MATH43032/63032 Exam Solutions, 2012[1]

A1.

$$\theta \mid\!\sim_{\vec{s}} \phi \iff \begin{cases} \forall i \ s_i \cap S_\theta = \emptyset & \text{or} \\ \exists i \ s_i \cap S_\theta \neq \emptyset & \text{and for the least such } i,\ s_i \cap S_\theta \subseteq S_\phi. \end{cases}$$

The Representation Theorem for Rational Consequence Relations: Every rational consequence relation on SL is of the form $\mid\!\sim_{\vec{s}}$ for some $\vec{s} = s_1, s_2, \ldots, s_m \subseteq \mathrm{At}^L$, and conversely every $\mid\!\sim_{\vec{s}}$ is a rational consequence relation.

(i) True, (ii) Not true (iii) True.

A2. (a) Assume that $\theta \wedge \neg\phi \mid\!\sim \phi$. Since $\theta \wedge \phi \models \phi$, by Supraclassicality, SCL, $\theta \wedge \phi \mid\!\sim \phi$. This with $\theta \wedge \neg\phi \mid\!\sim \phi$ gives by DIS $(\theta \wedge \phi) \vee (\theta \wedge \neg\phi) \mid\!\sim \phi$. Finally, since

$$(\theta \wedge \phi) \vee (\theta \wedge \neg\phi) \equiv \theta,$$

by LLE, $\theta \mid\!\sim \phi$. [Alternatively use CON on $\theta \wedge \neg\phi \mid\!\sim \phi$ to get $\theta \mid\!\sim (\neg\phi \to \phi)$ and then RWE with $\neg\phi \to \phi \models \phi$.]

(b) Letting $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be as usual $(p \wedge q), (p \wedge \neg q), (\neg p \wedge q), (\neg p \wedge \neg q)$ we have for $K = \{\, p \mid\!\sim \neg q,\ \neg q \mid\!\sim \neg p\}$

$$S_{\neg p \vee \neg q} = \{\alpha_2, \alpha_3, \alpha_4\}, \qquad S_{p \wedge \neg q} = \{\alpha_2\},$$
$$S_{\neg\neg q \vee \neg p} = S_{q \vee \neg p} = \{\alpha_1, \alpha_3, \alpha_4\}, \quad S_{\neg q \wedge \neg p} = \{\alpha_4\}.$$

Running the Z-algorithm we get:

$A_0 = \{\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \,\}$,
$K_0 = K = \{\, p \mid\!\sim \neg q,\ \neg q \mid\!\sim \neg p\}$,
$u_1 = A_0 \cap S_{\neg p \vee \neg q} \cap S_{\neg\neg q \vee \neg p}$
$\quad = \{\alpha_2, \alpha_3, \alpha_4\} \cap \{\alpha_1, \alpha_3, \alpha_4\} = \{\alpha_3, \alpha_4\}$.

$A_1 = A_0 - u_1 = \{\, \alpha_1, \alpha_2\}$,
$K_1 = \{\, p \mid\!\sim \neg q, \}$, since $S_{\neg q \wedge \neg p} \cap u_1 = \{\alpha_4\} \neq \emptyset$,
$S_{p \wedge \neg q} \cap u_1 = \{\alpha_2\} \cap \{\, \alpha_3,\ \alpha_4 \,\} = \emptyset$,
$u_2 = A_1 \cap S_{\neg p \vee \neg q} = \{\, \alpha_1, \alpha_2 \,\} \cap \{\, \alpha_2, \alpha_3, \alpha_4 \,\} = \{\, \alpha_2 \,\}$.

$A_2 = A_1 - u_2 = \{\, \alpha_1\}$,
$K_2 = \emptyset$ since $u_2 \cap S_{p \wedge \neg q} = \{\, \alpha_2 \,\} \neq \emptyset$.
$u_3 = A_2 = \{\, \alpha_1 \,\}$

Since all the atoms have now been used up we must have $u_4 = \emptyset$ so the rational closure of K is $\mid\!\sim_{\vec{u}}$ where

$$\vec{u} = u_1, u_2, u_3 = \{\, \alpha_3,\ \alpha_4 \,\},\ \{\, \alpha_2 \,\},\ \{\, \alpha_1 \,\}.$$

A3. *Completeness Theorem for B*: For $\Gamma \subseteq SML, \theta \in SML$,

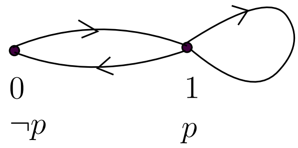
$$\Gamma \vdash^B \theta \iff \Gamma \models^B \theta.$$

[1] As usual these are more detailed than I would necessarily require from the students since they are also intended to serve an instructional purpose

[No need to define $\models^B$.]

(i) Let $\langle W, E, V\rangle$ be a B-frame, so E is symmetric, and let $i \in W$, $i \models \theta$. Suppose that $\langle i,j\rangle \in E$, so $\langle j,i\rangle \in E$ by symmetry and $j \models \Diamond\theta$. Now suppose that $\langle j,k\rangle \in E$, so by symmetry $\langle k,j\rangle \in E$ and $k \models \Diamond\Diamond\theta$. Since k was an arbitrary element of W such that $\langle j,k\rangle \in E$, $j \models \Box\Diamond\Diamond\theta$. Similarly since j was an arbitrary element of W such that $\langle i,j\rangle \in E$, $i \models \Box\Box\Diamond\Diamond\theta$, as required.

(ii) Let $\langle W, E, V\rangle$ be the symmetric frame



Then $1 \models \Diamond p$ so $0 \models \Diamond\Diamond p$ and $1 \models \Diamond\Diamond p$. Hence $1 \models \Box\Diamond\Diamond p$ so $0 \models \Box\Box\Diamond\Diamond p$. But $0 \not\models p$ so $\Box\Box\Diamond\Diamond p \not\models^B p$. By the Completeness Theorem for B then, $\Box\Box\Diamond\Diamond p \nvdash^B p$.

A4. A *proof* in T is a sequence of sequents $\Gamma_1|\theta_1, \Gamma_2|\theta_2, \ldots, \Gamma_m|\theta_m$, where the Γ_i are finite subsets of SML and the $\theta_i \in SML$, such that for each $i = 1, 2, \ldots, m$, either $\Gamma_i|\theta_i$ is an instance of an axiom of T or $\exists j_1, j_2, \ldots, j_s < i$ such that

$$\frac{\Gamma_{j_1}|\theta_{j_1}, \Gamma_{j_2}|\theta_{j_2}, \ldots, \Gamma_{j_s}|\theta_{j_s}}{\Gamma_i|\theta_i}$$

is an instance of a rule of proof of T.

$\Gamma \vdash^T \theta$ if $\exists$ a proof in T, $\Gamma_1|\theta_1, \Gamma_2|\theta_2, \ldots, \Gamma_m|\theta_m$, such that $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$.

Proof in T of $\vdash^T \Box(\Box\theta \to \theta)$:

1	$\Box\theta$	θ	T axiom
2		$(\Box\theta \to \theta)$	IMR, 1
3		$\Box(\Box\theta \to \theta)$	NEC, 2

A5. (i) Suppose that $\models^{Ł} \theta \to \phi$, Ⓢ, and let w be a $[0,1]$-valuation such that $w(\theta) = 1$. Then from Ⓢ, $w(\theta \to \phi) = 1$ so $w(\theta) \le w(\phi)$, giving that also $w(\phi) = 1$. Hence $\theta \models^{Ł} \phi$.

(ii) Let w be a $[0,1]$-valuation. If $w(\theta) \le w(\phi)$ then $w(\theta \to \phi) = 1$. If $w(\theta) > w(\phi)$ then

$$w(\neg\theta) = 1 - w(\theta) < 1 - w(\phi) = w(\neg\phi)$$

so $w(\neg\theta \to \neg\phi) = 1$. Either way then

$$1 = \max\{w(\theta \to \phi), w(\neg\theta \to \neg\phi)\} = w((\theta \to \phi)\underline{\vee}(\neg\theta \to \neg\phi)).$$

Hence $\models^{Ł} (\theta \to \phi)\underline{\vee}(\neg\theta \to \neg\phi)$.

(iii) Assume that

$$\vdash^{Ł} \theta \to \phi \;\; (\star), \qquad \vdash^{Ł} \phi \to \psi \;\; (\dagger).$$

From Ł2 (and Proposition 8, which as usual we assume throughout) we have

$$\vdash^{Ł} (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi)).$$

Modus Ponens with $(\star)$ now gives

$$\vdash^{\text{Ł}} (\phi \to \psi) \to (\theta \to \psi)$$

and now Modus Ponens with $(\dagger)$ gives as required that $\vdash^{\text{Ł}} \theta \to \psi$.

A6. Writing, as usual, just $x \wedge y$ for $F_\wedge(x, y)$:

(C1) $0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1$,
(C2) $\wedge$ is continuous,
(C3) $\wedge$ is increasing (not necessarily strictly) in each coordinate,
(C4) $\wedge$ is associative.

Mostert-Shields Theorem (for $F_\wedge$): Let $\wedge$ satisfy (C1-4) and let $A = \{\, x \in [0, 1] \,|\, x \wedge x = x \,\}$. Then for $x \in A$ and $0 \le z \le x \le y \le 1$,

$$z \wedge y = y \wedge z = z = \min\{y, z\},$$

and if $a < b$, $a, b \in A$ and $(a, b) \cap A = \emptyset$ then on $[a, b]$ either

$$\langle [a, b], \wedge, < \rangle \cong \langle [0, 1], \times, < \rangle \quad \text{or} \quad \langle [a, b], \wedge, < \rangle \cong \langle [0, 1], \max\{0, x + y - 1\}, < \rangle.$$

Let $0 \le x \le 1$. Since $0 = 0 \wedge 1 \le x \le 1 \wedge 1 = 1$ and the function $t \mapsto t \wedge 1$ is continuous by (C2) the IVT gives that there is some $0 \le t \le 1$ such that $x = t \wedge 1$. Hence

$$x = t \wedge 1 = t \wedge (1 \wedge 1) \text{ (by (C1))} \ = (t \wedge 1) \wedge 1 \text{ (by (C4))} \ = x \wedge 1.$$

For the last part, since $x \wedge x < x$ for $0 < x < 1$ the set A in the Mostert-Shields Theorem is just $\{0, 1\}$. Hence $\langle [0, 1], \wedge, < \rangle$ is isomorphic to one of $\langle [0, 1], \max\{x + y - 1\}, 0\}, < \rangle$, $\langle [0, 1], \times, < \rangle$. It cannot be the former since $x \wedge x > 0$ for $x > 0$, so it must be isomorphic to $\langle [0, 1], \times, < \rangle$.

B7. By the Representation Theorem it is enough to show $\theta \mid\!\sim_{\vec{s}} \phi \vee \psi$ that under the assumption that

$$\theta \wedge \neg\phi \mid\!\sim_{\vec{s}} \phi \vee \psi \qquad (\star)$$

If $s_i \cap S_\theta = \emptyset$ for all i then $\theta \mid\!\sim_{\vec{s}} \phi \vee \psi$ holds directly from the definition. So suppose that $s_i \cap S_\theta \neq \emptyset$ for some i and let k be the least such. Let $\alpha \in s_k \cap S_\theta$.

If $\alpha \notin S_{\neg\phi}$ then

$$\alpha \in (At^L - S_{\neg\phi}) = S_{\neg\neg\phi} = S_\phi \subseteq S_\phi \cup S_\psi = S_{\phi \vee \psi} \quad (\dagger).$$

If $\alpha \in S_{\neg\phi}$ then $\alpha \in S_\theta \cap S_{\neg\phi} = S_{\theta \wedge \neg\phi}$ so $s_k \cap S_{\theta \wedge \neg\phi} \neq \emptyset$. Indeed k must be the least such since if $j < k$ and $s_j \cap S_{\theta \wedge \neg\phi} \neq \emptyset$ then $s_j \cap S_\theta \neq \emptyset$ (since $S_{\theta \wedge \neg\phi} = S_\theta \cap S_{\neg\phi} \subseteq S_\theta$) contradicting the choice of k.

From $(\star)$ then $\alpha \in S_{\phi \vee \psi}$, again. With $(\dagger)$ this gives that $s_k \cap S_\theta \subseteq S_{\phi \vee \psi}$ so, as required, $\theta \mid\!\sim_{\vec{s}} \phi \vee \psi$.

Letting $L = \{p, q, r\}$, $\theta = p$, $\phi = q$, $\psi = r$ and

$$\vec{s} = s_1, s_2 = \{p \wedge q \wedge r\},\ \{p \wedge \neg q \wedge \neg r\}$$

we have that $\theta \mathrel{|\!\sim_{\vec{s}}} \phi \vee \psi$ (i.e. $p \mathrel{|\!\sim_{\vec{s}}} q \vee r$) but not $\theta \wedge \neg\phi \mathrel{|\!\sim_{\vec{s}}} \phi \vee \psi$ (i.e. not $p \wedge \neg q \mathrel{|\!\sim_{\vec{s}}} q \vee r$) so this 'rule' fails for $|\!\sim_{\vec{s}}$, and by the Representation Theorem $|\!\sim_{\vec{s}}$ is a rational consequence relation.

B8. To show the required Completeness Theorem for $\mathcal{R}$ there are two 'new' features that we have to check. Firstly we need to show that this additional axiom

$$| \Diamond(\theta \vee \neg\theta) \qquad (R)$$

is valid in serial frames, meaning that if $\langle W, E, V\rangle$ is a serial frame and $i \in W$ then $i \models \Diamond(\theta \vee \neg\theta)$. But since this frame is serial there is a $j \in W$ such that $\langle i, j\rangle \in E$. Clearly $j \models \theta \vee \neg\theta$ (since either $j \models \theta$ or $j \not\models \theta$ (– in which case $j \models \neg\theta$), so $i \models \Diamond(\theta \vee \neg\theta)$, as required.

Secondly we need to show that if Γ is a maximally $\mathcal{R}$-consistent frame then there is a maximally $\mathcal{R}$-consistent frame Δ such that

$$\{\, \phi \mid \Box\phi \in \Gamma\} \subseteq \Delta,$$

equivalently that $\{\, \phi \mid \Box\phi \in \Gamma\}$ is $\mathcal{R}$-consistent. Suppose not, say $\Box\phi_1, \Box\phi_2, \ldots, \Box\phi_m \in \Gamma$ and for some θ,

$$\phi_1, \phi_2, \ldots, \phi_m \vdash^{\mathcal{R}} \theta \wedge \neg\theta.$$

Then since $K \subseteq \mathcal{R}$,

$$\Box\phi_1, \Box\phi_2, \ldots, \Box\phi_m \vdash^{\mathcal{R}} \Box(\theta \wedge \neg\theta),$$

so

$$\Gamma \vdash^{\mathcal{R}} \Box(\theta \wedge \neg\theta) \qquad (\dagger)$$

since the $\Box\phi_i \in \Gamma$. But directly from (R),

$$\Gamma \vdash^{\mathcal{R}} \Diamond(\theta \vee \neg\theta) \qquad (\star).$$

Since

$$\begin{aligned} &\vdash^{SC} \neg(p \vee \neg p) \leftrightarrow (p \wedge \neg p),\\ \text{so} \quad &\vdash^{K} \neg(\theta \vee \neg\theta) \leftrightarrow (\theta \wedge \neg\theta),\\ \text{and} \quad &\vdash^{K} \neg\Box\neg(\theta \vee \neg\theta) \leftrightarrow \neg\Box(\theta \wedge \neg\theta). \end{aligned}$$

Hence with $(\star)$,

$$\Gamma \vdash^{\mathcal{R}} \neg\Box(\theta \wedge \neg\theta),$$

which with $(\dagger)$ shows Γ to be $\mathcal{R}$-inconsistent, and so yields the required contradiction.

Finally, combining this with the (identical!) Completeness Theorem for D,

$$\begin{aligned} \Gamma \vdash^{\mathcal{R}} \theta \quad &\Longleftrightarrow \quad \text{for all serial frames } \langle W, E, V\rangle \text{ and}\\ &\qquad\ i \in W, \text{ if } i \models \Gamma \text{ then } i \models \theta\\ &\Longleftrightarrow \quad \Gamma \vdash^{D} \theta. \end{aligned}$$

B9. *McNaughton's Theorem(for $L = \{p\}$)*: A function $F : [0,1] \longrightarrow [0,1]$ is of the form F_θ for some $\theta \in SL$, with $L = \{p\}$, iff there exist some $0 = \gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_{n-1} < \gamma_n = 1$ and $n_i, m_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n-1$, such that on each $[\gamma_i, \gamma_{i+1}]$ $F(x) = m_i + n_i x$ $(\in [0,1])$. [Recall that $F_\theta(x) = w(x)$ for that $[0,1]$-valuation w of L for which $w(p) = x$.]

Let $\theta \in SL$ and let the γ_i, n_i, m_i, $i = 0, 1, \ldots, m$ be as in McNaughton's Theorem for θ. On any one of the (finitely many) intervals $[\gamma_i, \gamma_{i+1}]$ either $n_i = 0$, in which case $F_\theta(x) = m_i \in \mathbb{Z}$ so never takes value $1/3$, or $n_i \neq 0$ in which case $F_\theta(x)$ is strictly monotone, and so can take the value $1/3$ at most once. Hence there are at most finitely many $x \in [0,1]$ for which $F_\theta(x) = 1/3$, equivalently only finitely many $[0,1]$-valuations w on L for which $w(\theta) = 1/3$.

$F_\theta(1) = w(\theta)$ where $w(p) = 1$. In this case $w(\phi) = V(\phi)$ for all $\phi \in SL$, where V is the Sentential Calculus $\{0,1\}$-valuation on L giving p value 1, so $F_\theta(1) = w(\theta) = V(\theta) \in \{0,1\}$. Similarly $F_\theta(0) \in \{0,1\}$. Hence if $F_\theta(0) \neq F_\theta(1)$ F_θ must take both the values $0, 1$, and so, being continuous, must take value $1/3$ by the IVT. So if F_θ does not take the value $1/3$ then either $F_\theta(0) = F_\theta(1) = 1$ or $F_\theta(0) = F_\theta(1) = 0$. In the first case both of the two valuations V on SL must give θ value 1, so $\models^{SC} \theta$. In the second case both valuations give θ value 0, so give $\neg\theta$ value 1, so $\models^{SC} \neg\theta$.