \Box (\theta \land \neg \theta), \]

which with \( (\dagger) \) shows \( \Gamma \) to be \( \mathcal{R} \)-inconsistent, and so yields the required contradiction.

Finally, combining this with the (identical!) Completeness Theorem for \( D \),

\[ \Gamma \vdash \mathcal{R} \theta \quad \iff \quad \text{for all serial frames } \langle W, E, V \rangle \text{ and } \]

\[ i \in W, \text{ if } i \models \Gamma \text{ then } i \models \theta \]

\[ \iff \quad \Gamma \vdash \mathcal{D} \theta. \]

**B9.** McNaughton’s Theorem(for \( L = \{ p \} \)): A function \( F : [0, 1] \longrightarrow [0, 1] \) is of the form \( F_\theta \) for some \( \theta \in SL \), with \( L = \{ p \} \), iff there exist some \( \theta = \gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_{n-1} < \gamma_n = 1 \) and \( n_i, m_i \in \mathbb{Z} \) for \( i = 1, 2, \ldots, n-1 \), such that on each \( [\gamma_i, \gamma_{i+1}] \) \( F(x) = m_i + n_i x \) (\( \in [0, 1] \)).

[Recall that \( F_\theta(x) = w(x) \) for that \( [0, 1] \)-valuation \( w \) of \( L \) for which \( w(p) = x \).]

Let \( \theta \in SL \) and let the \( \gamma_i, n_i, m_i, i = 0, 1, \ldots, m \) be as in McNaughton’s Theorem for \( \theta \). On any one of the (finitely many) intervals \( [\gamma_i, \gamma_{i+1}] \) either \( n_i = 0 \), in which case \( F_\theta(x) = m_i \in \mathbb{Z} \) so
never takes value $1/3$, or $n_i \neq 0$ in which case $F_\theta(x)$ is strictly monotone, and so can take the value $1/3$ at most once. Hence there are at most finitely many $x \in [0,1]$ for which $F_\theta(x) = 1/3$, equivalently only finitely many $[0,1]$-valuations $w$ on $L$ for which $w(\theta) = 1/3$.

$F_\theta(1) = w(\theta)$ where $w(p) = 1$. In this case $w(\phi) = V(\phi)$ for all $\phi \in SL$, where $V$ is the Sentential Calculus $\{0,1\}$-valuation on $L$ giving $p$ value 1, so $F_\theta(1) = w(\theta) = V(\theta) \in \{0,1\}$. Similarly $F_\theta(0) \in \{0,1\}$. Hence if $F_\theta(0) \neq F_\theta(1) F_\theta$ must take both the values 0,1, and so, being continuous, must take value 1/3 by the IVT. So if $F_\theta$ does not take the value 1/3 then either $F_\theta(0) = F_\theta(1) = 1$ or $F_\theta(0) = F_\theta(1) = 0$. In the first case both of the two valuations $V$ on $SL$ must give $\theta$ value 1, so $\models^SC \theta$. In the second case both valuations give $\theta$ value 0, so give $\neg \theta$ value 1, so $\models^SC \neg \theta$. 

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