MATH43032/63032, 2010-11, Solutions[1]

A1. $\mid\!\sim_{\vec{s}}$ is defined as follows: For $\theta, \phi \in SL$, $\theta \mid\!\sim_{\vec{s}} \phi$ if for each $i \in \{1, ..., m\}$ we have $S_\theta \cap s_i = \emptyset$, or the least index $i \in \{1, ..., m\}$ with $S_\theta \cap s_i \neq \emptyset$ satisfies $S_\theta \cap s_i \subseteq S_\phi$. Here S_θ denotes the set At^L of all atoms of L with $\alpha \models \theta$.

The Representation Theorem for rcr's says that a binary relation $\mid\!\sim$ on SL is a rcr if and only if $\mid\!\sim = \mid\!\sim_{\vec{s}}$ for some sequence $\vec{s} = s_1, ..., s_m$ with $s_i \subseteq At(L)$.

(i) is false. (ii) and (iii) are true.

A2. (a) In this case we can take $L = \{ p, q \}$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to be $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$, $\neg p \wedge \neg q$ respectively. Then

$$S_{\neg\neg(p\wedge q)\vee(p\vee q)} = S_{p\vee q} = \{\alpha_1, \alpha_2, \alpha_3\}, \qquad S_{\neg\neg q\vee\neg p} = S_{q\vee\neg p} = \{\alpha_1, \alpha_3, \alpha_4\},$$
$$S_{\neg(p\wedge q)\wedge(p\vee q)} = S_{(p\wedge\neg q)\vee(\neg p\wedge q)} = \{\alpha_2, \alpha_3\}, \quad S_{\neg q\wedge\neg p} = \{\alpha_4\}.$$

Running the Z-algorithm we get:

$A_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}$,
$K_0 = K = \{ \neg(p \wedge q) \mid\!\sim p \vee q, \neg q \mid\!\sim \neg p \}$,
$u_1 = A_0 \cap S_{\neg\neg(p\wedge q)\vee(p\vee q)} \cap S_{\neg\neg q\vee\neg p} = \{\alpha_1, \alpha_2, \alpha_3\} \cap \{\alpha_1, \alpha_3, \alpha_4\} = \{\alpha_1, \alpha_3\}$.

$A_1 = A_0 - u_1 = \{ \alpha_2, \alpha_4 \}$,
$K_1 = \{ \neg q \mid\!\sim \neg p \}$, since $S_{\neg(p\wedge q)\wedge(p\vee q)} \cap u_1 = \{\alpha_2, \alpha_3\} \cap \{\alpha_1, \alpha_3\} = \{\alpha_3\} \neq \emptyset$ and $S_{\neg q\wedge\neg p} \cap u_1 = \{\alpha_4\} \cap \{ \alpha_1, \alpha_3 \} = \emptyset$.
$u_2 = A_1 \cap S_{\neg\neg q\vee\neg p} = \{ \alpha_2, \alpha_4 \} \cap \{ \alpha_1, \alpha_3, \alpha_4 \} = \{ \alpha_4 \}$.

$A_2 = A_1 - u_2 = \{ \alpha_2 \}$,
$K_2 = \emptyset$, since $u_2 \cap S_{\neg q\wedge\neg p} = \{ \alpha_4 \} \cap \{\alpha_4\} = \{\alpha_4\} \neq \emptyset$,
$u_3 = A_2 \cap \bigcap_{\theta \mid\sim \phi \in K_2} S_{\neg\theta\vee\phi} = A_2 \cap At^L = \{\alpha_2\}$.

Since all the atoms have now been used up we must have $u_4 = \emptyset$ so the rational closure of K is $\mid\!\sim_{\vec{u}}$ where

$$\vec{u} = u_1, u_2, u_3 = \{ \alpha_1, \alpha_3 \}, \{\alpha_4\}, \{\alpha_2\}.$$

(b) From REF we have $\phi \mid\!\sim \phi$ so with $\theta \mid\!\sim \phi$ and DIS we obtain $\theta \vee \phi \mid\!\sim \phi$. With $\theta \vee \phi \mid\!\sim \theta$ and CMO we get $(\theta \vee \phi) \wedge \phi \mid\!\sim \theta$. But $((\theta \vee \phi) \wedge \phi) \equiv \phi$ so by LLE we obtain $\phi \mid\!\sim \theta$ as required.

A3. For $\Gamma \subseteq SML$ and $\theta \in SML$, $\Gamma \models^K \theta$ means that for every frame $\langle W, E, V \rangle$ and every world $i \in W$, if $i \models \gamma$ for all $\gamma \in \Gamma$ then $i \models \theta$.

[1] These solutions are more detailed than I would expect in the exam. That's because I want them to also serve an educational purpose when given with 'last year's paper' next year(!)

(i). Let $\langle W, E, V \rangle$ be a frame and let $i \in W$ with $i \models \Box(\theta \vee \phi)$. Suppose $i \not\models \Box\theta$. By definition, there is some $j \in W$ with $\langle i, j \rangle \in E$ such that $j \not\models \theta$. Since $i \models \Box(\theta \vee \phi)$, we have $j \models (\theta \vee \phi)$, thus $j \models \phi$. By definition of $i \models \Diamond\phi$, this gives $i \models \Diamond\phi$. Hence, either way, $i \models \Box\theta \vee \Diamond\phi$, as required.

(ii) $\langle W, E, V \rangle$ be the frame $W = \{1, 2\}$, $E = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle\}$ and let $V_1(p) = 1 = V_1(q)$, $V_2(p) = 0 = V_2(q)$. Then $1 \models \Diamond q$, since $\langle 1, 1 \rangle \in E$ and $1 \models q$, so $1 \models \Box p \vee \Diamond q$. However $\langle 1, 2 \rangle \in E$ and $2 \not\models p \vee q$ so $1 \not\models \Box(p \vee q)$ Thus $\Box p \vee \Diamond q \not\models^K \Box(p \vee q)$.

A4. A *proof* in D is a finite sequence of sequents

$$\Gamma_1 |\, \theta_1, \ldots, \Gamma_m |\, \theta_m,$$

where $\theta_i \in SML$ and $\Gamma_i \subseteq SML$ are finite, such that for each $i = 1, \ldots, m$, either $\Gamma_i |\, \theta_i$ is an instance of an axiom of D, or for some $j_1, \ldots, j_s < i$

$$\frac{\Gamma_{j_1} |\, \theta_{j_1} \quad \ldots \quad \Gamma_{j_s} |\, \theta_{j_s}}{\Gamma_i |\, \theta_i}$$

is an instance of one of the rules NEC or AND-MON from SC.

$\Gamma \vdash^D \theta \iff$ there is a proof $\Gamma_1 |\, \theta_1, \ldots, \Gamma_m |\, \theta_m$ in D such that $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$.

Formal proof of $\vdash^D \Box(\neg\Diamond\theta \rightarrow \neg\Box\theta)$:

1. $\Box\theta \,|\, \Diamond\theta$ D axiom
2. $\neg\Diamond\theta \,|\, \neg\Diamond\theta$ REF
3. $\Box\theta, \neg\Diamond\theta \,|\, \Diamond\theta$ MON 1
4. $\Box\theta, \neg\Diamond\theta \,|\, \neg\Diamond\theta$ MON 2
5. $\neg\Diamond\theta \,|\, \neg\Box\theta$ NIN 3,4
6. $|\, \neg\Diamond\theta \rightarrow \neg\Box\theta$ IMR 5
7. $|\, \Box(\neg\Diamond\theta \rightarrow \neg\Box\theta)$ NEC 6

A5. McNaughton's Theorem for $L = \{p\}$ says: A function $F : [0, 1] \longrightarrow [0, 1]$ is of the form F_θ for some $\theta \in SL$ iff there exist some $0 = \gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_{n-1} < \gamma_n = 1$ such that on each $[\gamma_{i-1}, \gamma_i]$, $F(x) = m_0 + m_1 x$ for some $m_0, m_1 \in \mathbb{Z}$.

Now assume $w(p) = \frac{1}{3}$. Let $S := \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. The functions $F_\rightarrow(x, y) = \min\{1, 1 - x + y\}$ and $F_\neg(x) = 1 - x$ of Łukasiewicz logic, restricted to the set S obviously have again values in S. Since F_θ is a composition of $F_\rightarrow$ and $F_\neg$, $F_\theta(S)$ also is contained in S. Since $w(\theta) = F_\theta(w(p))$ and $w(p) \in S$ we get $w(\theta) \in S$ as desired.

Alternatively, by McNaughton's theorem there are some $\gamma_i \leq 1/3 \leq \gamma_{i+1}$ and $m_0, m_1 \in \mathbb{Z}$ such that on $F_\theta(x) = m_0 + m_1 x \in [0,1]$ on $[\gamma_i, \gamma_{i+1}]$. Hence

$$w(\theta) = F_\theta(1/3) = m_0 + m_1/3 = (3m_0 + m_1)/3 \in \{0, 1/3, 2/3, 1\}$$

since $3m_0 + m_1 \in \mathbb{Z}$ and $(3m_0 + m_1)/3 \in [0,1]$.

A6. The desirable properties C1-C4 for a function $F_\wedge : [0,1]^2 \to [0,1]$ are the following:

(C1) $F_\wedge(0,1) = F_\wedge(1,0) = 0,\ F_\wedge(1,1) = 1$,
(C2) $F_\wedge$ is continuous,
(C3) $F_\wedge$ is increasing (not necessarily strictly) in each coordinate,
(C4) $F_\wedge$ is associative, i.e. $F_\wedge(x, F_\wedge(y,z)) = F_\wedge(F_\wedge(x,y), z)$ for $x, y, z \in [0,1]$.

Mostert-Shields Theorem: Let $F_\wedge$ satisfy (C1-4) and let $A = \{\, x \in [0,1] \mid F_\wedge(x,x) = x \,\}$. Then for $x \in A$ and $0 \leq z \leq x \leq y \leq 1$,

$$F_\wedge(z,y) = F_\wedge(y,z) = z = \min\{y, z\},$$

and if $a < b$, $a, b \in A$ and $(a,b) \cap A = \emptyset$ then on $[a,b]$ either

$$\langle [a,b], F_\wedge, < \rangle \cong \langle [0,1], \times, < \rangle$$

or

$$\langle [a,b], F_\wedge, < \rangle \cong \langle [0,1], \max\{0, x+y-1\}, < \rangle^2,$$

with the latter of these holding just if $F_\wedge(c,c) = a$ for some $a < c < b$.

Firstly if

$$G(x,x) = \tfrac{1}{2}\max\{x^2 + 2x - 1, 0\} = x$$

then either $x = 0$ or $x^2 + 2x - 1 = 2x \geq 0$ in which case $x^2 = 1$ and $x = 1$. Hence in the Mostert-Shields Theorem $A = \{0,1\}$ and $\langle [0,1], G, < \rangle$ is either isomorphic to $\langle [0,1], \max\{0, x+y-1\}, < \rangle$ or to $\langle [0,1], F_\wedge, < \rangle \cong \langle [0,1], \times, < \rangle$. It cannot be the later of these since in there we have $x \times x > 0$ whenever $x > 0$ whilst, for example, $G(\frac{1}{4}, \frac{1}{4}) = 0$. Hence

$$\langle [0,1], G, < \rangle \cong \langle [0,1], \max\{0, x+y-1\}, < \rangle$$

B7. In order to show

$$\frac{\mid\!\sim \theta \qquad \neg\phi \mid\!\sim \neg\theta}{\mid\!\sim \phi}$$

[2] In other words, on $[a,b]$ $F_\wedge$ either looks like a copy of the $F_\wedge$ of Product Logic, $\mathbb{F}^2$ (on [0,1]) or looks like a copy of the $F_\wedge$ of Łukasiewicz Logic, $\mathbb{F}^3$.

we use the Representation Theorem for rational consequence relations, so we may assume that $\mid\!\sim\, = \,\mid\!\sim_{\vec{s}}$, with $\vec{s} = s_1, ..., s_m$. Assume that $\mid\!\sim_{\vec{s}} \theta$, $\neg\phi \mid\!\sim_{\vec{s}} \neg\theta$. To confirm the rule we have to show that $\mid\!\sim_{\vec{s}} \phi$. If each $s_i = \emptyset$ then for η a tautology $s_i \cap \eta = \emptyset$ for each i so $\eta \mid\!\sim \phi$ as required. Otherwise, without loss of generality we may assume that $s_1 \neq \emptyset$. Then for any tautology η, $s_1 \cap S_\eta = s_1 \cap At^L = s_1 \neq \emptyset$ so, since $\mid\!\sim_{\vec{s}} \theta$, $s_1 \subseteq S_\theta$. We have to show $\mid\!\sim \phi$, in other words $s_1 \subseteq S_\phi$. If $s_1 \not\subseteq S_\phi$, then $s_1 \cap S_{\neg\phi} \neq \emptyset$, hence $\neg\phi \mid\!\sim \neg\theta$ implies $s_1 \cap S_{\neg\theta} \neq \emptyset$. As $S_{\neg\theta}$ is the complement of S_θ, this contradicts $s_1 \subseteq S_\theta$ and the result follows.

In order to show that the rule

$$\frac{\psi \mid\!\sim \theta \quad \neg\phi \mid\!\sim \neg\theta}{\psi \mid\!\sim \phi}$$

can not be derived from the Gabbay-Makinson conditions we provide a rational consequence relation that fails to satisfy it: Let $L = \{p, q, r\}$, $s_1 = \{\neg p \wedge \neg q \wedge \neg r\}$, $s_2 = \{\neg p \wedge q \wedge r\}$, $\vec{s} = s_1, s_2$. Then $r \mid\!\sim_{\vec{s}} q$ and $\neg p \mid\!\sim_{\vec{s}} \neg q$ but $r \not\mid\!\sim_{\vec{s}} p$ so the 'rule'

$$\frac{\psi \mid\!\sim \theta \quad \neg\phi \mid\!\sim \neg\theta}{\psi \mid\!\sim \phi}$$

fails for $\mid\!\sim_{\vec{s}}$ with $\theta = q,\ \phi = p,\ \psi = r$.

B8. $\Rightarrow$: We need to show that if $\langle W, E, V\rangle$ is a thin frame, $i \in W$ and $i \models \Diamond\theta$ then $i \models \Box\theta$. So assume that $i \in W$ and $i \models \Diamond\theta$. Then there must be some $j \in W$ such that $j \models \theta$ and $\langle i, j\rangle \in E$. Let $\langle i, k\rangle \in E$. Then by thinness $j = k$, so $k \models \theta$. Hence by definition $i \models \Box\theta$, as required.

$\Leftarrow$: Let $\Delta, \Gamma, \Omega \subseteq SML$ be maximally consistent in H and

$$\{\theta \mid \Box\theta \in \Delta\} \subseteq \Gamma, \Omega.$$

We must show that $\Gamma \subseteq \Omega$, since then by symmetry $\Omega \subseteq \Gamma$, so $\Gamma = \Omega$ and the 'canonical frame' is thin. So suppose that $\theta \in \Gamma$. If $\Box\theta \in \Delta$ then $\theta \in \Omega$, as required. So suppose that $\Box\theta \notin \Delta$. Then $\neg\Box\theta \in \Delta$ (because Δ is maximal) so $\Diamond\neg\theta \in \Delta$ since Δ is closed under proofs in H and $\neg\Box\theta \vdash^K \Diamond\neg\theta$. $\therefore$ by the axiom of H, $\Box\neg\theta \in \Delta$ and $\neg\theta \in \Gamma$. But then Γ must be inconsistent since also $\theta \in \Gamma$, contradiction. We conclude that the case $\Box\theta \notin \Delta$ is impossible and the required result follows.

Let $\langle W, E, V\rangle$ be the thin frame with $W = \{0\}$, $E = \emptyset$, $V_0(p) = 1$. Then $0 \models \Box p$ (trivially since there are no $j \in W$ such that $\langle 0, j\rangle \in E$) and similarly $0 \not\models \Diamond p$. Hence $\Box p \not\models^H \Diamond P$ so by the first part $\Box p \nvdash^H \Diamond p$.

B9.

(a) Let w be a $[0,1]$-valuation. If $w(q) \leq w(p)$ then $w(q \to p) = 1$ so $w((p \to q) \to p) \leq 1 = w(q \to p)$ so $w(((p \to q) \to p) \to (q \to p)) = 1$.

If $w(q) > w(p)$ then

$$w(q \to p) = \min\{1, 1 - w(q) + w(p)\} = 1 - w(q) + w(p)$$

and $w(p \to q) = \min\{1, 1 - w(p) + w(q)\} = 1$ so

$$w((p \to q) \to p) = \min\{1, 1 - w(p \to q) + w(p)\} = w(p).$$

Hence again

$$\begin{aligned} w(((p \to q) \to p) \to (q \to p)) &= \min\{1, 1 - w((p \to q) \to p) + w(q \to p)\} \\ &= \min\{1, 1 - w(p) + (1 - w(q) + w(p))\} \\ &= \min\{1, 2 - w(q)\} = 1 \quad \text{since } w(q) \leq 1, \end{aligned}$$

so $\models^{Ł} ((p \to q) \to p) \to (q \to p)$ follows.

(b) Let $w(p) = \frac{2}{3}$, $w(q) = \frac{1}{3}$. Then

$$w(p \to q) = \min\{1, 1 - \tfrac{2}{3} + \tfrac{1}{3}\} = \tfrac{2}{3}$$

so

$$w((p \to q) \to p) = \min\{1, 1 - \tfrac{2}{3} + \tfrac{2}{3}\} = 1$$

and since this is greater than $w(p) = \frac{2}{3}$ so $w((p \to q) \to p) \to p) < 1$ and $\not\models^{Ł} ((p \to q) \to p) \to p$ follows.

(c) A suitable proof of $\vdash^{Ł} ((p \to q) \to p) \to (q \to p)$ is:

1.	$\|q \to (p \to q)$	Ł1
2.	$\|(q \to (p \to q)) \to (((p \to q) \to p) \to (q \to p))$	Ł2
3.	$\|((p \to q) \to p) \to (q \to p)$	MP,1,2