MATH43032/63032 January 2010, Solutions

These solutions are intended also as a teaching aid for future cohorts. Consequently they often contain more detail than the students actually sitting the examination would be expected to give.

A1. *Representation Theorem for Monotone Consequence Relations:*
For every finite $\Delta \subseteq SL$, $\mid\!\sim_\Delta$ is a monotone consequence relation and every monotone consequence relation is of this form for some such Δ.

By the Representation Theorem let finite $\Delta \subseteq SL$ be such that $\mid\!\sim_0 = \mid\!\sim_\Delta$. Then

$$\begin{aligned}
\Gamma \mid\!\sim \phi &\iff \Gamma \mid\!\sim_0 \theta \to \phi \\
&\iff \Gamma \mid\!\sim_\Delta \theta \to \phi \\
&\iff \Gamma, \Delta \vdash (\theta \to \phi) \text{ by definition of } \mid\!\sim_\Delta \\
&\iff \Gamma, \Delta, \theta \vdash \phi \text{ by using IMR in the direction, } \Leftarrow \\
&\qquad \text{and } MP \text{ and } \Gamma, \Delta, \theta \vdash \theta \text{ in the } \Rightarrow \text{ direction} \\
&\iff \Gamma \mid\!\sim_{\Delta \cup \{\theta\}} \phi.
\end{aligned}$$

Hence since $\Delta \cup \{\theta\}$ is finite $\mid\!\sim_0$ is an mcr by the Representation Theorem.

A2

$$\theta \mid\!\sim_{\vec{s}} \phi \iff \begin{cases} \forall i \;\, s_i \cap S_\theta = \emptyset & \text{or} \\ \exists i \;\, s_i \cap S_\theta \neq \emptyset & \text{and for the least such } i,\ s_i \cap S_\theta \subseteq S_\phi. \end{cases}$$

(i) True, (ii) Not true (iii) True.

A3 In this case we can take $L = \{ p, q \}$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to be $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$, $\neg p \wedge \neg q$ respectively. Then

$$S_{\neg(p \to \neg q) \vee p} = S_p = \{\alpha_1, \alpha_2\}, \qquad S_{(p \to \neg q) \wedge p} = \{\alpha_2\},$$
$$S_{\neg\neg p \vee q} = S_{p \vee q} = \{\alpha_1, \alpha_2, \alpha_3\}, \qquad S_{\neg p \wedge q} = \{\alpha_3\}.$$

Running the Z-algorithm we get:

$A_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}$,
$K_0 = K = \{ (p \to \neg q) \mid\!\sim p,\ \neg p \mid\!\sim q \}$,
$u_1 = A_0 \cap S_{\neg(p \to \neg q) \vee p} \cap S_{\neg\neg p \vee q} = \{\alpha_1, \alpha_2\} \cap \{\alpha_1, \alpha_2, \alpha_3\} = \{\alpha_1, \alpha_2\}$.

$A_1 = A_0 - u_1 = \{ \alpha_3, \alpha_4 \}$,
$K_1 = \{ \neg p \mid\!\sim q \}$, since $S_{(p \to \neg q) \wedge p} \cap u_1 = \{\alpha_2\} \neq \emptyset$ and $S_{\neg p \wedge q} \cap u_1 = \{\alpha_3\} \cap \{ \alpha_1, \alpha_2 \} = \emptyset$.
$u_2 = A_1 \cap S_{\neg\neg p \vee q} = \{ \alpha_3, \alpha_4 \} \cap \{ \alpha_1, \alpha_2, \alpha_3 \} = \{ \alpha_3 \}$.

$A_2 = A_1 - u_2 = \{\, \alpha_4 \,\}$,
$K_2 = \emptyset$, since $u_2 \cap S_{\neg p \wedge q} = \{\, \alpha_3, \alpha_4 \,\} \cap \{\alpha_3\} = \{\alpha_3\} \neq \emptyset$,
$u_3 = A_4 \cap \bigcap_{\theta \mid\sim \phi \in K_2} S_{\neg\theta \vee \phi} = A_4 \cap At^L = \{\alpha_4\}$.

Since all the atoms have now been used up we must have $u_4 = \emptyset$ so the rational closure of K is $\mid\sim_{\vec{u}}$ where

$$\vec{u} = u_1, u_2, u_3 = \{\, \alpha_1, \alpha_2 \,\},\ \{\alpha_3\},\ \{\alpha_4\}.$$

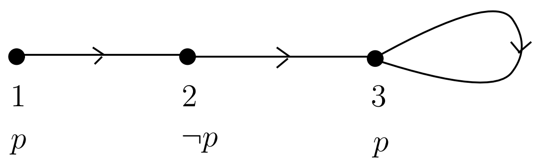
A4. $\Gamma \models^D \theta$ if for all serial frames $\langle W, E, V\rangle$ and $i \in W$, if $\langle W, E, V\rangle, i \models \Gamma$ then $\langle W, E, V\rangle, i \models \theta$.

Completeness Theorem for D: For $\Gamma \subseteq SML$ and $\theta \in SML$,

$$\Gamma \models^D \theta \iff \Gamma \vdash^D \theta.$$

(i) Let $\langle W, E, V\rangle$ be a serial frame and $i \in W$ such that $i \models \Box\Box\theta$. Since this frame is serial there are $j, k \in W$ such that $\langle i, j\rangle, \langle j, k\rangle \in E$. Since $i \models \Box\Box\theta$, $j \models \Box\theta$ and in turn $k \models \theta$. Hence $j \models \Diamond\theta$ and in turn $i \models \Diamond\Diamond\theta$. Hence $\Box\Box\theta \models^D \Diamond\Diamond\theta$.

(ii) Consider the frame:



This is a serial frame and $1 \models \Box\Box p$ but $1 \not\models \Diamond p$ so $\Box\Box p \not\models^D \Diamond p$ and by the Completeness Theorem for D, $\Box\Box p \nvdash^D \Diamond p$.

A formal proof in D of $\Box\Box\theta \vdash \Box\Diamond\theta$:

1. $\Box\theta \mid \Diamond\theta$ D axiom
2. $\mid (\Box\theta \to \Diamond\theta)$ IMR 1
3. $\mid \Box(\Box\theta \to \Diamond\theta)$ NEC 2
4. $\Box(\Box\theta \to \Diamond\theta) \| \Box\Box\theta \to \Box\Diamond\theta$ K axiom
5. $\mid \Box(\Box\theta \to \Diamond\theta) \to (\Box\Box\theta \to \Box\Diamond\theta)$ IMR 4
6. $\mid \Box\Box\theta \to \Box\Diamond\theta$ MP 3,5.

A5 (C1) $0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1$,
(C2) $\wedge$ is continuous,
(C3) $\wedge$ is increasing (not necessarily strictly) in each coordinate,
(C4) $\wedge$ is associative, i.e. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ for $x, y, z \in [0,1]$.

$F_\wedge(b,0) \leq F_\wedge(1,0) = 0$ by (C3),(C1) so $F_\wedge(b,0) = 0$. Hence if $F_\wedge(b,b) = b$, for $x \in [0,b]$, $F_\wedge(b,0) = 0 \leq x \leq b = F_\wedge(b,b)$. Therefore, since $F_\wedge$ is continuous by (C2), by the Intermediate Value Theorem $F_\wedge(b,t) = x$ for some $0 \leq t \leq b$. Hence by (C4),

$$F_\wedge(b,x) = F_\wedge(b, F_\wedge(b,t)) = F_\wedge(F_\wedge(b,b),t) = F_\wedge(b,t) = x,$$

as required.

In this case there cannot be $b \in (0,1)$ such that $F_\wedge(b,b) = b$, otherwise for any $x \in (0,b)$,

$$F_\wedge(b,x) = x \neq 0,$$

contradicting the assumption that for some such x, $F_\wedge(b,x) = 0$. Hence by the Mostert-Shield's Theorem $\langle [0,1], F_\wedge, < \rangle$ must be isomorphic to one of $\langle [0,1], \min\{x,y\}, < \rangle$, $\langle [0,1], \times, < \rangle$, $\langle [0,1], \max\{0, x+y-1\}, < \rangle$. But of these the only one with the given property that for all $y \in (0,1)$ there is an $x \in (0,y)$ such that $F_\wedge$ (for that structure) of $\langle y, x \rangle$ is zero is $\langle [0,1], \max\{0, x+y-1\}, < \rangle$. Hence $\langle [0,1], F_\wedge, < \rangle$ must be isomorphic to this (Łukasiewicz) structure.

A6 $\Gamma \models^{\text{Ł}} \phi$ if for all $[0,1]$-valuations V, if $V(\phi) = 1$ for all $\phi \in \Gamma$ then $V(\theta) = 1$.

Completeness Theorem for Ł: For finite $\Gamma \subseteq SL$ and $\theta \in SL$,

$$\Gamma \vdash^{\text{Ł}} \theta \iff \Gamma \models^{\text{Ł}} \theta.$$

(i) Let V be the $[0,1]$-valuation such that $V(p) = 1/2$, $V(q) = 0$. Then in Ł, $V(\neg p) = 1 - 1/2 = 1/2$ and

$$V(p \rightarrow q) = V(\neg p \rightarrow q) = min\{1, 1 - 1/2 + 0\} = 1/2$$

so

$$V((p \rightarrow q) \underline{\vee} (\neg p \rightarrow q)) = \max\{V(p \rightarrow q), V(\neg p \rightarrow q)\} = \max\{1/2, 1/2\} \neq 1.$$

Hence $\not\models^{\text{Ł}} (p \rightarrow q) \underline{\vee} (\neg p \rightarrow q)$ so by the Completeness Theorem $\not\vdash^{\text{Ł}} (p \rightarrow q) \underline{\vee} (\neg p \rightarrow q)$.

(ii) Let V be a $[0,1]$-valuation. Then

$$V(p \rightarrow q) = \min\{1, 1 - V(p) + V(q)\} \tag{1}$$

and

$$\begin{aligned} V(\neg p \to q) &= \min\{1, 1 - V(\neg p) + V(q)\} \\ &= \min\{1, 1 - (1 - V(p)) + V(q)\} \\ &= \min\{1, V(p) + V(q)\}. \end{aligned} \tag{2}$$

If

$$V((p \to q) \vee (\neg p \to q)) = \min\{1, V(p \to q) + V(\neg p \to q)) < 1 \tag{3}$$

then $V(p \to q), V(\neg p \to q) < 1$ so from (1) and (2),

$$V(p \to q) = 1 - V(p) + V(q), \quad V(\neg p \to q) = V(p) + V(q).$$

Hence

$$V(p \to q) + V(\neg p \to q) = 1 + 2V(q) \geq 1,$$

contradicting (3). Hence we must have

$$V((p \to q) \vee (\neg p \to q)) = 1$$

so

$$\models^{\text{Ł}} (p \to q) \vee (\neg p \to q).$$

B7. Let $\mid\!\sim$ be a rcr such that $\theta \mid\!\sim \phi$ and $\phi \wedge \neg\psi \mid\!\sim \psi$. By the Representation Theorem we may assume that $\mid\!\sim = \mid\!\sim_{\vec{s}}$ for some $\vec{s}$. We need to show that $\theta \mid\!\sim_{\vec{s}} \psi$.

If $s_i \cap S_\theta = \emptyset$ for all i then $\theta \mid\!\sim_{\vec{s}} \psi$, as required.

Otherwise let i be minimal such that $s_i \cap S_\theta \neq \emptyset$. Then

$$s_i \cap S_\theta \subseteq S_\phi \tag{4}$$

since by assumption $\theta \mid\!\sim_{\vec{s}} \phi$. Let $\alpha \in s_i \cap S_\theta$ and suppose that $\alpha \notin S_\psi$. Then $\alpha \in S_{\neg\psi}$ so with (4), $\alpha \in S_\phi \cap S_{\neg\psi} = S_{\phi\wedge\neg\psi}$. Hence there is a least j such that $s_j \cap S_{\phi\wedge\neg\psi} \neq \emptyset$ and since by assumption $\phi \wedge \neg\psi \mid\!\sim_{\vec{s}} \psi$,

$$s_j \cap S_{\phi\wedge\neg\psi} \subseteq S_\psi.$$

But since $S_{\phi\wedge\neg\psi} \subseteq S_{\neg\psi}$ this gives

$$\emptyset \neq s_j \cap S_{\phi\wedge\neg\psi} \subseteq S_\psi \cap S_{\neg\psi} = \emptyset,$$

contradiction. We conclude that $\alpha \in S_\psi$ and hence $s_i \cap S_\theta \subseteq S_\psi$, and $\theta \mid\!\sim_{\vec{s}} \psi$, as required.

Take $L = \{p, q, r\}$, $\theta = p$, $\phi = q$, $\psi = r$, $\vec{s} = s_1, s_2$ where $s_1 = \{\neg p \wedge q \wedge r\}$, $s_2 = \{p \wedge q \wedge \neg r\}$. Then $s_i \cap S_{p\wedge\neg q} = \emptyset$ for $i = 1, 2$ so $p \wedge \neg q \mid\!\sim_{\vec{s}} q$. Also $i = 1$

is minimal such that $s_i \cap S_q \neq \emptyset$ and $s_1 \cap S_q \subseteq S_r$ so $q \mathrel{|\!\sim_{\vec{s}}} r$. However $i = 2$ is minimal such that $s_i \cap S_p \neq \emptyset$ whilst $s_2 \cap S_p \nsubseteq S_r$ so $p \mathrel{|\!\not\sim_{\vec{s}}} r$. Hence the 'rule'

$$\frac{\theta \wedge \neg\phi \mathrel{|\!\sim} \phi, \quad \phi \mathrel{|\!\sim} \psi}{\theta \mathrel{|\!\sim} \psi}$$

fails for this rcr $\mathrel{|\!\sim_{\vec{s}}}$.

B8. Proof by induction on k such that $\theta \in SML_k$. If $k = 0$ then $\theta = p$ for some $p \in L$ so for any $i \in W$,

$$\langle W, E, V \rangle,\, i \models p \iff V_i(p) = 1 \iff V_i^+(p) = 1 \iff \langle W^+, E^+, V^+ \rangle,\, i \models p,$$

since V_i and V_i^+ agree on L.

Now assume the result for $\phi \in SML_k$ and let $\theta \in SML_{k+1} - SML_k$.
If $\theta = \neg\phi$, so $\phi \in SML_k$, then for any $i \in W$,

$$\begin{aligned} \langle W, E, V \rangle,\, i \vDash \theta &\iff \langle W, E, V \rangle,\, i \nvDash \phi \\ &\iff \langle W^+, E^+, V^+ \rangle,\, i \nvDash \phi \quad \text{by inductive hypothesis} \\ &\iff \langle W^+, E^+, V^+ \rangle,\, i \vDash \theta. \end{aligned}$$

The cases for the other connectives are exactly analogous. Finally suppose that $\theta = \Box\phi$, so again $\phi \in SML_k$. Then for $i \in W$,

$$\begin{aligned} \langle W, E, V \rangle,\, i \vDash \theta &\iff \langle W, E, V \rangle,\, i \vDash \Box\phi \\ &\iff \forall j \in W\, (\langle i, j \rangle \in E \Rightarrow \langle W, E, V \rangle,\, j \vDash \phi) \\ &\iff \forall j \in W^+\, (\langle i, j \rangle \in E^+ \Rightarrow \langle W^+, E^+, V^+ \rangle,\, j \vDash \phi) \\ &\qquad \text{by inductive hypothesis and since } W = W^+, E = E^+, \\ &\iff \langle W^+, E^+, V^+ \rangle,\, i \vDash \Box\phi \\ &\iff \langle W^+, E^+, V^+ \rangle,\, i \vDash \theta, \end{aligned}$$

as required.

For the second part suppose that $\vDash^K (q \to \theta)$ with $\theta \in SML$ but $\nvDash^K \theta$. Let $\langle W, E, V \rangle$ be a frame for L and $i \in W$ such that $i \nvDash \theta$, so $i \vDash \neg\theta$. Let $\langle W^+, E^+, V^+ \rangle$ be the frame for L^+ such that $W = W^+$, $E = E^+$, and for all $j \in W$, $V_j(p) = V_j^+(p)$ for $p \in L$ and $V_j^+(q) = 1$. Then

$$\langle W^+, E^+, V^+ \rangle,\, i \vDash q$$

and by the above,

$$\langle W^+, E^+, V^+ \rangle,\, i \vDash \neg\theta,$$

so

$$\langle W^+, E^+, V^+ \rangle,\, i \nvDash (q \to \theta),$$

contradicting $\models^K (q \to \theta)$. Hence $\models^K \theta$.

B9. *McNaughton's Theorem for* L_1: A function $F : [0,1] \longrightarrow [0,1]$ is of the form F_θ for some $\theta \in SL_1$ iff there exist some $0 = \gamma_1 < \gamma_2 < \gamma_3 < ... < \gamma_{n-1} < \gamma_n = 1$ such that on each $[\gamma_{i-1}, \gamma_i]$ $F(x) = m_0 + m_1 x$ ($\in [0,1]$) for some $m_0, m_1 \in \mathbb{Z}$.

Here $F_\theta(x) = V(\theta)$ for that $[0,1]$-valuation V such that $V(p) = x$.

Given $\theta \in SL_1$ with $V(\theta) = 1/2$ let $\gamma_1, \gamma_2, \dots, \gamma_n$ be as in McNaughton's Theorem for θ and let $V(p) = r$. Then for some $1 \leq i < n$, $\gamma_i \leq r \leq \gamma_{i+1}$. By the theorem

$$F_\theta(x) = m_0 + m_1 x \quad \text{ for } x \in [\gamma_i, \gamma_{i+1}]$$

for some integers m_0, m_1. Hence if $V(\theta) = F_\theta(r) = 1/2$, $m_0 + m_1 r = 1/2$, giving

$$r = \frac{1 - 2m_0}{2m_1}.$$

Since $0 \leq r \leq 1$ and $2m_1$ cannot exactly divide $1 - 2m_0 \neq 0$ it must be that $0 < |2m_1| < |1 - 2m_0|$, as required.

Conversely let $n \in \mathbb{N}$ be odd, $m \in \mathbb{N}$ be even with $0 < n < m$. We want a θ for which in some such interval $\gamma_i \leq n/m \leq \gamma_{i+1}$ we have $F_\theta(x) = r + sx$ with $r, s \in \mathbb{Z}$ and

$$r + s(n/m) = 1/2.$$

Since $m = 2m'$, $n = 2n' + 1$ this amounts to

$$1/2 = r + s\frac{(2n'+1)}{2m'} = r + \frac{sn'}{m'} + \frac{1}{2} \times \frac{s}{m'}.$$

Clearly an integer solution to this is $s = m'$, $r = -n'$. For this r, s we have $r + sx = 0$ when $x = n'/m'$ and $r + sx = 1$ when $x = (n'+1)/m'$. Also

$$0 \leq n'/m' < n/m < (n'+1)/m' \leq (n+1)/m \leq 1.$$

By McNaughton's Theorem there is a $\theta \in SL_1$ such that

$$F_\theta(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq n'/m', \\ r + sx & \text{for } n'/m' \leq x \leq (n'+1)/m', \\ 1 & \text{for } (n'+1)/m' \leq x \leq 1 \end{cases}$$

and this θ will give $F_\theta(n/m) = 1/2$.