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MATH43032/63032 Part 2

Modal Logic

A Motivating Example

Consider the following example of 'follows':

It is not possible that Bill and Monica are both telling the truth
Possibly Monica is telling the truth

$\therefore$ *It is not necessarily the case that Bill is telling the truth*

As with the previous sections on monotonic and nonmonotonic reasoning we want to formally capture in what sense this conclusion 'follow' from the premises, why we are justified in writing $\therefore$.

Clearly in this example the modalities (as they are called) 'possible' and 'necessary' matter. It is true that if we just ignored them then the conclusion would follow from the premises by the propositional (= sentential) calculus, however in doing this we would have changed the meaning of the statements: Saying 'Possibly Monica is telling the truth' is not the same as saying 'Monica is telling the truth'.

Nevertheless we might hope to somehow express them in terms of the connectives $\wedge, \vee, \neg, \rightarrow$ which we already have in the Propositional or Sentential Calculus, denoted SC in this part of the course. However that hope is easy dashed because clearly the *possibility* of Monica telling the truth isn't completely determined by whether or not she *actually* is telling the truth. In other words the truth value of *Possibly Monica is telling the truth* is not a function of the the truth value of *Monica is telling the truth*, that is 'possibility' is not *Truth Functional.* However the connectives of SC are truth functional, i.e. the truth values of $\theta \wedge \phi, \theta \vee \phi, \theta \rightarrow \phi, \neg\theta$ are completely determined by the truth values of θ and ϕ individually, and the same goes for any compound connective built up from these. So there is no way of understanding examples such as the above purely within SC, we need to enlarge our language, proof theory and semantics.

Syntax of Modal Logic

A *language* L for modal logic is just the same as for SC, hence a non-empty set of propositional variables.
From L we define the set SML of *modal sentences* of L using the *connectives* $\wedge$, $\vee$, $\rightarrow$, $\neg$ and the *modality* $\Box$ (read as 'necessarily' or 'box') and parentheses,(,) as follows:

M1 $\quad L \subseteq SML$

M2 $\quad$ If $\theta, \phi \in SML$ then $\neg\theta, (\theta \wedge \phi), (\theta \vee \phi), (\theta \rightarrow \phi),\ \Box\theta \in SML$

M3 $\quad \theta \in SML$ just if this can be derived on the basis of M1,M2.

As usual we define the length of $\theta \in SML$, $|\theta|$, as the number of symbols in θ (excluding commas) and can use proof by induction on the length of θ to show that some property holds for all $\theta \in SML$.

Notice that $SL \subseteq SML$. Again we have unique readability since the case of $\Box\theta$ is the same as for $\neg\theta$ in SC.

Abbreviation For $\theta \in SML$ we write

$$\Diamond\theta = \neg\Box\neg\theta \text{ (read "possibly } \theta\text{")}$$

Unlike the propositional calculus, for Modal Logic(s) it seems most natural to consider analyzing the "follows" relation initially by developing a proof theory and leave the semantics until later.

Proof systems of modal logic

Unlike with the Propositional Calculus, many systems of Modal Logic are studied. All have the REF-axiom[1] $\theta \,|\, \theta$ for $\theta \in SML$, and the rules of SC applied to sentences and subsets of SML, viz: For $\Gamma \subseteq SML$ finite etc.

And right:	$\dfrac{\Gamma \,|\, \theta \qquad \Delta \,|\, \phi}{\Gamma, \Delta \,|\, \theta \wedge \phi}$	AND
And left:	$\dfrac{\Gamma, \theta, \phi \,|\, \psi}{\Gamma, \theta \wedge \phi \,|\, \psi}$	ANL
And out:	$\dfrac{\Gamma \,|\, \theta \wedge \phi}{\Gamma \,|\, \theta}$ and $\dfrac{\Gamma \,|\, \theta \wedge \phi}{\Gamma \,|\, \phi}$	AO
Or right:	$\dfrac{\Gamma \,|\, \theta}{\Gamma \,|\, \phi \vee \theta}$ and $\dfrac{\Gamma \,|\, \theta}{\Gamma \,|\, \theta \vee \phi}$	ORR
Disjunction, Or left:	$\dfrac{\Gamma, \theta \,|\, \psi \qquad \Delta, \phi \,|\, \psi}{\Gamma, \Delta, \theta \vee \phi \,|\, \psi}$	DIS
Implies right:	$\dfrac{\Gamma, \theta \,|\, \phi}{\Gamma \,|\, \theta \to \phi}$	IMR
Modus Ponens:	$\dfrac{\Gamma \,|\, \theta \qquad \Delta \,|\, \theta \to \phi}{\Gamma, \Delta \,|\, \phi}$	MP
Not in:	$\dfrac{\Gamma, \theta \,|\, \phi \qquad \Delta, \theta \,|\, \neg\phi}{\Gamma, \Delta \,|\, \neg\theta}$	NIN
Not not out:	$\dfrac{\Gamma \,|\, \neg\neg\theta}{\Gamma \,|\, \theta}$	NNO
Monotonicity, or Weakening:	$\dfrac{\Gamma \,|\, \theta}{\Gamma, \Delta \,|\, \theta}$	MON

So our modal logics extend SC. All our systems of modal logic we consider also have the *necessity rule*,

$$\frac{|\, \theta}{|\, \Box\theta} \qquad \text{NEC}$$

Notice that the left hand sides here are empty.

After this the different systems are characterized by the remaining axioms where:

[1]The left hand side here should really be the *set* $\{\theta\}$ rather than just the sentence θ but it is a common abbreviation to drop the braces in such cases. Similarly we may write $\Gamma, \theta \,|\, \phi$ for $\Gamma \cup \{\theta\} \,|\, \phi$.

$$\begin{array}{llll}
K: & \Box(\theta \to \phi) \mid (\Box\theta \to \Box\phi) & & \\
D: & K + \Box\theta \mid \Diamond\theta & T: & K + \Box\theta \mid \theta \\
B: & K + \theta \mid \Box\Diamond\theta & S_4: & T + \Box\theta \mid \Box\Box\theta \\
S_5: & T + \Diamond\theta \mid \Box\Diamond\theta & &
\end{array}$$

Definition. A *proof* (in K, T, ...) is a finite sequence of sequents

$$\Gamma_1 | \, \theta_1, \ldots, \Gamma_m | \, \theta_m,$$

where $\theta_i \in SML$ and $\Gamma_i \subseteq SML$ are finite, such that for each $i = 1, \ldots, m$, either $\Gamma_i | \, \theta_i$ is an instance of an axiom (of K, T,...) or for some $j_1, \ldots, j_s < i$

$$\frac{\Gamma_{j_1} | \, \theta_{j_1} \quad \ldots \quad \Gamma_{j_s} | \, \theta_{j_s}}{\Gamma_i | \, \theta_i}$$

is an instance of one of the rules NEC or AND-MON from SC.

The natural number m here is called the *length* of the proof.

Definition. Let J be one of the systems $K, B, \ldots, S_5$ (a convention we shall adopt throughout). For $\theta \in SML$ and $\Gamma \subseteq SML$, possibly infinite, we define

$$\Gamma \vdash^J \theta \iff \text{there is a proof } \Gamma_1 | \, \theta_1, \ldots, \Gamma_m | \, \theta_m \text{ in } J \text{ such that } \Gamma_m \subseteq \Gamma \text{ and } \theta_m = \theta.$$

In this case $\Gamma_1 | \, \theta_1, \ldots, \Gamma_m | \, \theta_m$ is called a *proof*[2] *of* θ *from* Γ *in* J.

Example.
A formal proof of $\theta \vdash^T \Diamond\theta$:

1. $\Box\neg\theta \mid \neg\theta$ by T
2. $\theta \mid \theta$ by REF
3. $\theta, \Box\neg\theta \mid \neg\theta$ by MON applied to 1.
4. $\theta, \Box\neg\theta \mid \theta$ by MON applied to 2.
5. $\theta \mid \neg\Box\neg\theta$ by NIN (Not in) applied to 3. and 4.

Another example
A formal proof of $\Box(\theta \wedge \phi) \vdash^K \Box\theta$:

1. $\theta \wedge \phi \mid \theta \wedge \phi$ by REF
2. $\theta \wedge \phi \mid \theta$ by AO from 1
3. $\mid (\theta \wedge \phi) \to \theta$ by IMR from 2
4. $\mid \Box((\theta \wedge \phi) \to \theta)$ by NEC from 3
5. $\Box((\theta \wedge \phi) \to \theta) \mid \Box(\theta \wedge \phi) \to \Box\theta$ by the K axiom
6. $\mid \Box((\theta \wedge \phi) \to \theta) \to (\Box(\theta \wedge \phi) \to \Box\theta)$ by IMR from 5
7. $\mid \Box(\theta \wedge \phi) \to \Box\theta$ by MP from 4,6
8. $\Box(\theta \wedge \phi) \mid \Box(\theta \wedge \phi) \to \Box\theta$ by MON from 7
9. $\Box(\theta \wedge \phi) \mid \Box(\theta \wedge \phi)$ by REF
10. $\Box(\theta \wedge \phi) \mid \Box\theta$ by MP from 8, 9

[2]We insert the adjective 'formal' when we want to distinguish between a proof in this sense and an 'informal proof' such as we would give to justify a theorem.

Proposition 1 *Let J be one of the systems $K, B, \ldots, S_5$. Let $\Gamma \subseteq SML$ and $\theta \in SML$.*

(i) if $\theta \in \Gamma$, then $\Gamma \vdash^J \theta$.

(ii) if $\phi \in SML$ and $\phi|\theta$ is an axiom of the system J with $\phi \in \Gamma$, then $\Gamma \vdash^J \theta$.

(iii) if

$$\frac{\Gamma_1 \,|\, \theta_1,\ \ \Gamma_2 \,|\, \theta_2,\ \ \ldots,\ \ \Gamma_k \,|\, \theta_k}{\Gamma \,|\, \theta}$$

is an instance of a rule of proof, except that possibly the Γ_i, Γ are allowed to be infinite, and $\Gamma_i \vdash^J \theta_i$ for $i = 1, 2, \ldots, k$ then $\Gamma \vdash^J \theta$.

Proof (i) follows from REF: $\theta \,|\, \theta$ is a proof of θ from Γ.

(ii). If $\phi \,|\, \theta$ is an axiom of the system J and $\phi \in \Gamma$ then, by definition, $\phi \,|\, \theta$ is a proof of θ from ϕ in J.

(iii). We do the case where the rule is AND. So suppose $\Gamma \vdash^J \theta$ and $\Gamma \vdash^J \phi$, say $\Lambda_1 \,|\, \theta_1, \ldots, \Lambda_m \,|\, \theta_m$ is a proof of $\Gamma \vdash^J \theta$ and $\Delta_1 \,|\, \phi_1, \ldots, \Delta_h \,|\, \phi_h$ is a proof of $\Gamma \vdash^J \phi$. So $\theta_m = \theta$, $\phi_h = \phi$ and Λ_m, Δ_h are finite subsets of Γ. Then it is easy to check that

$$\Lambda_1 \,|\, \theta_1, \ldots, \Lambda_m \,|\, \theta_m, \Delta_1 \,|\, \phi_1, \ldots, \Delta_h \,|\, \phi_h, \Lambda_m \cup \Delta_h \,|\, \theta_m \wedge \phi_h$$

is a proof (the justification for the new step being AND) of
$\Gamma \vdash^J \theta \wedge \phi$ since $\theta_m \wedge \phi_h = \theta \wedge \phi$ and $\Lambda_m \cup \Delta_h$ is a finite subset of Γ.

The other rules are similar. [See also the appended Example Sheet.] ■

Notation. We write $L^{(n)}$ for the finite language $\{p_1, \ldots, p_n\}$.

Proposition 2 *Let J be one of the systems $K, B, \ldots, S_5$, let $\phi_1, \ldots, \phi_n \in SML$ and let $\theta \in SML^{(n)}$. Let $\theta^{\vec{\phi}}$ be the result of simultaneously replacing each occurrence of p_i in θ by ϕ_i for $1 \leq i \leq n$. Then*

(i) $\theta^{\vec{\phi}} \in SML$

(ii) If $\Gamma_1 \,|\, \theta_1,\ \ldots, \Gamma_m \,|\, \theta_m$ is a proof in J with $\Gamma_i \subseteq SML^{(n)}$, $\theta_i \in SML^{(n)}$, then $\Gamma_1^{\vec{\phi}} \,|\, \theta_1^{\vec{\phi}},\ \ldots, \Gamma_m^{\vec{\phi}} \,|\, \theta_m^{\vec{\phi}}$ is a proof in J.

iii If $\Gamma \vdash^J \theta$ then $\Gamma^{\vec{\phi}} \vdash^J \theta^{\vec{\phi}}$.

Proof (i) is a straightforward induction on $|\theta|$ for $\theta \in SML^{(n)}$.
(ii) is a straightforward induction on m.
(iii) follows directly from (ii). ■

A useful consequence this proposition is that since any SC-proof for $L^{(n)}$ is also a proof for $K, B, \ldots, S_5$, we may directly lift proofs and derivability results from SC to these systems. For example, since $\models^{SC} (p_1 \to p_2) \leftrightarrow (\neg p_2 \to \neg p_1)$, by the Completeness Theorem for SC, $\vdash^{SC} (p_1 \to p_2) \leftrightarrow (\neg p_2 \to \neg p_1)$, and hence we have $\vdash^K (p_1 \to p_2) \leftrightarrow (\neg p_2 \to \neg p_1)$ and by the above proposition $\vdash^K (\phi_1 \to \phi_2) \leftrightarrow (\neg \phi_2 \to \neg \phi_1)$ for all $\phi_1, \phi_2 \in SML$.

Some provability results worth noting

In any of the systems J, by REF, IMR, MP,

$$\Gamma, \theta \vdash^J \phi \iff \Gamma \vdash^J \theta \to \phi.$$

If $\vdash^J (\theta \to \phi)$ then $\vdash^J (\Box\theta \to \Box\phi)$ and $\vdash^J (\Diamond\theta \to \Diamond\phi)$. More generally for $\Box$, if $\theta_1, \theta_2, \ldots, \theta_n \vdash^J \phi$ then $\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_n \vdash^J \Box\phi$, see **3** on the Example Sheet.

If $\theta \vdash^J \phi$ holds then so does $\theta \vdash^{J'} \phi$ for any extension J' of J.

By AO and AND,[3]

$$\vdash^J (\theta \leftrightarrow \phi) \iff \vdash^J (\theta \to \phi) \text{ and } \vdash^J (\phi \to \theta) \iff \theta \vdash^J \phi \text{ and } \phi \vdash^J \theta$$

by MON, REF, MP. From this it follows that the relation $\vdash^J (\theta \leftrightarrow \phi)$ is an equivalence relation, called *provable equivalence (in J).*

$$\begin{array}{llll}
\vdash^K: & (k1)\ \ \Box(\theta \wedge \phi) \leftrightarrow (\Box\theta \wedge \Box\phi) & & (k_2)\ \ \Box\theta \leftrightarrow \neg\Diamond\neg\theta \\
 & (k3)\ \ \Box\neg\theta \leftrightarrow \neg\Diamond\theta & & (k4)\ \ \neg\Box\theta \leftrightarrow \Diamond\neg\theta \\
\\
\vdash^T: & (t0)\ \ \Box\theta \to \theta & & (t1)\ \ \theta \to \Diamond\theta \\
\\
\vdash^{S_4}: & (s1)\ \ \Box\theta \leftrightarrow \Box\Box\theta & & (s2)\ \ \Diamond\theta \leftrightarrow \Diamond\Diamond\theta \\
 & (s3)\ \ \Box\Diamond\theta \leftrightarrow \Box\Diamond\Box\Diamond\theta & & (s4)\ \ \Diamond\Box\theta \leftrightarrow \Diamond\Box\Diamond\Box\theta \\
\\
\vdash^{S_5}: & (\sigma0)\ \ \Box\theta \leftrightarrow \Box\Box\theta & & (\sigma1)\ \ \Diamond\theta \leftrightarrow \Box\Diamond\theta \\
 & (\sigma2)\ \ \Box\theta \leftrightarrow \Diamond\Box\theta & & (\sigma3)\ \ \Box(\theta \vee \Box\phi) \leftrightarrow (\Box\theta \vee \Box\phi) \\
 & (\sigma4)\ \ \Box(\theta \vee \Diamond\phi) \leftrightarrow (\Box\theta \vee \Diamond\phi) & & (\sigma5)\ \ \Diamond\theta \leftrightarrow \Diamond\Diamond\theta.
\end{array}$$

We leave these as a fun exercises for you to do. [See the Examples Sheet at the end of these notes. Of course Propositions 1 and 2 are useful here.] As usual once we have the Completeness Theorems we will have much more straightforward ways to show these. On the other hand we actually need some of these to prove the Completeness Theorems in the first place!

Corollary 3 *S_5 extends S_4 (in fact strictly).*

Proof By ($\sigma0$) there is a proof in S_5 whose last sequent is $\mid (\Box\theta \to \Box\Box\theta)$ and hence a proof $\mathcal{P}_\theta$ with last sequent $\Box\theta \mid \Box\Box\theta$. Hence if $\mathcal{Q}$ is a proof in S_4 we can obtain a proof in S_5 with the same final sequent by replacing every occurrence of the S_4 axiom, $\Box\theta \mid \Box\Box\theta$, in $\mathcal{Q}$ by $\mathcal{P}_\theta$. (We'll see later that this extension is strict.)

■

Corollary 4 *Let M_i be $\Box$ or $\Diamond$ for $i = 1, 2, \ldots, n$, $n \geq 1$. Then in S_4 $M_1 M_2 \ldots M_n\theta$ is provably equivalent to one of*

$$\Box\theta,\ \ \Diamond\theta,\ \ \Box\Diamond\theta,\ \ \Diamond\Box\theta,\ \ \Box\Diamond\Box\theta,\ \ \Diamond\Box\Diamond\theta.$$

Corollary 5 *Let M_i be $\Box$ or $\Diamond$ for $i = 1, 2, \ldots, n$, $n \geq 1$. Then in S_5, $M_1 M_2 \ldots M_n\theta$ is provably equivalent to $M_n\theta$ (so one of $\Box\theta$, $\Diamond\theta$)*

[3] As usual $\theta \leftrightarrow \phi$ is an abbreviation for $(\theta \to \phi) \wedge (\phi \to \theta)$.

Semantics for the modal logic K

Definition. A *frame* for a language L is a triple $\langle W, E, V\rangle$, such that

1. $\langle W, E\rangle$ is a directed graph, i.e. W is a nonempty set and $E \subseteq W \times W$.[4][5]

2. $V = (V_i)_{i\in W}$ where $V_i : L \to \{0, 1\}$ $(i \in W)$.

The elements $i \in W$ are called *worlds*.

Definition. Let $\langle W, E, V\rangle$ be a frame for the language L. For $i \in W$, we define $\langle W, E, V\rangle,\ i \models \theta$ (or just $i \models \theta$ if the frame is clear from the context) for $\theta \in SML$ by induction on $|\theta|$ using the equivalences:

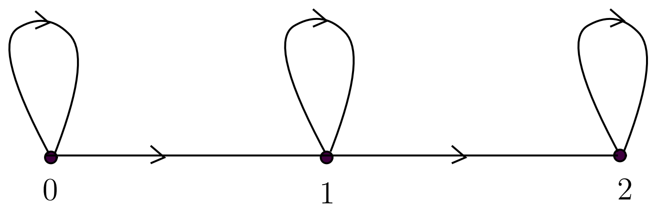
$$\begin{aligned}
i \models p &\iff V_i(p) = 1\\
i \models \neg\phi &\iff i \not\models \phi\\
i \models (\phi \wedge \psi) &\iff i \models \phi \text{ and } i \models \psi\\
i \models (\phi \vee \psi) &\iff i \models \phi \text{ or } i \models \psi\\
i \models (\phi \to \psi) &\iff i \not\models \phi \text{ or } i \models \psi\\
i \models \Box\phi &\iff \forall j \in W\ \ ((i, j) \in E \Rightarrow j \models \phi)
\end{aligned}$$

By unique readability this definition is correct.

Remark. By definition we have $i \models \Box\neg\theta \iff \forall j \in W\ ((i, j) \in E \Rightarrow j \not\models \theta)$. Hence

$$i \models \Diamond\theta \iff \exists j \in W\ ((i, j) \in E \text{ and } j \models \theta).$$

For example in the frame $W = \{0, 1, 2\}$, $E = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2)\}$, $L = \{p\}$, $V_0(p) = V_1(p) = 1$, $V_2(p) = 0$, diagrammatically:



p	p	$\neg p$
$0 \models \Box p$	$1 \not\models \Box p$	$2 \models \Box\neg p$
$0 \not\models \Box\Box p$	$1 \not\models \Box\neg p$	$2 \not\models \Box p$
$0 \models \Box p \wedge \neg\Box\Box p$		

[4] E is sometimes referred to as an *accessibility relation*. So if $(i, j) \in E$ we say that j is *accessible* from i.

[5] For ordered pairs, triples etc. both round and angle brackets are used synonymously in this part of the course(!).

Definition. For $\Gamma \subseteq SML$ and $\theta \in SML$ we say that θ is a logical consequence of Γ in K, denoted

$$\Gamma \models^K \theta,$$

if for every frame $\langle W, E, V\rangle$ and every $i \in W$ we have[6]

$$i \models \Gamma \ \Rightarrow i \models \theta.$$

Theorem 6 (Correctness Theorem for K) *If $\Gamma \subseteq SML$ and $\theta \in SML$, then*

$$\Gamma \vdash^K \theta \ \Rightarrow \ \Gamma \models^K \theta.$$

Proof Let $\Gamma_1|\theta_1, \ldots, \Gamma_m|\theta_m$ be a proof of $\Gamma \vdash^K \theta$. Hence $\Gamma_m \subseteq \Gamma$ and $\theta_m = \theta$ and it is enough to show by induction on k for $k = 1, 2, \ldots, m$ that $\Gamma_k \models^K \theta_k$.

CASE 1. $\Gamma_k|\theta_k$ is an axiom of K.

If $\Gamma_k|\theta_k$ is an instance of REF, then clearly $\Gamma_k \models^K \theta_k$. Otherwise $\Gamma_k|\theta_k$ is an instance of K, i.e. of the form $\Box(\psi \to \phi) \mid (\Box\psi \to \Box\phi)$. Let $\langle W, E, V\rangle$ be a frame of L and let $i \in W$ with $i \models \Box(\psi \to \phi)$. We have to show $i \models (\Box\psi \to \Box\phi)$, i.e. we may assume $i \models \Box\psi$ and we need to show $i \models \Box\phi$. Suppose $j \in W$ with $(i, j) \in E$ but $j \not\models \phi$. Since $i \models \Box\psi$ we have $j \models \psi$, hence $j \not\models (\psi \to \phi)$. Since $(i, j) \in E$, this contradicts our assumption $i \models \Box(\psi \to \phi)$.

CASE 2. $\Gamma_k|\theta_k$ is obtained by applying a rule of K to some $\Gamma_{r_1}|\theta_{r_1}, \ldots, \Gamma_{r_s}|\theta_{r_s}$, where $r_1, \ldots, r_s < k$.

By the induction hypothesis we know $\Gamma_{r_n} \models^K \theta_{r_n}$ for $n = 1, \ldots, s$.

We do each rule separately. Except for the necessity rule, all follow as in the propositional calculus, since the verification always remains within one world. It remains to show that $\models^K$ respects the necessity rule

$$\frac{|\theta}{|\Box\theta}$$

for every $\theta \in SML$. In other words we may assume $\models^K \theta$ and we must show $\models^K \Box\theta$.

Let $\langle W, E, V\rangle$ be a frame of L and $i \in W$. Since $\models^K \theta$ we know $j \models \theta$ for every $j \in W$. Therefore in particular $j \models \theta$ for every $j \in W$ with $(i, j) \in W$. This shows $i \models \Box\theta$ as desired. ■

The Correctness Theorem is very useful because it enables us to show results of the form $\Gamma \nvdash^K \phi$ by showing $\Gamma \not\models^K \phi$, and this simply requires us to exhibit a suitable frame $\langle W, E, V\rangle$ and $i \in W$ such that $i \vDash \Gamma$ but $i \nvDash \phi$.

For example in the frame $W = \{0, 1, 2\}$, $E = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2)\}$, $L = \{p\}$, $V_0(p) = V_1(p) = 1$, $V_2(p) = 0$ given earlier we have $0 \vDash \Box p$ but $0 \nvDash \Box\Box p$ so $\Box p \nvdash^K \Box\Box p$, and hence K must be strictly weaker that S_4.

Definition $\Gamma \subseteq SML$ is *J-inconsistent* if $\Gamma \vdash^J \theta \wedge \neg\theta$ for some $\theta \in SML$,
– equivalently $\Gamma \vdash^J \theta$ and $\Gamma \vdash^J \neg\theta$ for some $\theta \in SML$,
– equivalently $\Gamma \vdash^J \phi$ for any $\phi \in SML$.

Γ is *J-consistent* if not *J-inconsistent*.[7]

As a corollary to Theorem 6 we have:

[6] $i \vDash \Gamma$ stands for $i \vDash \phi$ for all $\phi \in \Gamma$.
[7] We shall see later that this notion is independent of L.

Corollary 7 *K is consistent, i.e. we cannot have $\vdash^K \phi$ and $\vdash^K \neg\phi$ for any sentence ϕ. Equivalently $\emptyset$ is consistent in K.*

Proof If not we would have $i \vDash \phi$ and $i \vDash \neg\phi$ for any world i in any frame for K. But clearly (by unique readability) this is impossible. ■

Completeness for the modal logic K

Our aim in this section is to show the Completeness Theorem for K, that is that $\vdash^K$ and $\vDash^K$ are actually the same thing. The structure of the proof is similar to that for SC (or the Predicate Calculus), as you will see.

Lemma 8 *(i) If $\Delta \subseteq SML$ is consistent in K and $\theta \in SML$, then (at least) one of $\Delta \cup \{\theta\}$ or $\Delta \cup \{\neg\theta\}$ is consistent in K.*

(ii) Suppose $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$ are K-consistent subsets of SML. Then $\bigcup_{n=0}^{\infty} \Gamma_n$ is K-consistent. More generally the union of a set of K-consistent subsets of SML, which is totally ordered by inclusion, is K-consistent.

Proof The proof is the same as for SC:

(i) If $\Delta \cup \{\theta\}$ is K-inconsistent then $\Delta, \theta \vdash^K \phi$ and $\Delta, \theta \vdash^K \neg\phi$ for (any) ϕ so $\Delta \vdash^K \neg\theta$ by NIN. Hence if $\Delta \cup \{\neg\theta\}$ is also K-inconsistent, $\Delta \vdash^K \neg\neg\theta$ so Δ is also K-inconsistent, contradiction.

(ii) Suppose on the contrary $\Delta_1 \,|\, \phi_1, \ldots, \Delta_m \,|\, \phi_m$ was a proof of $\theta \wedge \neg\theta$ from $\bigcup_{n=0}^{\infty} \Gamma_n$. Then $\Delta_m \subseteq \bigcup_{n=0}^{\infty} \Gamma_n$ so because Δ_m is finite it must actually be a subset of some Γ_j. To see this suppose $\Delta_m = \{\psi_1, \ldots, \psi_r\}$. Each $\psi_i \in \bigcup_{n=0}^{\infty} \Gamma_n$ so for each i there is some n_i such that $\psi_i \in \Gamma_{n_i}$. But then since the Γ_n are increasing each ψ_i for $i = 1, 2, \ldots, r$ is in Γ_n where n is the maximum of the n_i, so $\Delta_1 \,|\, \phi_1, \ldots, \Delta_m \,|\, \phi_m$ is a proof of $\theta \wedge \neg\theta$ from Γ_n, contradicting the K-consistency of Γ_n. ■

Definition. A subset Ω of SML is called *maximal K-consistent* if Ω is K-consistent and every $\Omega' \subseteq SML$ properly containing Ω is K-inconsistent

Lemma 9 *Let $\Omega \subseteq SML$ be maximal K-consistent. Then for $\theta, \phi \in SML$*

1. $\Omega \vdash^K \theta \iff \theta \in \Omega$.
2. $\theta \notin \Omega \iff \neg\theta \in \Omega$.
3. $\theta \wedge \phi \in \Omega \iff \theta \in \Omega$ *and* $\phi \in \Omega$.
4. $\theta \vee \phi \in \Omega \iff \theta \in \Omega$ *or* $\phi \in \Omega$.
5. $(\theta \to \phi) \in \Omega \iff \theta \notin \Omega$ *or* $\phi \in \Omega$.

Proof (1) If $\theta \notin \Omega$ then by maximal consistency $\Omega \cup \{\theta\}$ must be inconsistent, so $\Omega, \theta \vdash^K \phi \wedge \neg\phi$ (any) ϕ. But then $\Omega \vdash^K \theta \to (\phi \wedge \neg\phi)$ so if $\Omega \vdash^K \theta$ them $\Omega \vdash^K \phi \wedge \neg\phi$ and Ω itself would be K-inconsistent, contradiction. Hence $\Omega \nvdash^K \theta$.

Conversely if $\theta \in \Omega$ then $\Omega \vdash^K \theta$ by Proposition 1(i).

(2) If $\theta \notin \Omega$ then $\Omega \cup \{\theta\}$ must be K-inconsistent so $\Omega, \theta \vdash^K \phi$ and $\Omega, \theta \vdash^K \neg\phi$. By NIN $\Omega \vdash^K \neg\theta$ so by (1) $\neg\theta \in \Omega$.

Conversely, if $\theta \in \Omega$ we cannot also have $\neg\theta \in \Omega$, otherwise $\Omega \vdash^K \theta$ and $\Omega \vdash^K \neg\theta$ and Ω would be K-inconsistent.

(3) This follows from (1) since $\Omega \vdash^K \theta \wedge \phi$ iff $\Omega \vdash^K \theta$ and $\Omega \vdash^K \phi$.

(4) Right to left follows from (1) since $\Omega \vdash^K \phi$ implies $\Omega \vdash^K \theta \vee \phi$ etc. In the other direction suppose $\theta \notin \Omega$ and $\phi \notin \Omega$. Then by (2) and $\neg\theta, \neg\phi \in \Omega$ and by (1) and AND $\Omega \vdash^K \neg\theta \wedge \neg\phi$. Hence $\Omega \vdash^K \neg(\theta \vee \phi)$, since $\vdash^K (\neg\theta \wedge \neg\phi) \leftrightarrow \neg(\theta \vee \phi)$, so $\theta \vee \phi \notin \Omega$ by (1) and (2).

(5) Since $\vdash^K (\theta \to \phi) \leftrightarrow (\neg\theta \vee \phi)$, $\Omega \vdash^K (\theta \to \phi)$ iff $\Omega \vdash^K \neg\theta \vee \phi$ and so this case follows from (4) with the help of (1) and (2). ■

Lemma 10 *Every K-consistent subset of SML is contained in a maximal K-consistent subset of SML.*

Proof Assume for the moment that L is countable. Then we can enumerate the sentences of SML as $\theta_1, \theta_2, \theta_3, \ldots$ (exercise!). Let Δ be a K-consistent subset of SML and set $\Gamma_0 = \Delta$,

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\theta_{n+1}\} & \text{if this is K}-\text{consistent} \\ \Gamma_n & \text{otherwise.} \end{cases}$$

By induction on n each Γ_n is consistent so $\bigcup_{n=0}^{\infty} \Gamma_n$ is consistent and extends $\Delta = \Gamma_0$. It is also maximal consistent for if θ ($= \theta_{n+1}$ say) could be added to this set and still preserve consistency then it would already have been added in to Γ_n so would already be included in $\bigcup_{n=0}^{\infty} \Gamma_n$.

[An essentially similar proof works for languages which are not countable but, as usual, that requires more Set Theory than this course assumes.] ■

Definition. A subset Γ of SML is said to be *satisfiable* (in K) or K-*satisfiable*, if there is a frame $\langle W, E, V\rangle$ and some $i \in V$ such that $i \models \Gamma$.

Note that this notion is independent of L.

The Completeness Theorem for K will follow as an easy corollary of our next theorem.

Theorem 11 *$\Gamma \subseteq SML$ is K-satisfiable if and only if Γ is K-consistent.*

Proof First suppose Γ is satisfiable in K. Let $\langle W, E, V\rangle$ be a frame and let $i \in V$ such that $i \models \Gamma$. Suppose Γ is not consistent in K, hence $\Gamma \vdash^K \theta$ and $\Gamma \vdash^K \neg\theta$ for some $\theta \in SML$. By the Correctness Theorem for K we have $\Gamma \models^K \theta$ and $\Gamma \models^K \neg\theta$. Since $i \models \Gamma$ we get $i \models \theta$ and $i \models \neg\theta$, a contradiction.

Conversely suppose Γ is consistent in K. Let W be the set of all maximal K-consistent subsets of SML. Define[8]

$$E = \{(\Omega, \Lambda) \in W \times W \mid \Lambda \supseteq \{\,\theta \mid \Box\theta \in \Omega\}\}$$

and for each $\Omega \in W$ define a map $V_\Omega : L \longrightarrow \{0, 1\}$ by

$$V_\Omega(p) = 1 \iff p \in \Omega.$$

Then with $V = (V_\Omega)_{\Omega \in W}$, $\langle W, E, V\rangle$ is a frame and we claim that

$$(\dagger) \qquad \Omega \models \theta \iff \theta \in \Omega$$

[8]It is useful to notice that the requirement that $\Lambda \supseteq \{\,\theta \mid \Box\theta \in \Omega\}$ is equivalent to $\Omega \supseteq \{\,\Diamond\theta \mid \theta \in \Lambda\,\}$, see question 13 on the Examples Sheet.

for all $\theta \in SML$. We prove this by induction on $|\theta|$. If $\theta = p \in L$, the claim holds true by definition.

Using Lemma 9 and the induction hypothesis that the result holds for all ϕ with $1 \leq |\phi| < |\theta|$ we have for each $\Omega \in W$:

$$\begin{array}{lll}
\text{if } \theta = \neg\phi, & \Omega \models \theta & \iff \Omega \not\models \phi \iff \phi \notin \Omega \iff \theta \in \Omega \\
\text{if } \theta = (\phi \wedge \psi), & \Omega \models \theta & \iff \Omega \models \phi \text{ and } \Omega \models \psi \iff \phi \in \Omega \text{ and } \psi \in \Omega \\
& & \iff (\phi \wedge \psi) \in \Omega \\
\text{if } \theta = (\phi \vee \psi), & \Omega \models \theta & \iff \Omega \models \phi \text{ or } \Omega \models \psi \iff \phi \in \Omega \text{ or } \psi \in \Omega \\
& & \iff (\phi \vee \psi) \in \Omega \\
\text{if } \theta = (\phi \to \psi), & \Omega \models \theta & \iff \Omega \not\models \phi \text{ or } \Omega \models \psi \iff \phi \notin \Omega \text{ or } \psi \in \Omega \\
& & \iff (\phi \to \psi) \in \Omega
\end{array}$$

It remains to show (†) in the case $\theta = \Box\phi$. If $\theta \in \Omega$, then by definition of E, whenever $(\Omega, \Lambda) \in E$ we have $\phi \in \Lambda$, i.e. $\Lambda \models \phi$ by induction. This proves $\Omega \models \theta$.

Conversely suppose $\theta \notin \Omega$. We must find some $\Lambda \in W$ with $(\Omega, \Lambda) \in E$ such that $\Lambda \not\models \phi$, i.e. $\phi \notin \Lambda$ by induction. By Lemma 10 and the definition of E, it is enough to show that $\{\neg\phi\} \cup \{\psi \in SML \mid \Box\psi \in \Omega\}$ is K-consistent.

Otherwise, there are $\psi_1, \ldots, \psi_k, \eta \in SML$ with $\Box\psi_i \in \Omega$, and $\neg\phi, \psi_1, \ldots, \psi_k \vdash^K \eta \wedge \neg\eta$. By AO, NIN and NNO we get $\psi_1, \ldots, \psi_k \vdash^K \phi$ and hence by Q3 on the Examples sheet $\Box\psi_1, \ldots, \Box\psi_k \vdash^K \Box\phi$. Since $\Box\psi_1, \ldots, \Box\psi_k \in \Omega$, this gives $\Omega \vdash^K \Box\phi$ and hence $\Box\phi = \theta \in \Omega$, the required contradiction. Hence (†) is proved.

To complete the proof notice that since Γ is K-consistent, there is some $\Omega \in W$ containing Γ. By (†) we have $\Omega \models \theta$ for all $\theta \in \Gamma$, thus $\langle W, E, V\rangle, \Omega \models \Gamma$. ■

Corollary 12 (The Completeness Theorem for K) *Let $\Gamma \subseteq SML$ and $\theta \in SML$. Then*

$$\Gamma \vdash^K \theta \iff \Gamma \models^K \theta.$$

Proof $\Gamma \vdash^K \theta$ is equivalent to $\Gamma \cup \{\neg\theta\}$ being K-inconsistent (exercise!). $\Gamma \models^K \theta$ is equivalent to $\Gamma \cup \{\neg\theta\}$ being not satisfiable (another exercise!). Hence the corollary is a reformulation of Theorem 11. ■

Of course since proofs are finite we also get the

Theorem 13 (Compactness Theorem for K) *$\Gamma \subseteq SML$ is K-satisfiable if and only if every finite subset of Γ is K-satisfiable.*

A nice property for a modal logic to possess is that it has the *finite model property*:

Definition J has the *finite model property* if every finite J-satisfiable subset Γ of SML is satisfiable in a frame for J with only a finite set of worlds.

Just for interest we mention that:

Theorem 14 *K has the finite model property.*

Semantics for other modal logics

By putting further restriction on the directed graph $\langle W, E\rangle$, we obtain frames suitable for semantics of the other modal systems.

Definition. Let $\langle W, E, V\rangle$ be a frame. We says that $\langle W, E, V\rangle$ is a

- *D-frame*, if E is *serial*, i.e. $\forall i \in W\ \exists j \in W\ (i, j) \in E$.
- *T-frame*, if E is *reflexive*, i.e. $\forall i \in W\ (i, i) \in E$.
- *B-frame*, if E is *symmetric*, i.e. $\forall i, j \in W\ [(i, j) \in E \Rightarrow (j, i) \in E]$.
- S_4*-frame*, if E is reflexive and *transitive*, i.e. $\forall i, j, k \in W\ [(i, j), (j, k) \in E \Rightarrow (i, k) \in E]$.
- S_5*-frame*, if E is an equivalence relation, i.e. E is reflexive, symmetric and transitive.

As usual let J be one of the systems K, B, D, T, S_4 or S_5. For $\Gamma \subseteq SML$ and $\theta \in SML$ we define

$$\Gamma \models^J \theta \iff \text{for every } J\text{-frame } \langle W, E, V\rangle \text{ and each } i \in W, \text{ if } i \models \Gamma \text{ then } i \models \theta.$$

We say that $\Gamma \subseteq SML$ is satisfiable in J, or J-satisfiable, if there is a J-frame and a world i of that frame satisfying Γ, i.e. $i \models \Gamma$.

Theorem 15 *For $J = K, T, B, D, S_4, S_5$ and $\Gamma \subseteq SML$, Γ is J-satisfiable iff Γ is J-consistent.*

Proof In each case the proof is just a refinement of the proof of Theorem 11. We illustrate this with some key steps:

$\Leftarrow$ $J = T$: Suppose Γ is satisfied at $i \in W$ in the reflexive frame $\langle W, E, V\rangle$ but $\Gamma \vdash^T \theta \wedge \neg\theta$. Then there is a proof $\Gamma_1 \,|\, \theta_1, \Gamma_2 \,|\, \theta_2, \ldots, \Gamma_n \,|\, \theta_n$ in T of $\theta \wedge \neg\theta$ from Γ (so $\Gamma_n \subseteq \Gamma$, $\theta_n = (\theta \wedge \neg\theta)$). We show by induction on m for $1 \leq m \leq n$ that $\Gamma_m \vDash^T \theta_m$.

This divides into cases according to the justification for $\Gamma_m \,|\, \theta_m$ being in the proof. The cases for the rules and axioms of K are exactly as before so all we need to check is the T axiom, i.e. we need to show $\Box\theta \vDash^T \theta$. So let $\langle W', E', V'\rangle$ be any reflexive frame and $j \in W'$. If it is not the case that $j \models \Box\theta \Rightarrow j \models \theta$ then $j \vDash \Box\theta$ and $j \nvDash \theta$. But then since $(j, j) \in E$ (because E is reflexive) we obtain also $j \vDash \theta$ from $j \nvDash \theta$, contradiction! Hence $j \models \Box\theta \Rightarrow j \models \theta$ and $\Box\theta \vDash^T \theta$ as required.

Having shown $\Gamma_n \vDash^T \theta_n$ we see that since $i \models \Gamma_n$ (because $i \models \Gamma$ and $\Gamma_n \subseteq \Gamma$), $i \models \theta_n$, i.e. $i \models \theta \wedge \neg\theta$, so $i \models \theta$ and $i \models \neg\theta$ which is impossible. Hence Γ must be T-consistent.

The proofs for the other values of J are exactly similar, we just have to check in each case that the 'new' J axioms are valid in the J-frames.

$\Rightarrow$ We construct a frame just as for K but with J in place of K everywhere. So the worlds are the maximal J-consistent subsets of SML (so satisfying (1)-(5) of Lemma 9 with $\vdash^J$ in place of $\vdash^K$) and we again define

$$V_\Gamma(p) = 1 \iff p \in \Gamma,$$

$$(\Psi, \Lambda) \in E \iff \Lambda \supseteq \{\, \eta \,|\, \Box\eta \in \Psi\}.$$

The only additional feature is that we must show that the graph $\langle W, E\rangle$ we get out has the appropriate structure (reflexive, serial, etc) for J. To demonstrate we'll show this for a couple of values of J:

Case 1 $J = T$. We must show the graph is reflexive. Let $\Omega \in W$. Then since $\vdash^T \Box\theta \to \theta$, $\Box\theta \in \Omega \Rightarrow \Omega \vdash^T \Box\theta \Rightarrow \Omega \vdash^T \theta \Rightarrow \theta \in \Omega$ by (1). Therefore $\Omega \supseteq \{\,\theta \,|\, \Box\theta \in \Omega\,\}$ so $(\Omega, \Omega) \in E$, as required.

Case 2 $J = D$. We must show that $\langle W, E\rangle$ is serial. Let $\Omega \in W$. It is enough to show that $\{\,\eta \,|\, \Box\eta \in \Omega\,\}$ is D-consistent since then it can be extended to a maximal D-consistent subset Λ of SML and we shall have $(\Omega, \Lambda) \in E$. So assume on the contrary that this set is not D-consistent, say $\eta_1, \ldots, \eta_j \in \{\,\eta \,|\, \Box\eta \in \Omega\,\}$ and $\eta_1, \ldots, \eta_j \vdash^D \phi$ and $\eta_1, \ldots, \eta_j \vdash^D \neg\phi$ for some ϕ. Then by **3** on the Examples Sheet, $\Box\eta_1, \ldots, \Box\eta_j \vdash^D \Box\phi$ and $\Box\eta_1, \ldots, \Box\eta_j \vdash^D \Box\neg\phi$. Hence $\Omega \vdash^D \Box\phi$ and $\Omega \vdash^D \Box\neg\phi$ since $\Box\eta_1, \ldots, \Box\eta_j \in \Omega$. Using now the D axiom (plus IMR, MP etc) we obtain from $\Omega \vdash^D \Box\phi$ that $\Omega \vdash^D \neg\Box\neg\phi$, which with $\Omega \vdash^D \Box\neg\phi$ shows Ω to be D-inconsistent, contradiction.

Case 3 $J = S_5$. We must show that E is an equivalence relation on W. Since S_5 extends T E is certainly reflexive so it is enough to show that if $(\Omega, \Lambda), (\Omega, \Psi) \in E$ then $(\Lambda, \Psi) \in E$ since with reflexivity this implies both transitivity and symmetry. So suppose $\Omega, \Lambda, \Psi \in W$ and

$$\Lambda \supseteq \{\,\eta \,|\, \Box\eta \in \Omega\,\}, \quad \Psi \supseteq \{\,\eta \,|\, \Box\eta \in \Omega\,\}.$$

Let $\Box\phi \in \Lambda$. It is enough to show that $\phi \in \Psi$. But if it isn't then $\Box\phi \notin \Omega$ so by (2) $\neg\Box\phi \in \Omega$. $\therefore$ by (k4) $\Diamond\neg\phi \in \Omega$ so $\Box\Diamond\neg\phi \in \Omega$ by using the S_5 axiom and (1). This means that $\Diamond\neg\phi \in \Lambda$, so by (k4) $\neg\Box\phi \in \Lambda$, which contradicts the consistency of Λ and the result follows. ■

Exactly as in the case of K we now have T, B, D, S_4, S_5 are consistent and:

Corollary 16 (Completeness Theorem for J) *For $\Gamma \subseteq SML$, $\theta \in SML$ and J one of K, T, D, B, S_4, S_5,*

$$\Gamma \vdash^J \theta \iff \Gamma \vDash^J \theta.$$

Corollary 17 (Compactness Theorem for J) *$\Gamma \subseteq SML$ is J-satisfiable iff every finite subset of Γ is satisfiable.*

For interest we mention that we also have:

Theorem 18 (Finite Model Property for J) *K, T, D, B, S_4, S_5 each satisfy the finite model property.*

Again Corollary 16 in the forward direction is useful in showing certain non-provability results, all one needs to do is construct a suitable frame of the appropriate type. We give some examples:

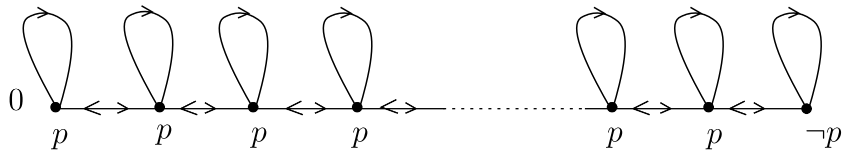
Example 1 S_4 properly extends T by the previous example of a reflexive frame $W = \{0, 1, 2\}$, $E = \{(0,0), (0,1), (1,1), (1,2), (2,2)\}$, $V_0(p) = V_1(p) = 1$, $V_2(p) = 0$ since $0 \vDash \Box p$, $0 \nvDash \Box\Box p$ so $\nvdash^T (\Box p \to \Box\Box p)$ whilst $\vdash^{S_4} (\Box p \to \Box\Box p)$.

Example 2 S_4 does not extend B (in terms of what is derivable in it) since $\vdash^B (p \to \Box\Diamond p)$ but $\neg(p \to \Box\Diamond p)$ is consistent with S_4 since it holds at vertex 0 in the reflexive and transitive frame $W = \{0, 1\}$, $E = \{(0,0), (0,1), (1,1)\}$, $V_0(p) = 1$, $V_1(p) = 0$.

Also B does not extend S_4 since $\vdash^{S_4} (\Box p \to \Box\Box p)$ whilst $\neg(\Box p \to \Box\Box p)$ holds at the vertex 0 in the symmetric frame $W = \{0, 1\}$, $E = \{(0,1), (1,0)\}$, $V_0(p) = 0$, $V_1(p) = 1$.

Notice that since $\theta \vdash^{S_5} \Box\Diamond\theta$ and $\Box\theta \vdash^{S_5} \Box\Box\theta$ the system S_5 properly extends both B and S_4. [Of course now we have the Completeness Theorem for S_5 it is easy to derive these last two, by simply considering equivalence frames notice that they hold with $\vDash^{S_5}$ in place of $\vdash^{S_5}$.]

Example 3 In the following reflexive, serial and symmetric graph with $n+2$ vertices



at world 0 we have $0 \vDash \Box^n p \wedge \neg \Box^m p$ for every $m > n$, where $\Box^n p$ is p preceded by n copies of $\Box$. This shows that all the modalities $\Box^n$ are inequivalent in T, D, B.

Other Interpretations

Although historically $\Box$ was introduced to represent necessity, where the logic S_5 seems to best capture its properties, there are now many other interpretations studied for which the same or similar axioms to T, S_4, S_5 etc. are appropriate. For example treating the propositional variables as sentences of some language for the Predicate Calculus and interpreting $\Box p$ to mean p is provable in some axiom system, and similarly for sentences built up from these p.

Some other interpretations of $\Box\theta$ are

θ *will be true at some time in the future*[9]	θ *is obligatory*[10]
θ *is against the law*	θ *is believed*
θ *is known*[11]	

and for each of these various axioms, frames etc. are judged to be appropriate.

Logics are also studied which use multiple modalities, a currently particularly popular instance being in the analysis of *multi-agent systems*. One example here is where we have agents $1, \ldots, n$ and $\Box_i\theta$ stands for 'agent i knows θ'. In addition to the usual axioms and rules of SC the following axioms and rules seem appropriate here:

For $i = 1, \ldots, n$ axioms	$\Box_i(\theta \to \phi) \mid \Box_i\theta \to \Box_i\phi$	and rules
	$\Box_i\theta \mid \theta$	NEC $\dfrac{\mid \theta}{\mid \Box_i\theta}$
	$\Box_i\theta \mid \Box_i\Box_i\theta$	
	$\neg\Box_i\theta \mid \Box_i\neg\Box_i\theta$	i.e. S_5 for each agent

A suitable frame for this calculus is of the form $\langle W, E_1, E_2, \ldots, , E_k, V\rangle$ where each $\langle W, E_i, V\rangle$ is a frame for S_5 (i.e. E_i is an equivalence relation on W). So now given maximal consistent Ω, Λ we put

$$(\Omega, \Lambda) \in E_i \iff \Lambda \supseteq \{\, \phi \mid \Box_i\phi \in \Omega \,\}.$$

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1. Let $\mathcal{A}$ be the modal logic K augmented with the axiom

$$\Diamond\Diamond\Box\theta \mid \theta$$

Call a frame $\langle W, E, V\rangle$ an $\mathcal{A}$-frame if whenever $\langle i, j\rangle, \langle j, k\rangle \in E$ then $\langle k, i\rangle \in E$. Let $\Gamma \models^{\mathcal{A}} \phi$ be short for:

> For all $\mathcal{A}$-frames $\langle W, E, V\rangle$ and $i \in W$, if $i \models \psi$ for all $\psi \in \Gamma$ then $i \models \phi$.

Give proofs of the key new features (new in the sense that they do not already essentially appear in the corresponding proof for K) that

$$\Gamma \models^{\mathcal{A}} \phi \iff \Gamma \vdash^{\mathcal{A}} \phi.$$

[8 Marks]

Show that $\Diamond\Box p \nvdash^{\mathcal{A}} p$. [2 Marks]

Example Sheet on Modal Logic

1. Show that

$$\frac{\mid \neg\theta}{\mid \neg\Diamond\theta}$$

is a derived rule of K.

2. Show that for any of the systems J if $\vdash^J (\theta \to \phi)$ then $\vdash^J (\Box\theta \to \Box\phi)$ and $\vdash^J (\Diamond\theta \to \Diamond\phi)$.

3. Show that if $\theta_1, \theta_2, \ldots, \theta_n \vdash^J \phi$ then $\Box\theta_1, \Box\theta_2, \ldots, \Box\theta_n \vdash^J \Box\phi$

4. Show Proposition 1(iii) in the case the rule is IMR i.e. show that if $\Gamma, \theta \vdash^J \phi$ then $\Gamma \vdash^J \theta \to \phi$.

5. Show that the following are provable in K

$$(k1)\ \ \Box(\theta \wedge \phi) \leftrightarrow (\Box\theta \wedge \Box\phi) \qquad (k_2)\ \ \Box\theta \leftrightarrow \neg\Diamond\neg\theta$$

$$(k3)\ \ \Box\neg\theta \leftrightarrow \neg\Diamond\theta \qquad (k4)\ \ \neg\Box\theta \leftrightarrow \Diamond\neg\theta$$

6. Show that the following are provable in S_4

$$(s1)\ \ \Box\theta \leftrightarrow \Box\Box\theta \qquad (s2)\ \ \Diamond\theta \leftrightarrow \Diamond\Diamond\theta$$

$$(s3)\ \ \Box\Diamond\theta \leftrightarrow \Box\Diamond\Box\Diamond\theta \qquad (s4)\ \ \Diamond\Box\theta \leftrightarrow \Diamond\Box\Diamond\Box\theta$$

[Hint: For (s3) use (t1) with $\Box\Diamond\theta$ in place of θ.]

7. Show that the following are provable in S_5

$$(\sigma0)\ \ \Box\theta \leftrightarrow \Box\Box\theta \qquad (\sigma1)\ \ \Diamond\theta \leftrightarrow \Box\Diamond\theta$$

$$(\sigma2)\ \ \Box\theta \leftrightarrow \Diamond\Box\theta \qquad (\sigma3)\ \ \Box(\theta \vee \Box\phi) \leftrightarrow (\Box\theta \vee \Box\phi)$$

$$(\sigma4)\ \ \Box(\theta \vee \Diamond\phi) \leftrightarrow (\Box\theta \vee \Diamond\phi)$$

8. Analyze the reasoning used in the original Bill-Monica example.

9. Show that provability (in J) is independent of the particular overlying language chosen by showing that if

$$\Gamma_1 \mid \theta_1, \Gamma_2 \mid \theta_2, \ldots, \Gamma_n \mid \theta_n$$

is a proof in J and $\Gamma_n \subseteq SM\mathcal{L}$, $\theta_n \in SM\mathcal{L}$ then there is a proof

$$\Delta_1 \mid \phi_1, \Delta_2 \mid \phi_2, \ldots, \Delta_k \mid \phi_k$$

in J such that $\Delta_k \mid \phi_k = \Gamma_n \mid \theta_n$ and each $\Delta_i \subseteq SM\mathcal{L}$, and each $\phi_i \in SM\mathcal{L}$. [In other words if $\Gamma \vdash^J \theta$ then there is a proof of this which never mentions any proposition variable not already mentioned in Γ or θ.]

10. Let $W = \{\, a, b, c, \,\}$, $E = \{\, (a,b), (b,b), (a,c), (c,a) \,\}$, $L = \{p\}$, $V_a(p) = V_b(p) = 1$, $V_c(p) = 0$. Which of the following hold at a in the frame $\langle W, E, V \rangle$?

$$\text{(i) } \Box p, \quad \text{(ii) } \Diamond p \quad , \text{(iii) } \Box\Box p, \quad \text{(iv) } \Diamond\Box\Diamond\neg p, \quad \text{(v) } \Box(p \to \Box p)$$

11. Show that none of the following are provable in K (i.e. from the empty set):

(i) $p \to \Diamond p)$,
(ii) $\Box(p \vee q) \to (\Box p \vee \Box q)$,
(iii) $\Box\Box p \to \Box p$,
(iv) $\Diamond p \to p$,
(v) $\Diamond p \to \Diamond\Diamond p$.

12. By showing that $\models^K \theta$ and using the Completeness Theorem for K show that $\vdash^K \theta$ for each of the following sentences θ:

(i) $\Diamond(\theta \wedge \phi) \to (\Diamond\theta \wedge \Diamond\phi)$,
(ii) $\Box(\theta \to \phi) \to (\Diamond\theta \to \Diamond\phi)$,
(iii) $(\Box\theta \wedge \Diamond\phi) \to \Diamond(\theta \wedge \phi)$.

13. Let Λ, Ω be maximal K-consistent subsets of SML. Show that

$$\Lambda \supseteq \{\, \theta \,|\, \Box\theta \in \Omega \} \iff \Omega \supseteq \{\, \Diamond\theta \,|\, \theta \in \Lambda \,\}.$$

14. By using the Completeness Theorems for D and B show that $\vdash^D \neg(\Box p \wedge \Box\neg p)$ but $\nvdash^B \neg(\Box p \wedge \Box\neg p)$.

Find a sentence θ such that $\vdash^B \theta$ but $\nvdash^D \theta$. [So this question shows that $B \nsubseteq D$ and $D \nsubseteq B$.]

15. Let $\langle W, E, V \rangle$ be a frame for K, let $w \in W$, $w' \notin W$ and let $\langle W', E', V' \rangle$ be a frame with $W' = W \cup \{w'\}$ $E = E \cup \{(w', w)\}$ and $V_u = V'_u$ for $u \in W$. Show that for $\theta \in SML$ and $u \in W$,

$$\langle W, E, V \rangle, u \models \theta \iff \langle W, E, V \rangle, u \models \theta.$$

Hence show that if $\vdash^K \Box\theta$ then $\vdash^K \theta$.

Let $\overline{K}$ be the result of adding the rule

$$\frac{|\,\Box\theta}{|\,\theta}$$

to K. Show that

$$\vdash^{\overline{K}} \theta \iff \vdash^K \theta.$$

[Notice that this does not imply that $K = T$!]

16. Let $K4$ be K augmented with the axiom schema $\Box\theta \,|\, \Box\Box\theta$. Write $\Gamma \models^{K4} \theta$ if for all *transitive* frames $\langle W, E, V\rangle$ and $i \in W$, if $i \models \Gamma$ then $i \models \theta$. Outline the key new steps (i.e. beyond those used in proving the corresponding result for K) of a proof that

$$\Gamma \models^{K4} \theta \iff \Gamma \vdash^{K4} \theta.$$

17. Show that there is a *not-transitive* frame $\langle W, E, V\rangle$ such that for any $i \in W$ and $\theta \in SML$,

$$i \models \Box\theta \to \Box\Box\theta.$$

[Compare this with the result in **15.**]

18. Let $\phi \in SML$ and let $\langle W, E, V\rangle$ be a frame with $W = \mathbb{N}'$ $E = \{\, (n, m) \,|\, n < m \in \mathbb{N} \,\}$. Show that for any $i \in \mathbb{N}$,

$$i \models \Diamond\Box\phi \to (\Box(\Box\phi \to \phi) \to \Box\phi).$$

Show that if we replace $\mathbb{N}$ here by the set of rational numbers $\mathbb{Q}$ then this schema no longer holds.

19. Let K_3 be K augmented with the axiom schema

$$\theta \,|\, \Diamond\Diamond\Diamond\theta.$$

By considering a suitable set of frames or otherwise show that $p \nvdash^{K_3} \Diamond\Diamond p$.

20. Let $L = \{p\}$ and let $\langle W, E, V\rangle$ be the frame $W = \mathbb{N}$,
$E = \{\, (n, m) \,|\, n \le m + 1 \,\}$, $V_0(p) = 0$, $v_n(p) = 1$ for $n > 0$. Show that in this frame

(i) $2 \models \Box p \wedge \neg\Box\Box p$.

(ii) For any $\theta \in SML$ and $n \in \mathbb{N}$, if $n \not\models \Box\theta \to \Box\Box\theta$ then $n \not\models \Box(\Box\Box\theta \to \Box\Box\Box\theta)$.

Let J be the system T augmented with the axiom schema

$$|\, \Box(\Box\Box\theta \to \Box\Box\Box\theta) \to (\Box\theta \to \Box\Box\theta).$$

Show that $\neg(\Box p \to \Box\Box p)$ is satisfiable in some frame for J (i.e. a frame in which the sentences

$$\Box(\Box\Box\theta \to \Box\Box\Box\theta) \to (\Box\theta \to \Box\Box\theta) \quad \text{and} \quad \Box\theta \to \theta$$

are true at all worlds in this frame) but is not satisfied in any *finite* frame for J.
[Bit harder than the average question. Compare this with the situation for K, T, B, S_4, S_5 where the finite model property does hold.]

Solutions to Modal Logic Examples

1.
1. $\Box\neg\theta \mid \Box\neg\theta,$ REF
2. $\neg\Box\neg\theta \mid \neg\Box\neg\theta,$ REF
3. $\Box\neg\theta, \neg\Box\neg\theta \mid \Box\neg\theta,$ MON 1
4. $\Box\neg\theta, \neg\Box\neg\theta \mid \neg\Box\neg\theta,$ MON 2
5. $\Box\neg\theta, \mid \neg\neg\Box\neg\theta,$ NIN 3,4
6. $\mid \Box\neg\theta \to \neg\Diamond\theta,$ IMR 5

Hence if we have proof of $\mid \neg\theta$ then NEC gives $\mid \Box\neg\theta$ and by MP with the above proof we obtain $\mid \neg\Diamond\theta$.

2. Assume $\vdash^J \theta \to \phi$. By Proposition 1(iii) (which we will henceforth use without mention) and NEC $\vdash^J \Box(\theta \to \phi)$. Using the K axiom $\Box(\theta \to \phi) \vdash^J \Box\theta \to \Box\phi$ and IMR $\vdash^J \Box(\theta \to \phi) \to (\Box\theta \to \Box\phi)$ so $\vdash^J (\Box\theta \to \Box\phi$ by MP. Similarly for θ, ϕ transposed so with AND $\vdash^J \Box\theta \leftrightarrow \Box\phi$.
Also if $\vdash^J \theta \to \phi$ then since

$$\vdash^J (\theta \to \phi) \to (\neg\phi \to \neg\theta),$$

(by Proposition 2 since $\vdash^{\mathrm{SC}} (p_1 \to p_2) \to (\neg p_2 \to \neg p_1))$, $\vdash^J \neg\phi \to \neg\theta$. By the first part $\vdash^J \Box\neg\phi \to \Box\neg\theta$ and by the above trick again $\vdash^J \neg\Box\neg\theta \to \neg\Box\neg\phi$, as required.

3. From $\theta_1, \ldots, \theta_n \vdash^J \phi$ IMR repeatedly gives

$$\vdash^J (\theta_1 \to (\theta_2 \to \ldots (\theta_n \to \phi) \ldots)).$$

By NEC

$$\vdash^J \Box(\theta_1 \to (\theta_2 \to \ldots (\theta_n \to \phi) \ldots)).$$

By the K axiom

$$\vdash^J \Box(\theta_1 \to (\theta_2 \to \ldots (\theta_n \to \phi) \ldots)) \to (\Box\theta_1 \to \Box(\theta_2 \to (\theta_3 \to \ldots (\theta_n \to \phi) \ldots))$$

so by MP,

$$\vdash^J (\Box\theta_1 \to \Box(\theta_2 \to (\theta_3 \to \ldots (\theta_n \to \phi) \ldots)).$$

MON, REF and MP now give

$$\Box\theta_1 \vdash^J \Box(\theta_2 \to (\theta_3 \to \ldots (\theta_n \to \phi) \ldots).$$

The result follows by repeating this step for each of $\theta_2, \theta_3, \ldots, \theta_n$.

4. Suppose that $\Gamma_1 \mid \theta_1, \ldots, \Gamma_m \mid \theta_m$ is a proof of $\Gamma, \theta \vdash^J \phi$, so $\theta_m = \phi$ and $\Gamma_m \subseteq \Gamma \cup \{\theta\}$. Then

$$\Gamma_1 \mid theta_1, \ldots, \Gamma_m \mid \theta_m, \Gamma \cup \{\theta\} \mid \theta_m, \Gamma_m \cup \{\theta\} - \{\theta\} \mid \theta \to \phi$$

is also a proof (the last two sequents being justified by MON and IMR) and si furthermore a proof of $\Gamma \vdash^J \theta \to \phi$ since $\theta \to \theta_m = \theta \to \phi$ and

$$\Gamma_m \cup \{\theta\} - \{\theta\} \subseteq \Gamma - \{\theta\} \subseteq \Gamma.$$

5. (k1) It is enough to show $\Box(\theta \wedge \phi) \vdash^K \Box\theta \wedge \Box\phi$ and $\Box\theta \wedge \Box\phi \vdash^K \Box(\theta \wedge \phi)$. By the notes $\Box(\theta \wedge \phi) \vdash^K \Box\theta$ and similarly $\Box(\theta \wedge \phi) \vdash^K \Box\phi$ so the first conclusion follows by AND. For the second $\theta \vdash^K \phi \to (\theta \wedge \phi)$ (using Proposition 2 on $p_1 \vDash^{\mathrm{SC}} p_2 \to (p_1 \wedge p_2)$) so $\Box\theta \vdash^K \Box(\phi \to (\theta \wedge \phi))$ by **2** above. Using K, REF and MP, we now get $\Box\theta \vdash^K \Box\phi \to \Box(\theta \wedge \phi)$. REF, MON and MP this gives $\Box\theta, \Box\phi \vdash^K \Box(\theta \wedge \phi)$ and ANL completes the job.

(k2) Since $\theta \vdash^K \neg\neg\theta$ (because $p \vdash^{\mathrm{SC}} \neg\neg p$), $\Box\theta \vdash^K \Box\neg\neg\theta$ (by **2** above). $\therefore \Box\theta, \Diamond\neg\theta \vdash^K \Box\neg\neg\theta$ and $\Box\theta, \Diamond\neg\theta \vdash^K \neg\Box\neg\neg\theta$ by REF and MON (notice $\neg\Box\neg\neg\theta = \Diamond\neg\theta$) so by NIN $\Box\theta \vdash^K \neg\Diamond\neg\theta$. In the other direction, since $\neg\neg\theta \vdash^K \theta$, $\Box\neg\neg\theta \vdash^K \Box\theta$. Also $\neg\neg\Box\neg\neg\theta (= \neg\Diamond\neg\theta) \vdash^K \Box\neg\neg\theta$ so $\neg\Diamond\neg\theta \vdash^K \Box\theta$ by REF, MON, IMR, MP.

(k3) By (k2) with $\neg\theta$ in place of θ, $\vdash^K \Box\neg\theta \leftrightarrow \neg\Diamond\neg\neg\theta$. Using the facts that $\vdash^K \theta \leftrightarrow \neg\neg\theta$, $\vdash^K \Diamond\theta \leftrightarrow \Diamond\neg\neg\theta$, the required $\vdash^K \Box\neg\theta \leftrightarrow \neg\Diamond\theta$ follows from the fact that 'provable equivalence (in K)' is an equivalence relation.

From (k2) $\vdash^K \Box\theta \leftrightarrow \neg\Diamond\neg\theta$ and with Proposition 2 and $\vdash^{\mathrm{SC}} (p_1 \leftrightarrow p_2) \to (\neg p_1 \leftrightarrow \neg p_2)$ etc. we obtain $\vdash^K \neg\Box\theta \leftrightarrow \Diamond\neg\theta$.

6. (s1) $\Box\theta \vdash^{S_4} \Box\Box\theta$ from the S_4 axiom and $\Box\Box\theta \vdash^{S_4} \Box\theta$ from the T axiom which gives it.

(s2) $\vdash^{S_4} \Box \neq \theta \leftrightarrow \Box\Box\neg\theta$ by (s1) so $\vdash^{S_4} \neg\Box\neg\theta \leftrightarrow \neg\Box\Box\neg\theta$. By (k4), $\vdash^{S_4} \neg\Box\Box\neg\theta \leftrightarrow \Diamond\neg\Box\neg\theta$ and the result follows by provable equivalence (in S_4).

(s3) By (t1) $\vdash^{S_4} \Box\Diamond\theta \to \Diamond\Box\Diamond\theta$. By **4** above $\vdash^{S_4} \Box\Box\Diamond\theta \to \Box\Diamond\Box\Diamond\theta$ and using the S_4 axiom, $\vdash^{S_4} \Box\Diamond\theta \to \Box\Box\Diamond\theta$, we obtain $\vdash^{S_4} \Box\Diamond\theta \to \Box\Diamond\Box\Diamond\theta$. For the converse, by (t1), $\vdash^{S_4} \Box\neg\theta \to \Diamond\Box\neg\theta$. By **2** and S_4, $\vdash^{S_4} \Box\neg\theta \to \Box\Diamond\Box\neg\theta$. $\therefore \vdash^{S_4} \neg\Box\Diamond\Box\neg\theta \to \neg\Box\neg\theta$, so by **2** $\vdash^{S_4} \Box\neg\Box\Diamond\Box\neg\theta \to \Box\neg\Box\neg\theta$, i.e. $\vdash^{S_4} \Box\Diamond\Box\Diamond\theta \to \Box\Diamond\theta$, as required.

(s4) This follows directly from (s3) by replacing θ by $\neg\theta$ and putting $\neg$ in front of both equivalents.

7. ($\sigma0$) $\vdash^{S_5} \Box\Box\theta \to \Box\theta$ is as in the proof of (s1). For the converse, from S_5, $\Diamond\neg\theta \vdash^{S_5} \Box\Diamond\neg\theta$, so $\neg\Box\Diamond\neg\theta \vdash^{S_5} \neg\Diamond\neg\theta$, giving by (k2) and **2** that $\Diamond\Box\theta \vdash^{S_5} \Box\theta^*$. By (t1), $\Box\theta \vdash^{S_5} \Box\Diamond\Box\theta$ so using S_5, $\Box\theta \vdash^{S_5} \Box\Diamond\Box\theta$. But by $*$ with **4**, $\Box\Diamond\Box\theta \vdash^{S_5} \Box\Box\theta$, so $\Box\theta \vdash^{S_5} \Box\Box\theta$ as required.

($\sigma1$) $\vdash^{S_5} \Box\Diamond\theta \to \Diamond\theta$ follows from T whilst by S_5 $\vdash^{S_5} \Diamond\theta\Box\Diamond\theta$.

($\sigma2$) This follows easily from ($\sigma1$) by replacing θ by $\neg\theta$ and using (k2), (k4).

($\sigma3$) Using Proposition 2 with $p_1 \vee p_2, \neg p_2 \vdash^{\mathrm{SC}} p_1$ we get $\theta \vee \Box\phi, \neg\Box\phi \vdash^{S_5} \theta$. By **3**, $\Box(\theta \vee \Box\phi), \Box\neg\Box\phi \vdash^{S_5} \Box\theta$, $\therefore \Box(\theta \vee \Box\phi) \vdash^{S_5} \Box\neg\Box\phi \to \Box\theta$. By ($\sigma1$) and **2** $\vdash^{S_5} \neg\Box\phi \to \Diamond\neg\phi$, $\vdash^{S_5} \Diamond\neg\phi \to \Box\Diamond\neg\phi$, $\vdash^{S_5} \Box\Diamond\neg\phi \to \Box\neg\Box\phi$ giving $\Box(\theta \vee \Box\phi) \vdash^{S_5} \neg\Box\phi \to \Box\theta$ and hence $\Box(\theta \vee \Box\phi) \vdash^{S_5} \Box\theta \vee \Box\phi$ by using Proposition 2 and $\vdash^{\mathrm{SC}} (\neg p_1 \to p_2) \leftrightarrow (p_2 \vee p_1)$.

For the other direction, using Proposition 2, $\vdash^{S_5} \theta \to (\theta \vee \Box\phi)$, so $\vdash^{S_5} \Box(\theta \to (\theta \vee \Box\phi))$ and with K, $\Box\theta \vdash^{S_5} \Box(\theta \vee \Box\phi)$. Similarly $\Box\Box\phi \vdash^{S_5} \Box(\theta \vee \Box\phi)$, so $\Box\phi \vdash^{S_5} \Box(\theta \vee \Box\phi)$ by using ($\sigma0$), **2** etc. $\therefore$ by DIS and Proposition 2, $\Box\theta \vee \Box\phi \vdash^{S_5} \Box(\theta \vee \Box\phi)$, as required.

($\sigma4$) This is proved exactly like ($\sigma3$) but using ($\sigma1$) in place of ($\sigma0$).

8. Using b for 'Bill is telling the truth' and m for 'Monica is telling the truth' the validity of the argument

It is not possible that Bill and Monica are both telling the truth
Possibly Monica is telling the truth

$\therefore$ *It is not necessarily the case that Bill is telling the truth* amounts to the validity of $\neg\Diamond(b \wedge m), \Diamond M \vdash^J \neg\Box b$. In fact this follows with $J = K$ since using Proposition **??** $\vdash^K \neg(b \wedge m) \to (b \to \neg m)$ so by NEC (and some sentential stuff)

$$\vdash^K \Box\neg(b \wedge m) \to \Box(b \to \neg m) \tag{1}$$

Since $\vdash^K \neg\Diamond(b \wedge m) \leftrightarrow \Box\neg(b \wedge m)$ we get $\neg\Diamond(b \wedge m) \vdash^K \Box\neg(b \wedge m)$ and using (1) we obtain $\neg\Diamond(b \wedge m) \vdash^K \Box(b \to \neg m)$. Applying the K axiom (plus some usual sentential stuff) gets us to $\neg\Diamond(b \wedge m) \vdash^K \Box b \to \Box\neg m$ and hence to $\neg\Diamond(b \wedge m) \vdash^K \neg\Box\neg m \to \neg\Box b$ and finally to $\neg\Diamond(b \wedge m), \neg\Box\neg m \vdash^K \neg\Box b$ which is what we want since $\neg\Box\neg m = \Diamond m$.

9. Suppose that $\Gamma_1 \,|\, \theta_1, \dots, \Gamma_n \,|\, \theta_n$ was a proof which mentioned some propositional variable $p \notin \mathcal{L}$, where $\Gamma_n \subseteq SM\mathcal{L}$ and $\theta_n \in SM\mathcal{L}$. Let $q \in \mathcal{L}$ and for a sentence ϕ let ϕ^* be the result of replacing p everywhere in ϕ by q. By induction on $|\phi|$ for $\phi \in SML$ it is easy to see that ϕ^* is also a sentence. Furthermore $\Gamma_1^* \,|\, \theta_1^*, \dots, \Gamma_n^* \,|\, \theta_n^*$ is still a proof. The reason for this, as you can easily convince yourself, is that if $\phi \,|\, \phi$ is an instance of REF then so is $\phi^* \,|\, \phi^*$ whilst if

$$\frac{\Omega_1 \,|\, \psi_1, \dots, \Omega_t \,|\, \psi_t}{\Omega \,|\, \psi}$$

is an instance of a rule then

$$\frac{\Omega_1^* \,|\, \psi_1^*, \dots, \Omega_t^* \,|\, \psi_t^*}{\Omega^* \,|\, \psi^*}$$

is an instance of that same rule. Also since Γ_n, θ_n don't mention p, $\Gamma_n^* \,|\, \theta_n^* = \Gamma_n \,|\, \theta_n$. So $\Gamma_1^* \,|\, \theta_1^*, \dots, \Gamma_{n-1}^* \,|\, \theta_{n-1}^*, \Gamma_n \,|\, \theta_n$ is still a proof with the same final sequent which mentions one less propositional variable not in $\mathcal{L}$ than $\Gamma_1 \,|\, \theta_1, \dots, \Gamma_n \,|\, \theta_n$. Repeating for each of the finitely many (because the Γ_i are finite) mentioned propositional not in $\mathcal{L}$ enables us to remove them all and obtain a proof which remains entirely within the language $\mathcal{L}$.

10. (i) No, because $(a, c) \in E$ but $c \nvDash p$.

(ii) Yes, because $(a, b) \in E$ and $b \vDash p$.
(iii) Yes, because $c, b \vDash \Box p$.
(iv) Yes, because $c \vDash \neg p$, $a \vDash \Diamond\neg p$, $c \vDash \Box\Diamond\neg p$, $a \vDash \Diamond\Box\Diamond\neg p$.
(v) Yes, because $c \vDash p \to \Box p$ and $b \vDash p \to \Box p$.

11. By the Correctness (or Completeness) Theorem for K it is enough to find a frame $\langle W, E, V\rangle$ and a world $i \in W$ at which these sentences fail. Suitable frames are:

(i) $W = \{i\}$, $E = \emptyset$, $V_i(p) = 1$

(ii) $W = \{i, j, k\}$, $E = \{(i, j), (i, k)\}$, $V_i(p) = V_i(q) = 1'$ $V_j(p) = 1$, $V_j(q) = 0$, $V_k(p) = 0$, $V_k(q) = 1$

(iii) $W = \{i, j\}$, $E = \{(i, j)\}$, $V_i(p) = 1$, $V_j(p) = 0$

(iv) $W = \{i, j\}$, $E = \{(i, j)\}$, $V_i(p) = 0$, $V_j(p) = 1$

(v) $W = \{i, j, k\}$, $E = \{(i, j), (j, k)\}$, $V_i(p) = V_j(p) = 1$, $V_k(p) = 0$

12. Let $\langle W, E, V\rangle$ be a frame and $i \in W$ throughout.

(i) If $i \nvDash \Diamond(\theta \wedge \phi)$ then $i \vDash \Diamond(\theta \wedge \phi) \to (\Diamond\theta \wedge \Diamond\phi)$ as required. So suppose $i \vDash \Diamond(\theta \wedge \phi)$. [We'd normally take this previous step as read, indeed I'll do so from now on.] Then $j \vDash \theta \wedge \phi$ for some $(i, j) \in E$. $\therefore j \vDash \theta$ and $j \vDash \phi$ so, since $(i, j) \in E$, $i \vDash \Diamond\theta$ and $i \vDash \Diamond\phi$. $\therefore i \vDash \Diamond\theta \wedge \Diamond\phi$ so again $i \vDash \Diamond(\theta \wedge \phi) \to (\Diamond\theta \wedge \Diamond\phi)$. Since the frame and world i in the frame were arbitrary this shows that $\models^K \Diamond(\theta \wedge \phi) \to (\Diamond\theta \wedge \Diamond\phi)$ and by the Completeness Theorem for K this also holds for $\vdash^K$ in place of $\models^K$. [In the examples which follow we will miss out this common last step of appealing to the Completeness Theorem etc..]

(ii) Suppose $i \models \Box(\theta \to \phi)$ and $i \models \Diamond\theta$. Then $j \models \theta$ for some $(i,j) \in E$ and also for this j, $j \models \theta \to \phi$ since $i \models \Box(\theta \to \phi)$ and $(i,j) \in E$. $\therefore$ $j \models \phi$. $\therefore$ $i \models \Diamond\phi$, again since $(i,j) \in E$. $\therefore$ $i \models \Box(\theta \to \phi) \to (\Diamond\theta \to \Diamond\phi)$, as required.

(iii) Suppose $i \models \Box\theta \wedge \Diamond\phi$. Then $i \models \Box\theta$ and $i \models \Diamond\phi$ so $j \models \phi$ for some $(i,j) \in E$. $\therefore$ also $j \models \theta$ since $i \models \Box\theta$, so $j \models \theta \wedge \phi$. $\therefore$ $i \models \Diamond(\theta \wedge \phi)$. $\therefore$ $i \models (\Box\theta \wedge \Diamond\phi) \to \Diamond(\theta \wedge \phi)$, as required.

13. Assume that $\Omega \supseteq \{\, \Diamond\theta \,|\, \theta \in \Lambda \,\}$ and $\Box\theta \in \Omega$. We need to show that $\theta \in \Lambda$. Suppose not. Then since Λ is maximal $\neg\theta \in \Lambda$. $\therefore$ $\Diamond\neg\theta \in \Omega$, from our initial assumption, so $\Omega \vdash^K \neg\Box\theta$ since $\Diamond\neg\theta \vdash^K \neg\Box\theta$. But also $\Omega \vdash^K \Box\theta$ since $\Box\theta \in \Omega$, so Ω is inconsistent, contradiction. We conclude that $\theta \in \Lambda$, as required.

In the other direction suppose that $\Lambda \supseteq \{\, \theta \,|\, \Box\theta \in \Omega\}$ and $\theta \in \Lambda$. We need to show that $\Diamond\theta \in \Omega$. Suppose not. Then since Ω is maximal, $\neg\Diamond\theta \in \Omega$. Hence $\Omega \vdash^K \Box\neg\theta$ since $\neg\Diamond\theta \vdash^K \Box\neg\theta$. It follows that $\Box\neg\theta \in \Omega$ (otherwise $\neg\Box\neg\theta \in \Omega$ and Ω will be inconsistent). Hence by our initial assumption $\neg\theta \in \Lambda$ and since also $\theta \in \Lambda$ this means Λ is inconsistent, contradiction. We conclude that, as required, $\Diamond\theta \in \Omega$.

14. Let $\langle W, E, V\rangle$ be a serial frame and $i \in W$. By seriality(!) $(i,j) \in E$ for some $j \in W$. Either $j \models p$ or $j \models \neg p$. $\therefore$ either $j \not\models \neg p$ or $l \not\models p$ so since $(i,j) \in E$, either $i \not\models \Box\neg p$ or $i \not\models \Box p$. $\therefore$ $i \not\models \Box p \wedge \Box\neg p$. $\therefore$ by the Completeness Thm for D, $\vdash^D \neg(\Box p \wedge \Box\neg p)$.

But in the symmetric frame with $W = \{i\}$, $E = \emptyset$, $V_i(p) = 1$ (or 0). $i \models \Box p$ and $i \models \Box\neg p$ so $i \models \Box p \wedge \Box\neg p$. $\therefore$ $\not\models^B \neg(\Box p \wedge \Box\neg p)$ so by the Completeness Thm for B, $\nvdash^B \neg(\Box p \wedge \Box\neg p)$.

For the converse it is clearly enough to show that $\nvdash^D p \to \Box\Diamond p$ since $\vdash^B p \to \Box\Diamond p$. So by the Completeness Thm for D it is enough to exhibit a serial frame $\langle W, E, V\rangle$ and $i \in W$ such that $i \not\models p \to \Box\Diamond p$. In this case a suitable frame is $W = \{0,1\}$, $E = \{\,(0,1),(1,1)\}$, $V_0(p) = 1$, $V_1(p) = 0$.

15. The proof is by induction on $|\theta|$. If $\theta = p$ then, since $u \in W$, $V_u(p) = V'_u(p)$ so the result is true. Assume now it holds for all $|\phi| < |\theta|$. If $\theta = \neg\phi$ then

$$\begin{aligned}
\langle W, E, V\rangle,\, u \models \neg\phi \quad &\Longleftrightarrow \quad \langle W, E, V\rangle,\, u \not\models \phi, \\
&\Longleftrightarrow \quad \langle W', E', V'\rangle,\, u \not\models \phi \quad \text{by IH} \\
&\Longleftrightarrow \quad \langle W', E', V'\rangle,\, u \models \neg\phi, \quad \text{as required,}
\end{aligned}$$

and similarly for the other connectives.

Finally if $\theta = \Box\phi$ then

$$\begin{aligned}
\langle W, E, V\rangle,\, u \models \Box\phi \quad &\Longleftrightarrow \quad \forall t,\ (u,t) \in E \Rightarrow \langle W, E, V\rangle,\, t \models \phi \\
&\Longleftrightarrow \quad \forall t,\ (u,t) \in E' \Rightarrow \langle W, E, V\rangle,\, t \models \phi \\
&\qquad\quad \text{since because } w' \neq u \in W,\ (u,t) \in E' \iff (u,t) \in E, \\
&\Longleftrightarrow \quad \forall t,\ (u,t) \in E' \Rightarrow \langle W', E', V'\rangle,\, t \models \phi \\
&\qquad\quad \text{by IH since } (u,t) \in E' \Rightarrow t \in W, \\
&\Longleftrightarrow \quad \langle W', E', V'\rangle,\, u \models \Box\phi, \quad \text{as required.}
\end{aligned}$$

To show the second part suppose that $\vdash^K \Box\theta$ but $\nvdash^K \theta$. Then by the Completeness thm for K there is a frame $\langle W, E, V\rangle$ and $w \in W$ such that $w \models \neg\theta$. Let $\langle W', E', V'\rangle$, w' be as above.

Since $\vdash^K \Box\theta$, by Completeness $\langle W', E', V'\rangle$, $w' \vDash \Box\theta$ so $\langle W', E', V'\rangle$, $w \vDash \theta$ since $(w', w) \in E'$, contradicting the first part above. $\therefore \vdash^K \theta$.

For the last part it is clear that since $\overline{K}$ extends K if $\vdash^K \theta$ then $\vdash^{\overline{K}} \theta$. In the other direction suppose that $\vdash^{\overline{K}} \theta$, say that $\Gamma_1 \,|\, \theta_1, \dots, \Gamma_n \,|\, \theta_n$ was a proof in $\overline{K}$ of $\vdash^{\overline{K}} \theta$. We show by induction on m for $m = 1, 2, \dots, n$ that $\Gamma_m \vdash^K \theta_m$ (which gives the result). Suppose that $\Gamma_j \vdash^K \theta_j$ for $j < m$. If $\Gamma_m \,|\, \theta_m$ is justified in this prof by one of the axioms or rules of K then $\Gamma_m \vdash^K \theta_m$ follows as usual by Proposition 1. Otherwise the rule must be the new one, $\Gamma_m \,|\, \theta_m$ must be $|\, \theta_m$ and some earlier $\Gamma_j \,|\, \theta_j$ must be $|\, \Box\theta_m$. By IH $\vdash^K \Box\theta_m$ so by the second part $\vdash^K \theta_m$, i.e $\Gamma_m \vdash^K \theta_m$, as required.

16. $\Leftarrow$ Suppose that $\Gamma_1 \,|\, \theta_1, \dots, \Gamma_n \,|\, \theta_n$ was a proof of $\Gamma \vdash^{K4} \theta$. We prove by induction on $m = 1, 2, \dots, n$ that if $\langle W, E, V\rangle$ is a transitive frame and $i \in W$ and $i \vDash \Gamma_m$ then $i \vDash \theta_m$, i.e. $\Gamma_m \vDash^{K4} \theta_m$. So assume that $\Gamma_j \vDash^{K4} \theta_j$ for $j < m$. If $\Gamma_m \,|\, \theta_m$ is justified in this proof either by being an axiom or rule of K then just the same argument as was used for K shows that $\Gamma_m \vDash^{K4} \theta_m$. Thus the only new case is when $\Gamma_m \,|\, \theta_m$ is an instance of the new axiom $\Box\theta \,|\, \Box\Box\theta$. In this case suppose that $i \vDash \Box\theta$. To show that $i \vDash \Box\Box\theta$ we must show $j \vDash \Box\theta$ when $(i, j) \in E$. So let $(j, r) \in E$. By transitivity $(i, r) \in E$ so $r \vDash \theta$ (since $i \vDash \Box\theta$). $\therefore j \vDash \Box\theta$ and hence $i \vDash \Box\Box\theta$. $\therefore \Gamma_m \vDash^{K4} \theta_m$. This completes the induction. Finally, since $\Gamma \supseteq \Gamma_n$ and $\theta = \theta_n$ we obtain $\Gamma \vDash^{K4} \theta$, as required.

$\Rightarrow$ To prove this direction we assume $\Gamma \nvdash^{K4} \theta$ and construct a frame $\langle W, E, V\rangle$ and $i \in W$ such that $i \vDash \Gamma$ but $i \nvDash \theta$. The construction follows exactly the course as in the notes for K but with $\vdash^{K4}$ in place of $\vdash^K$. The only new feature is that we must check that the resulting frame is transitive. This amounts to checking that if $\Omega, \Lambda, \Psi \in W$ are maximal $K4$-consistent subsets of SML and $(\Omega, \Lambda), (\Lambda, \Psi) \in E$, i.e.

$$\{\, \eta \,|\, \Box\eta \in \Omega \,\} \subseteq \Lambda, \quad \{\, \eta \,|\, \Box\eta \in \Lambda \,\} \subseteq \Psi, \tag{2}$$

then $(\Omega, \Psi) \in E$, i.e. $\{\, \eta \,|\, \Box\eta \in \Omega \,\} \subseteq \Psi$. So assume (2) and let $\theta \in \{\, \eta \,|\, \Box\eta \in \Omega \,\}$. Then $\Box\theta \in \Omega$ so $\Omega \vdash^{K4} \theta$. But also $\Box\theta \vdash^{K4} \Box\Box\theta$ (using the new axiom) so $\Omega \vdash^{K4} \Box\Box\theta$ and by the first of the proven properties of max.con. sets, $\Box\Box\theta \in \Omega$. θ $\Box\theta \in \Lambda$ and $\theta \in \Psi$ by (2). We have shown that $\{\, \eta \,|\, \Box\eta \in \Omega \,\} \subseteq \Psi$, as required.

17. Let $\langle W, E, V\rangle$ be a frame with $W = \{a, b\}$, $E = \{(a, b), (b, a)\}$ and $V_a = V_b$. Clearly this is not-transitive since $(a, b), (b, a) \in E$ but $(a, a) \notin B$. Clearly also, by symmetry, for any $\phi \in SML$, $a \vDash \phi \iff b \vDash \phi$. $\therefore$ if $a \vDash \Box\phi$ then $b \vDash \Box\phi$ so $a \vDash \Box\Box\phi$. $\therefore a \vDash \Box\phi \to \Box\Box\phi$ and similarly for b.

18. Suppose $i \in \mathbb{N}$ and $i \vDash \Diamond\Box\phi$ and $i \vDash \Box(\Box\phi \to \phi)$. Then, from the first of these and the definition of E, $m \vDash \Box\phi$ for some $i < m \in \mathbb{N}$, so $n \vDash \phi$ for all $m < n \in \mathbb{N}$, whilst from the second $\forall j > i$, $j \vDash \Box\phi \to \phi$. Hence $m \vDash \Box\phi to\phi$ so $m \vDash \phi$. $\therefore k \vDash \phi$ for all $k \geq m$. $\therefore$ $m - 1 \vDash \Box\phi$. If $i = m - 1$ then $i \vDash \Box\phi$, showing in general, that

$$i \vDash \Diamond\Box\phi \to (\Box(\Box\phi \to \phi) \to \Box\phi).$$

On the other hand if $i < m - 1$ then repeating the argument again with $m - 1$ in place of m shows that $m - 2 \vDash \Box\phi$. Clearly, by the well foundedness of the usual ordering on $\mathbb{N}$, we can continue this descent until we do reach i, and hence $i \vDash \Box\phi$, giving the required results as above.

This proof relied on the well foundedness of $\mathbb{N}$. It fails with $\mathbb{Q}$ in place of $\mathbb{N}$ because, e.g. $\{\, r \in \mathbb{Q} \,|\, \sqrt{2} < q \,\}$ has no glb in $\mathbb{Q}$. E.g. take $V_q(p) = 0$ if $q < \sqrt{2}$, $V_q(p) = 1$ if $\sqrt{2} < q$ and $\phi = p$, $i = 0$.

19. Let $\mathcal{J}$ be the set of serial frames $\langle W, E, V\rangle$ such that if $(i,j),(j,k) \in E$ then $(k,i) \in E$. Write $\Gamma \vDash^{K_3} \theta$ if for all frames $\langle W, E, V\rangle \in \mathcal{J}$ and $i \in W$, if $i \vDash \Gamma$ then $i \vDash \theta$. Claim that for any $\langle W, E, V\rangle \in \mathcal{J}$ and $i \in W$, if $i \vDash \theta$ then $i \vDash \Diamond\Diamond\Diamond\theta$. Since, by seriality there are some $j, k \in W$ such that $(i,j),(j,k) \in E$, and hence by definition of $\mathcal{J}$, $(k,i) \in E$. So from $i \vDash \theta$, $k \vDash \Diamond\theta$, $j \vDash \Diamond\Diamond\theta$ and $i \vDash \Diamond\Diamond\Diamond\theta$, as required. $\therefore$ the schema $\theta \,|\, \Diamond\Diamond\Diamond\theta$ is satisfied in all frames in $\mathcal{J}$.

Now suppose that $p \vdash K_3\Diamond\Diamond p$, say $\Gamma_1 \,|\, \theta_1, \ldots, \Gamma_m \,|\, \theta_m$ was a proof of this. We show by induction on $k = 1, 2, \ldots, m$ that $\Gamma_k \vDash^{K_3} \theta_k$. If $\Gamma_k \,|\, \theta_k$ is an axiom of K or rule of K then the result follows as in the course notes. The only essentially new case is when $\Gamma_k \,|\, \theta_k$ is an instance of the new axiom $\theta \,|\, \Diamond\Diamond\Diamond\theta$. But in this case the above argument has shown that $\theta \vDash^{K_3} \Diamond\Diamond\Diamond\theta$. By induction then $\Gamma_k \vDash^{K_3} \theta_m$, so $p \vDash^{K_3} \Diamond\Diamond p$. But this is refuted at vertex 1 in the frame (which is in $\mathcal{J}$) with $W = \{1,2,3\}$, $E = \{(1,2),(2,3),(3,1)\}$ $V_1(p) = 1$, $V_2(P) = V_3(p) = 0$. $\therefore p \nvDash^{K_3} \Diamond\Diamond p$.

20. (i) $0 \nvDash p$ so $1 \nvDash \Box p$ (since $(1,0) \in E$) and $2 \nvDash \Box\Box p$ (since $(2,1) \in E$). Also

$$(2,n) \in E \Rightarrow n \in \{1,2,3,\ldots\} \Rightarrow n \vDash p$$

so $2 \vDash \Box p$. $\therefore 2 \vDash \Box p \wedge \neg\Box\Box p$, i.e. $2 \nvDash \Box p \rightarrow \Box\Box p$.

(ii) Suppose that $n \nvDash \Box\theta \rightarrow \Box\Box\theta$, so $n \vDash \Box\theta$, $n \nvDash \Box\Box\theta$, but $n \vDash \Box(\Box\Box\theta \rightarrow \Box\Box\Box\theta)$. First notice that if $n \leq m$ then $(n,i) \in E \Rightarrow (m,i) \in E$ so since $n \vDash \Box\theta$, also $m \vDash \Box\theta$. $\therefore n{+}1 \vDash \Box\Box\theta$ since —$(n+1,i) \in E \Rightarrow n \leq i$. Since $n \vDash \Box(\Box\Box\theta \rightarrow \Box\Box\Box\theta)$, $n+1 \vDash \Box\Box\theta \rightarrow \Box\Box\Box\theta$ so $n+1 \vDash \Box\Box\Box\theta$. $\therefore n \vDash \Box\Box\theta$, contradiction.

From (i),(ii) and the fact that the frame $\langle W, E, V\rangle$ is reflexive (so $\Box\theta \rightarrow \theta$ is true at all worlds) we see that this is a frame for J which satisfies $\neg(\Box p \rightarrow \Box\Box p)$ at some world. However $\neg(\Box p \rightarrow \Box\Box p)$ cannot be satisfied in any finite frame $\langle W', E', V'\rangle$ for J. Since suppose it was, at world $a_1 \in W'$ say. By (ii) then $a_1 \nvDash \Box(\Box\Box p \rightarrow \Box\Box\Box p)$ so there must be a world $a_2 \in W'$ such that $(a_1, a_2) \in E$ and $a_2 \nvDash \Box\Box p \rightarrow \Box\Box\Box p$. Applying (ii) again now at a_2 with $\Box p$ in place of p shows the existence of a world $a_3 \in W'$ such that $(a_2, a_3) \in E$ and $a_3 \nvDash \Box\Box\Box p \rightarrow \Box\Box\Box\Box p$. Continuing in this way we obtain worlds $a_1, a_2, a_3, a_4, \ldots$ such that $a_1 \vDash \Box p$, $a_1 \nvDash \Box^2 p$, $a_2 \vDash \Box^2 p$, $a_2 \nvDash n^3 p$, $a_3 \vDash \Box^3 p$, $a_3 \nvDash \Box^4 p$, $a_4 \vDash \Box^4 p, \ldots$. But since the T axiom is true at every world in W', if $a_i \vDash \Box^{i+1} p$ then $a_i \vDash \Box^j p$ for all $j \leq i+1$. Hence the $a_1, a_2, a_3, \ldots$ must all be distinct, contradicting the finiteness of $\langle W', E', V'\rangle$.